Compatibility conditions from multipartite entanglement measures

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We consider an arbitrary $d_1 \otimes d_2 \otimes \cdots \otimes d_N$ composite quantum system and find necessary conditions for general $m$-party subsystem states to be the reduced states of a common $N$-party state. These conditions will lead to various monogamy inequalities for bipartite quantum entanglement and partial disorder in multipartite states. Our results are tightly connected with the measures of multipartite entanglement.

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I. INTRODUCTION

The quantum analog of the marginal distributions of a joint probability distribution is to determine whether a given set of subsystem states $\{\rho_1, \rho_2, \cdots, \rho_{12}, \cdots, \rho_{12-j}\}$ comes from a single multipartite state. This kind of compatibility problem is closely related to condensed matter physics and chemical physics, where the significative conditions will enable us to design powerful variational methods to substantially simplify the computation of many physical variants. People have utilized different ideas and techniques to attack the tough problem of compatibility of quantum states and obtained several important partial results [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. However, it is still very difficult to present compatibility conditions for a set of general $m$-party reduced states and a multipartite state. The difficulty comes from the phenomenon of multipartite entanglement. Since the number of local invariants in $N$-party density matrices will increase exponentially with the particle number $N$ [14], how to quantify the measure and characterize the structure for multipartite entanglement becomes obscure. Nevertheless, due to the substantial significance for distributed quantum information processing and strongly correlated physics [14, 15], many efforts for exploring multipartite entanglement have been done [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

In Refs. [17, 18], Mintert and co-workers define multipartite concurrence with a single factorizable observable, which can effectively characterize quantum correlations in an $N$-party pure state. From the information viewpoint, we propose a polynomial stochastic local operations and classical communication (SLOCC) [29] invariant $E_{12-\cdots-N}$ in [28], and show that $E_{1234}$ satisfies the necessary conditions for a natural entanglement measure. By investigating the relations among different local invariants, it is possible for us to gain insight into the nature and complex structure of multipartite entanglement, as well as its connection with the general compatibility problem.

In this paper, we find several necessary conditions for general $k$-party density matrices to be the reduced states of a common arbitrary $d_1 \otimes d_2 \otimes \cdots \otimes d_N$ multipartite state. The main idea is to establish a direct connection between the information-theoretic measure of multiqubit entanglement and the monogamy of pure states [30] and partial disorder in multipartite states. Our results reveal explicitly the relationship between multipartite entanglement measures, the general compatibility problem and the monogamous nature of entanglement.

The structure of this paper is as follows. In Sec. II we first establish the relationship between information-theoretic entanglement measure and multipartite concurrence, from which we derive compatibility conditions for general $d_1 \otimes d_2 \otimes \cdots \otimes d_N$ multipartite states in Sec. III. As applications of these compatibility conditions, in Sec. IV we present various monogamy inequalities for bipartite quantum entanglement and partial disorder in multipartite states. In Sec. V are the conclusions.

II. INFORMATION-THEORETIC ENTANGLEMENT MEASURE AND MULTIPARTITE CONCURRENCE

Consider a pure state $|\psi_N\rangle$ of $N$ qubits, labeled as $1, 2, \cdots, N$; generally we can write $|\psi_N\rangle = |\psi\rangle_1 \otimes \cdots \otimes |\psi\rangle_M$, where $|\psi\rangle_m$ are non-product pure states, $m = 1, 2, \cdots, M$, and the qubits of different $|\psi\rangle_m$ have no intersection. Here, we consider the nontrivial case $|A| = 1$, i.e. $|\psi_N\rangle$ itself is not a product pure state. Any bipartite partition $P = A|B$ will divide all the qubits into two subsets $A$ and $B$. We denote the number of qubits contained in $A, B$ as $|A|$ and $|B|$ respectively. For $N \in \text{even}$, there are two different kinds of bipartite partitions $P_T$ and $P_{TT}$. If $|A|, |B| \in \text{odd}$, $P \in P_T$, otherwise if $|A|, |B| \in \text{even}$, $P \in P_{TT}$. Using the linear entropy [31], the mutual information between $A$ and $B$ is $S_{A|B} = S_A + S_B - S_{AB}$, where $S_Y = 1 - Tr\rho_Y$, $Y = A, B, AB$. Since $|\psi_N\rangle$ is a pure state, we can write $S_{A|B} = 2(1 - Tr\rho_A^2) = 2(1 - Tr\rho_B^2)$, where $\rho_A$ and $\rho_B$ are the reduced density matrices. The information-theoretic measure of multi-qubit entangle-
ment is defined as 28

$$\mathcal{E}_{12\ldots N} = \sum_{P \in \mathcal{P}_I} S_P - \sum_{P \in \mathcal{P}_{II}} S_P$$

(1)

where $S_P$ is the mutual information for bipartite partition $P$. It has shown been that $\mathcal{E}_{12\ldots N}$ is a polynomial SLOCC invariant, and is unchanged under permutations of qubits, i.e., it represents a collective property of all the $N$ qubits 28. In the following, we will connect $\mathcal{E}_{12\ldots N}$ with a single factorizable observable, and as a product, we prove that $\mathcal{E}_{12\ldots N} \geq 0$ is satisfied for pure states of an arbitrary even number qubits.

We can see that the bipartite mutual information $S_P$ is a quadratic polynomial function of the elements of the reduced density matrices. For any $n$th-degree polynomial function $f$ of the elements of a density matrix $\rho$, one could always find an observable $A$ on $n$ copies of $\rho$, without quantum state tomography 22. $f(\rho) = Tr(A\rho^n)$. Several important experimental schemes for measuring nonlinear properties of quantum states through multiple copies of quantum states have been also proposed 33, 34. In particular, Mintert and co-workers show that it is possible to measure the multipartite concurrence of pure states by detecting a single factorizable observable on two copies of the composite states 17, 18, 35.

Given an $N$-partite pure state

$$|\psi_N\rangle \in \mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

where $\mathcal{H}_i$ is the Hilbert space associated to the $i$th subsystem and $N \in \text{even}$. We can define two operators $P^1_i$ and $P^2_i$ as the projectors onto the symmetric and antisymmetric subspace of the Hilbert space $\mathcal{H}_i \otimes \mathcal{H}_i$, which describes the two copies of the $i$th subsystem. Using these two kinds of projectors, a set of factorizable observables are introduced 17, 18

$$A_{s_1s_2\ldots s_N} = P^1_{s_1} \otimes P^2_{s_2} \otimes \cdots \otimes P^N_{s_N}$$

(2)

where $s_1, s_2, \ldots, s_N = \pm, \mp$. The expectation value of such an observable $A_{s_1s_2\ldots s_N}$ with respect to $|\psi_N\rangle \otimes |\psi_N\rangle$ is represented as

$$\langle A_{s_1s_2\ldots s_N} \rangle = \langle \psi_N | \langle \psi_N | A_{s_1s_2\ldots s_N} | \psi_N \rangle | \psi_N \rangle$$

(3)

For simplicity, we denote the number of $-\$ in $\{s_1, s_2, \ldots, s_N\}$ as $N_a(\{s_1, s_2, \ldots, s_N\})$. If $N_a(\{s_1, s_2, \ldots, s_N\}) \in \text{odd}$, then $\langle A_{s_1s_2\ldots s_N} \rangle = 0$. For general $N$-party mixed states described by the density matrix $\rho$, the purity of $\rho$ can be expressed through the expectation values of different observables 17, 18

$$Tr\rho^2 = 1 - 2 \sum_{N_a(\{s_1, s_2, \ldots, s_N\}) \in \text{odd}} Tr(A_{s_1s_2\ldots s_N} \rho^{\otimes 2})$$

(4)

With this knowledge in hand, we present the following lemma about the information-theoretic multi-qubit entanglement measure $\mathcal{E}_{12\ldots N}$ for general even number $N$.

**Lemma 1** For pure states of even number $N$ qubits, the information-theoretic measure of multi-qubit entanglement $\mathcal{E}_{12\ldots N}$ defined in Eq.(1) satisfies

$$\mathcal{E}_{12\ldots N} = 2^N \langle P^1 \otimes P^2 \otimes \cdots \otimes P^N \rangle \geq 0$$

(5)

**Proof.** The mutual information of a bipartite partition $P = A|B$ is $S_{A|B} = 2(1 - Tr\rho^2_A)$. We denote the index set $A$ and $B$ as $\{a_1, a_2, \ldots, a_{|A|}\}$, $\{b_1, b_2, \ldots, b_{|B|}\}$ respectively. According to Eq.(4), $S_{A|B}$ can be expressed as follows

$$S_{A|B} = 4 \sum_{\substack{N_a(\{s_1|\in A\}) \in \text{odd}, \ N_a(\{s_1|\in B\}) \in \text{odd}}} \langle A_{s_1s_2\ldots s_N} \rangle$$

where $A_{s_1s_2\ldots s_N}$ are the factorizable observables defined in Eq.(2) on $\mathcal{H} \otimes \mathcal{H}$. In the above derivation, we have used two important properties that $P^1_i + P^2_i = I$ and $\langle A_{s_1s_2\ldots s_N} \rangle = 0$ when $N_a(\{s_1, s_2, \ldots, s_N\}) \in \text{odd}$ for pure states. If the expectation value of some observable $\langle A_{s_1s_2\ldots s_N} \rangle$ contributes to the mutual information for some bipartite partition $P = A|B \in \mathcal{P}_{II}$, there must exist one minimum index, denoted as $X$ that $s_X = \mp$. If $X \in A$, $\langle A_{s_1s_2\ldots s_N} \rangle$ will also contribute to the mutual information for some bipartite partition $P' = A + \{X\} \otimes |B\rangle \rangle \in \mathcal{P}_{II}$. Otherwise if $X \in B$, $\langle A_{s_1s_2\ldots s_N} \rangle$ will contribute to the mutual information for some bipartite partition $P' = A + |X\rangle \rangle \otimes |B\rangle \rangle \in \mathcal{P}_{II}$. Conversely, if the expectation value of some observable $\langle A_{s_1s_2\ldots s_N} \rangle$ contributes to the mutual information for some bipartite partition $P \in \mathcal{P}_{II}$, as long as $A_{s_1s_2\ldots s_N} \neq P^1_i \otimes P^2_i \otimes \cdots \otimes P^N_i$, it will also contribute to the mutual information for some bipartite partition $P \in \mathcal{P}_{II}$. According to this corresponding relation, and noting that the number of bipartite partitions in $\mathcal{P}_{II}$, $\mathcal{P}_{II}$ is $2^{N-2}$ and $2^{N-2} - 1$ respectively, we can easily obtain that

$$\mathcal{E}_{12\ldots N} = \sum_{P \in \mathcal{P}_{II}} S_P - \sum_{P \in \mathcal{P}_{II}} S_P = 2^N \langle P^1 \otimes P^2 \otimes \cdots \otimes P^N \rangle.$$
out in the special concurrence defined through $16 \langle P_1 \otimes P_2 \otimes P_3 \otimes P_4 \rangle$ for pure states of four qubits can effectively characterize separability properties independent of any pairing of subsystems, i.e., vanishes for any state where at least one subsystem is uncorrelated with all other system components. This also supports our proposed information-theoretic measure for multi-qubit entanglement. Moreover, our measure comes from the information-theoretic viewpoint, thus the relation in Eq. (5) shows that Mittent concurrence also reflects the entanglement measure and multipartite concurrence. We should note that lemma 1 is only valid for an even number of subsystems. Nevertheless, from the physical point of view, there is nothing fundamentally different, as for what concerns entanglement, between systems with an even number of subsystems and those with an odd number. This kind of limitation stems from the mathematics foundation of the information-theoretic measure for multi-qubit entanglement. If $N$ is an odd number, two different kinds of bipartite partitions $P_T$ and $P_{HT}$ do not exist anymore, while $\langle P_1 \otimes P_2 \otimes \cdots \otimes P_N \rangle$ will always vanish in coincidence.

III. COMPATIBILITY CONDITIONS FOR MULTIPARTITE STATES

The compatibility problem can be formulated as in Eq. (13): Given states of all proper subsystems of a multipartite quantum system, what are the necessary and sufficient conditions for these subsystem states to be compatible with a single entire system? A number of partial results have been obtained through different ideas and approaches. Higuchi et al. find the necessary and sufficient conditions for the possible one-qubit reduced states of a pure multi-qubit state. This result is then generalized for $3 \otimes 3 \otimes 3$ and for $2 \otimes 2 \otimes 4$ systems. An interesting necessary condition for an odd $n$-party state has also been proposed in Eq. (12). In Eq. (8), Christandl et al. establish a connection between the compatibility problem and the representation theory of the symmetric group. Most recently, Hall has attack the compatibility problem from a novel angel, i.e., utilizing the ideas of convexity. However, there are very few necessary criteria for the general form of the compatibility problem. In this section, we will derive a set of compatibility conditions for general $d_1 \otimes d_2 \otimes \cdots \otimes d_N$ multipartite states from the relationship between the information-theoretic entanglement measure and multipartite concurrence.

In Eq. (5), the invariant $E_{12\ldots N}$ is directly determined by the properties of reduced density matrices, therefore based on the above lemma, it is easy for us to get a simple necessary condition for the compatibility problem of arbitrary $d_1 \otimes d_2 \otimes \cdots \otimes d_N$ composite systems from multi-qubit entanglement measures.

**Theorem 1** Given a set of density matrices \{ρ₁, · · · , ρₙ, ρ₁₂..N−1, · · · , ρ₁₂..N−1, · · · , ρ₂..N−1\}, if they come from one common $N$-party pure states $|ψ_N\rangle$, the following inequality should be satisfied

$$\sum_{|A|\in \text{odd}} Trρ_A^2 - \sum_{|A|\in \text{even}} Trρ_A^2 \leq 1$$

(6)

where

$\mathcal{A} \subseteq \mathcal{N} = \{1, 2, \cdots, N\}$ and $\mathcal{A} \neq \emptyset$

**Proof.** First, we assume that $N \in \text{even}$, using the analysis in lemma 1, we can write the mutual information for a bipartite partition $\mathcal{P} = \mathcal{A}|\mathcal{B}$, with $\mathcal{A}, \mathcal{B} \neq \emptyset$, as $S_{\mathcal{A}|\mathcal{B}} = 2 - Trρ_A^2 - Trρ_B^2$. Therefore, $\sum_{P \in \mathcal{P}_I} S_P = 2 \cdot 2^{N-2} - \sum_{|A|\in \text{odd}} Trρ_A^2$ and $\sum_{P \in \mathcal{P}_I} S_P = 2 \cdot (2^{N-2} - 1) - \sum_{|A|\in \text{even}, \mathcal{A} \neq \mathcal{N}} Trρ_A^2$. This results in that $E_{12\ldots N} = 2 - \sum_{|A|\in \text{odd}} Trρ_A^2 + \sum_{|A|\in \text{even}, \mathcal{A} \neq \mathcal{N}} Trρ_A^2$. According to the result in Eq.(5) that $E_{12\ldots N} \geq 0$, and note that if $\mathcal{A} = \mathcal{N}$, $Trρ_A^2 = 1$, we prove that the above necessary condition is satisfied for general even number $N$. If $N \in \text{odd}$, it is obvious that $\sum_{|A|\in \text{odd}} Trρ_A^2 - \sum_{|A|\in \text{even}} Trρ_A^2 = 1$, thus we finish the proof of theorem 1.

The equality in theorem 1 will be satisfied when $E_{12\ldots N} = 0$, e.g. for general $N$-qubit W states $|W_N\rangle = \frac{1}{\sqrt{N}}(|00\cdots 01| + |00\cdots 010| + \cdots + |10\cdots 00\rangle)$. In the above derivation of theorem 1, the necessary condition for the compatibility problem follows from the fact that $\langle P_1 \otimes P_2 \otimes \cdots \otimes P_N \rangle \geq 0$. However, there are other similar factorizable observables such as $\langle P_{s_1} \otimes P_{s_2} \otimes \cdots \otimes P_{s_N} \rangle \geq 0$, with $N_a(s_1, s_2, \cdots, s_N) \in \text{even}$. Therefore, following the same idea, we can generalize the above compatibility condition to general multipartite mixed states.

**Theorem 2** Given an $N$-party state $\rho$, if $N = 2k$ is even, the reduced density matrices should satisfy the following inequalities

$$\sum_{|A|\in \text{odd}} Trρ_A^2 - \sum_{|A|\in \text{even}} Trρ_A^2 \leq 1$$

(7)

where

$\mathcal{A} \subseteq \mathcal{N} = \{1, 2, \cdots, 2k\}$ and $\mathcal{A} \neq \emptyset$

**Proof.** We could always find an ancillary system $\mathcal{R}$ and a pure state $|ψ\rangle_{\mathcal{N}|\mathcal{R}}$, such that $Tr_{\mathcal{R}} (|ψ\rangle_{\mathcal{N}|\mathcal{R}} \langle ψ|) = \rho$. In the similar way as the proof of theorem 1, using the expression of purity for reduced density matrices based on the expectation values in Eq.(4), and after some straightforward calculations, we can get

$$1 - \sum_{|A|\in \text{odd}} Trρ_A^2 + \sum_{|A|\in \text{even}} Trρ_A^2 = 2^{|N|} \langle P_1 \otimes P_2 \otimes \cdots \otimes P_{2k} \otimes P_{\mathcal{R}} \rangle,$$

where $P_{\mathcal{R}}$ is the projector onto the globally symmetric subspace of the Hilbert space $\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{R}$. Since the expectation value $\langle P_1 \otimes P_2 \otimes \cdots \otimes P_{2k} \otimes P_{\mathcal{R}} \rangle$ is always a non-negative value, we thus prove the necessary conditions in Eq.(7).
The compatibility conditions in theorem 1 and 2 directly come from the properties of multipartite entanglement measures, this basic idea is much different from previous works. In addition, as discussed in the beginning of this section, most known results on the compatibility problem are about one-party or two-party reduced states. However, our compatibility conditions are about general density matrices of all proper subsystems of an N-particle quantum system, including one-party, two-partite, ⋯, and (N − 1)-party states. Therefore, these compatibility conditions will be more powerful in the situation of general k-party reduced states. We could construct a simple example similar to the one in Eq. (10) to explicitly demonstrate the strength of our criteria. Consider the following states of all proper subsystems of a four-qubit system:

\[ \rho_1 = \rho_2 = \rho_3 = \rho_4 = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & \frac{2}{5} \end{pmatrix} \]

\[ \rho_{12} = \rho_{13} = \rho_{14} = \rho_{23} = \rho_{24} = \rho_{34} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \]

\[ \rho_{123} = \rho_{143} = \rho_{234} = |W\rangle\langle W| \]

\[ |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \] (8)

which are not compatible with an overall four-qubit state \( \rho_{1234} \) (not necessarily a pure state). In this example, the previous results can only determine whether one-party and two-party states come from a single four-qubit state. However, there are no known conditions for three-party reduced states. If we check the necessary condition in Eq. (7), it can be seen that \((1 \times 4 + \frac{3}{5} \times 4) - \left(\frac{3}{5} \times 6 + Tr\rho_{1234}^2\right) \leq 1\) will always not be satisfied. Therefore, theorem 2 confirms the incompatibility of the density matrices in Eq. (8).

IV. APPLICATIONS OF COMPATIBILITY CONDITIONS

Based on the above compatibility conditions, we could obtain a new kind of monogamy inequalities for bipartite quantum entanglement [30] and partial disorder in multipartite states. It is known that the entanglement measure for arbitrary dimension bipartite pure states, i.e. the square of I-concurrence [30], is also relevant to the purity of marginal density matrices. For a bipartite pure state \( |\Psi_{AB}\rangle \), the square of I-concurrence is defined as

\[ C_{AB}^2 = 2(1 - Tr\rho_A^2) = 2(1 - Tr\rho_B^2) \] (9)

where \( \rho_A \) and \( \rho_B \) are reduced density matrices. Therefore, according to theorem 1 and 2, we can derive a class of monogamy inequalities [30] for bipartite entanglement in general multipartite pure states as follows.

Corollary 1 Given an N-particle pure states \( |\psi_N\rangle \), there exist

\[ \sum_{|\Delta| \in \text{odd}} C_{A|N-\Delta}^2 \geq \sum_{|\Delta| \in \text{even}} C_{A|N-\Delta}^2 \] (10)

where \( I = \{i_1, i_2, \cdots, i_{2^n}\} \subseteq N \) and \( A \subseteq I \). Here, we assume that \( C_{A|N-\Delta}^2 = 0 \) when \( A = \emptyset \) or \( N \).

The above monogamy inequalities put new constrains on the distributed entanglement. It is valid not only for multipartite states of qubits, but also for arbitrary dimensions. The monogamous nature of entanglement is much relevant to quantum cryptography [30]. In the context of condensed matter physics, the monogamy property gives rise to some interesting effects, e.g. frustration in quantum spin systems. Therefore, these monogamy inequalities for bipartite entanglement may be valuable in many-body physics.

As another explicit application of our results, we consider a general \( d_1 \otimes d_2 \) bipartite mixed state \( \rho_{12} \). The disorder of a quantum state described by the density matrix \( \rho \) can be characterized by the mixedness \( D(\rho) = 1 - Tr\rho^2 \) [30], in which we neglect the normalization factor for simplicity. When \( \rho \) is a maximally mixed state, \( D(\rho) \) takes the maximum value. According to theorem 2, it is obvious that \( Tr\rho_1^2 + Tr\rho_2^2 - Tr\rho_{12}^2 \leq 1 \). Therefore, we can obtain a relation between the global disorder and local disorder

\[ D(\rho_{12}) \leq D(\rho_1) + D(\rho_2) \] (11)

This inequality demonstrates that the global disorder is always no larger than the sum of local disorder, which is an analog to the subadditivity of von Neumann entropy \( S(A, B) \leq S(A) + S(B) \). However, it is difficult to generalize this subadditivity based on von Neumann entropy to arbitrary multipartite mixed states. If we adopt \( D(\rho) \) as the measure of disorder and following the result in theorem 2, it is possible to achieve more general relations between global and local disorder. For example, for a general \( d_1 \otimes d_2 \otimes d_3 \otimes d_4 \) mixed state \( \rho_{1234} \), there exists the following relation between globally and locally disorder

\[ D(\rho_{1234}) + \sum_{i,j=1}^{4} D(\rho_{ij}) \leq \sum_{i=1}^{4} D(\rho_i) + \sum_{i,j,k=1}^{4} D(\rho_{ijk}) \] (12)

It is well known that many multipartite entanglement measures are polynomial invariants. In this paper, we link two such quantities defined from completely different viewpoints, the extension of which will help to clarify and unify the understanding of multipartite entanglement. Similarly, the connections of other multipartite entanglement measures will also lead to interesting results about general multipartite states.

V. CONCLUSION

In summary, we present necessary conditions for general \( m \)-party subsystem states to be the reduced states of
a single multipartite state of arbitrary $d_1 \otimes d_2 \otimes \cdots \otimes d_N$ composite systems. Our method is based on the properties of multipartite entanglement measures, rather than directly investigating the reduced density matrices as in previous work. These results clearly demonstrate the close connection between multipartite entanglement and the general compatibility problem. As a consequence, we get some interesting monogamy inequalities for bipartite entanglement and partial disorder in general multipartite states.

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