EQ-algebras from the point of view of generalized algebras with fuzzy equalities

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Abstract

EQ-algebras introduced by Novák are algebras of truth values for a higher-order fuzzy logic (fuzzy type theory). In this paper, the compatibility of multiplication w.r.t. the fuzzy equality in an arbitrary EQ-algebra is examined. Particularly, an example indicates that the compatibility axiom does not always hold, and then a class of EQ-algebras satisfying the compatibility axiom is characterized by introducing a residuated integral-meet-semilattice-ordered-monoid-valued fuzzy algebra called a generalized algebra with a fuzzy equality.

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1. Introduction

Every many-valued logic is uniquely determined by the algebraic properties of the structure of its truth values. At present, it is generally accepted that in fuzzy logic, the algebraic structure should be a residuated lattice, possibly fulfilling some additional properties. MTL-algebras, BL-algebras, MV-algebras, \(\Pi\)-algebras, G-algebras, and NM-algebras which are called R\(_0\)-algebras in [19], are the best known classes of residuated lattices [7,8,18]. In a residuated lattice, the basic operations are join, meet, multiplication and its residuum. However, the bi-residuum operation, which interprets the logical equivalence, is a derived operation.

Fuzzy type theory [13–16] whose basic connective is a fuzzy equality was developed as a counterpart of the classical higher-order logic (type theory in which identity is a basic connective, see [1]). Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra [12] for fuzzy type theory was proposed. This kind of corresponding algebra expresses certain fundamental properties of fuzzy equality and, at the same time, serves as an algebra of truth values. Its main primitive operations are meet, multiplication and a fuzzy equality. Its residuum and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattices. In [5,6], great importance was given to the study of good EQ-algebras. Particularly, in [5], they were represented as subalgebras of products of linearly ordered good EQ-algebras, and in [6], it is shown that EQ-algebras

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could be generalized by excluding the commutativity and associativity of the multiplication showing that nothing is lost, and the “goodness” property is necessary for reasonably behaving algebras. Some in-depth works have been done in literature [5,6,12].

In [2,3], algebras with fuzzy equalities were used as structures for equational fragments of first-order fuzzy logic, and then a syntactic-semantically complete logic for reasoning about fuzzy equalities was developed. A principle is stated that if arguments of a function are pairwise similar, then the results are similar as well. From a logical point of view, the situation is described by a logical formula that is traditionally being called the compatibility axiom.

Since fuzzy equalities play an important role in both EQ-algebras and algebras with fuzzy equalities, a natural question is whether the compatibility axiom holds in EQ-algebras? What is the connection between the two logical algebras? We will give an answer here. This paper, dedicated to the study of characterizations of EQ-algebras by generalized algebras with fuzzy equalities (GLE-algebras for short), is structured as follows: In Section 2, algebras with fuzzy equalities and EQ-algebras are briefly recalled. In Section 3, GLE-algebras are introduced with fuzzy equalities based on residuated integral-meet-semilattice-ordered-monoids (imso-monoids for short), and some properties are obtained. In Section 4, the compatibility of multiplication w.r.t. a fuzzy equality in an arbitrary EQ-algebra is investigated, and the notion of compatible EQ-algebras (CEQ-algebra for short) is introduced. Then CEQ-algebras and lattice CEQ-algebras are characterized by special GLE-algebras.

2. Preliminaries

Here, we have a brief introduction to algebras with fuzzy equalities and EQ-algebras.

2.1. Algebras with fuzzy equalities

In this subsection, $L$ denotes an arbitrary residuated lattice. A fuzzy set in a universe set $U$ is a mapping $A : U \rightarrow L$, $A(u) \in L$. Analogously, an $n$-ary fuzzy relation on a universe set $U$ is a fuzzy set on the universe $U^n$, e.g. a binary fuzzy relation $R$ on $U$ is a mapping $R : U \times U \rightarrow L$. Recall that a fuzzy equivalence relation $E$ on $U$ is a mapping $E : U \times U \rightarrow L$ satisfying the following properties:

(F1) reflexivity: $E(u, u) = 1$, for all $u \in U$,
(F2) symmetry: $E(u, v) = E(v, u)$, for all $u, v \in U$,
(F3) transitivity: $E(u, v) \otimes E(v, w) \leq E(u, w)$, for all $u, v, w \in U$.

A fuzzy equivalence relation $E$ on $U$ is called a fuzzy equality [2,4,9] if it holds that

(F4) separability: $E(u, v) = 1$ implies $u = v$, for all $u, v \in U$.

A mapping $f : U^n \rightarrow U$, $n \in \mathbb{N}$ is said to be compatible w.r.t. a binary fuzzy relation $R$ on $U$ if for arbitrary $u_1, v_1, \ldots, u_n, v_n \in U$ it holds that

$$R(u_1, v_1) \otimes \cdots \otimes R(u_n, v_n) \leq R(f(u_1, \ldots, u_n), f(v_1, \ldots, v_n)).$$

The compatibility has a natural description. It says that if $u_1$ and $v_1$ are $R$-related, $\ldots$, and $u_n$ and $v_n$ are $R$-related, then $f(u_1, \ldots, u_n)$ and $f(v_1, \ldots, v_n)$ are $R$-related. For further information on fuzzy equalities, compatibility of fuzzy relations and their roles in fuzzy reasoning, please refer to [8,10,11].

A type is a triplet $(\sim, F, \sigma)$ where $\sim \notin F$ and $\sigma$ is a mapping $\sigma : F \cup \{\sim\} \rightarrow \mathbb{N}$ with $\sigma(\sim) = 2$ and $\sigma(\diamond)$ denotes the arity of function $\diamond \in F$ on $U$.

Definition 2.1. (See [2,3]) A triplet $\mathcal{A} = (A, \sim, F)$ is an algebra with a fuzzy equality (LE-algebra for short) if it holds that

1. $(A, F)$ is an algebra,
2. $\sim$ is a fuzzy equality on $A$ such that each $\diamond \in F$ is compatible w.r.t. $\sim$.

More on LE-algebras, please see [2,3].
2.2. EQ-algebras

In this subsection, we present the definition of EQ-algebras and some basic properties of them.

Definition 2.2. (See [6].) A commutative EQ-algebra $E$ is an algebra of type $(2, 2, 2, 0)$, i.e.,

$$E = \langle E, \wedge, \otimes, \sim, 1 \rangle$$

where for all $a, b, c, d \in E$ it holds that

(E1) $\langle E, \wedge, 1 \rangle$ is a $\wedge$-semilattice with a top element 1. We set $a \leq b$ iff $a \wedge b = a$, as usual,

(E2) $\langle E, \otimes, 1 \rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. $\leq$,

(E3) $a \sim a = 1$,

(E4) $(a \wedge b) \otimes (b \sim a) \leq (a \sim c) \sim (b \sim d),$

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d),$

(E6) $(a \sim b \wedge c) \sim a \leq (a \sim b) \sim a$,

(E7) $a \otimes b \leq a \sim b$.

Note 2.3. In [12], the axiom

(E8) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$

was also introduced. It was shown in [6], however, that it can be deduced from the other ones.

The operation $\wedge$ is called meet and the operation $\otimes$ is called multiplication. As it will be shown further on, $\sim$ is a fuzzy equality (in fact, a fuzzy equivalence relation because separability (F4) does not always hold in every EQ-algebra). Axiom (E3) expresses reflexivity, (E4) is the substitution axiom, (E5) is the congruence axiom, (E6) is the monotonicity axiom and (E7) is the boundedness axiom.

Theorem 2.4. (See [12].) Let $E$ be an EQ-algebra. Then for all $a, b, c \in E$ it holds that

(1) $a \sim b = b \sim a$,

(2) $(a \sim b) \otimes (b \sim c) \leq a \sim c$.

Remark 2.5. $\sim$ is a fuzzy equality on $E$ from axiom (E3) and Theorem 2.4. As an operation, $\sim$ is compatible w.r.t. itself by axiom (E5).

Theorem 2.6. (See [6].) Let $E$ be an EQ-algebra. Then for all $a, b, c, d \in E$ it holds that

(1) $a \sim b \leq (a \wedge c) \sim (b \wedge c),$

(2) $(a \sim b) \otimes (c \sim d) \leq (a \wedge c) \sim (b \wedge d)$.

Remark 2.7. $\wedge$ is compatible w.r.t. $\sim$ from Theorem 2.6(2), and item (1) is equivalent to item (2) in Theorem 2.6.

Corollary 2.8. (See [6].) Let $E$ be an EQ-algebra. Then for all $a, b, c, d \in E$ it holds that

(1) $a \sim b \leq (a \sim c) \sim (b \sim c),$

(2) $((a \sim b) \sim c) \otimes (a \sim d) \leq c \sim (d \sim b)$.

Remark 2.9. Item (1) in Corollary 2.8 is equivalent to axiom (E5).

Definition 2.10. (See [12].) An EQ-algebra $E$ is called

(1) a lattice-ordered EQ-algebra if it has a lattice reduct;
(2) a lattice EQ-algebra if it is a lattice-ordered EQ-algebra in which the following substitution axiom holds, for 
$a, b, c, d \in E$:

\[(E9) \quad ((a \lor b) \sim c) \otimes (d \sim a) \leq c \sim (b \lor d);\]

(3) a good EQ-algebra if for all $a \in E$ it holds that $a \sim 1 = a$;

(4) a residuated EQ-algebra if for all $a, b, c \in E$ it holds that $(a \otimes b) \land c = a \otimes b$ iff $a \land ((b \land c) \sim b) = a$.

We put

\[a \rightarrow b = (a \land b) \sim a,\]

for $a, b \in E$.

Then an EQ-algebra $E$ is residuated if and only if it holds that $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

**Theorem 2.11.** (See [6].) Let $E$ be an EQ-algebra. Then $E$ is residuated if and only if for all $a, b \in E$ it holds that

1. $E$ is good,
2. $a \leq b \rightarrow (a \otimes b)$.

**Theorem 2.12.** (See [6].) Let $E$ be a lattice EQ-algebra. Then for all $a, b, c, d \in E$ it holds that

1. $a \sim b \leq (a \lor c) \sim (b \lor c)$,
2. $(a \sim b) \otimes (c \sim d) \leq (a \lor c) \sim (b \lor d)$.

**Remark 2.13.** $\lor$ is compatible w.r.t. $\sim$ from Theorem 2.12, and item (1) is equivalent to item (2).

3. Residuated imso-monoids and GLE-algebras

In this section, we shall develop some properties of residuated imso-monoids, which will give us a motivation to generalize LE-algebras.

3.1. Residuated imso-monoids

An ordered semigroup is a semigroup together with a compatible order. We shall use the term monoid to indicate a semigroup with an identity, and the term integral ordered monoid to represent an ordered monoid such that its identity is the greatest element. To avoid cumbersome terminology, in what follows, imso is meant to remind of integral-meet-semilattice-ordered, and we suppose that $\otimes$ is commutative.

**Definition 3.1.** $\mathcal{L} = \langle L, \land, \otimes, \rightarrow, 1 \rangle$ is called a residuated imso-monoid if for all $a, b, c \in L$ it holds that

1. $(L, \land, 1)$ is a commutative idempotent monoid (i.e. a $\land$-semilattice with an order defined as $a \leq b$ iff $a \land b = a$),
2. $(L, \otimes, 1)$ is a monoid and $\otimes$ is isotone w.r.t. $\leq$,
3. $a \otimes b \leq c$ iff $b \leq a \Rightarrow c$.

**Lemma 3.2.** Let $\mathcal{L} = \langle L, \land, \otimes, \rightarrow, 1 \rangle$ be a residuated imso-monoid. Then for all $a, b, c, d \in L$ it holds that

1. $a \otimes b \leq a \land b$,
2. $a \otimes (a \Rightarrow b) \leq b$,
3. $(a \Rightarrow b) \land (c \Rightarrow d) \leq (a \land c) \Rightarrow (b \land d)$,
4. $(a \Rightarrow b) \otimes (c \Rightarrow d) \leq (a \otimes c) \Rightarrow (b \otimes d)$,
5. $(b \Rightarrow a) \otimes (c \Rightarrow d) \leq (a \Rightarrow c) \Rightarrow (b \Rightarrow d)$.
Proof.

(1) It follows immediately from the isotonicity of $\otimes$.

(2) By axiom (S3), the inequality $a \Rightarrow b \leq a \Rightarrow b$ yields that $a \otimes (a \Rightarrow b) \leq b$.

(3) Applying the isotonicity of $\otimes$ and (2), we obtain $(a \land c) \otimes ((a \Rightarrow b) \land (c \Rightarrow d)) \leq a \otimes (a \Rightarrow b) \leq b$, as well as $(a \land c) \otimes ((a \Rightarrow b) \land (c \Rightarrow d)) \leq c \otimes (c \Rightarrow d) \leq d$, i.e. $(a \land c) \otimes ((a \Rightarrow b) \land (c \Rightarrow d)) \leq b \land d$. Using axiom (S3), it holds that $(a \Rightarrow b) \land (c \Rightarrow d) \leq (a \land c) \Rightarrow (b \land d)$.

(4) Using the associativity, commutativity of $\otimes$ and (2), it follows that $(a \land c) \otimes ((a \Rightarrow b) \land (c \Rightarrow d)) \leq (a \land c) \Rightarrow (b \Rightarrow d)$.

(5) Applying (2) three times, it holds that $b \otimes (a \Rightarrow c) \otimes (b \Rightarrow a) \otimes (c \Rightarrow d) \leq d$. Using axiom (S3) twice, it yields that $(b \Rightarrow a) \otimes (c \Rightarrow d) \leq (a \Rightarrow c) \Rightarrow (b \Rightarrow d)$.

In this paper, we consider a biresiduum defined by

$$a \Leftrightarrow b := (a \Rightarrow b) \land (b \Rightarrow a).$$

The following results are obvious.

**Theorem 3.3.** Let $\mathcal{L} = \langle L, \land, \otimes, \Rightarrow, 1 \rangle$ be a residuated imso-monoid. Then for all $a, b, c \in L$ it holds that

1. $a \Leftrightarrow b = 1$ iff $a = b$,
2. $a \Leftrightarrow b = b \Leftrightarrow a$,
3. $(a \Leftrightarrow b) \otimes (b \Leftrightarrow c) \leq a \Leftrightarrow c$.

**Note 3.4.** $\Leftrightarrow$ is a fuzzy equality on $\mathcal{L}$.

**Theorem 3.5.** Let $\mathcal{L} = \langle L, \land, \otimes, \Rightarrow, 1 \rangle$ be a residuated imso-monoid and $\diamond \in \{ \land, \otimes, \Rightarrow, \Leftrightarrow \}$. Then for all $a, b, c, d \in L$ it holds that

$$(a \Leftrightarrow b) \otimes (c \Leftrightarrow d) \leq (a \diamond c) \Leftrightarrow (b \diamond d).$$

**Proof.** It is similar to that given in [17] for residuated lattices. □

Here, we expand the notion of residuated imso-monoids to residuated lattices.

**Theorem 3.6.** (See [17].) Let $\mathcal{L}$ be a residuated lattice and $\diamond \in \{ \lor, \land, \otimes, \Rightarrow, \Leftrightarrow \}$. Then for all $a, b, c, d \in L$ it holds that

$$(a \Leftrightarrow b) \otimes (c \Leftrightarrow d) \leq (a \diamond c) \Leftrightarrow (b \diamond d).$$

**Remark 3.7.** $\lor, \land, \otimes, \Rightarrow$ and $\Leftrightarrow$ are compatible w.r.t. $\Leftrightarrow$, which inspires us to generalize the concept of LE-algebras.

### 3.2. GLE-algebras

Here, we shall slightly generalize LE-algebras in order to characterize EQ-algebras. Let $\mathcal{L} = \langle L, \land, \otimes, 1 \rangle$ be an imso-monoid, and $A$ be a universe set.

**Definition 3.8.** A triplet $\mathfrak{A} = \langle A, \sim, F \rangle$ is a generalized LE-algebra (GLE-algebra for short) if it holds that

1. $\langle A, F \rangle$ is an algebra,
2. $\sim$ is a fuzzy equality on $A$ such that each $\diamond \in F$ is compatible w.r.t. $\sim$.
Example 3.9. Let $\mathcal{L} = (L, \wedge, \otimes, \Rightarrow, 1)$ be a residuated imso-monoid. Assume that $\sim = \iff$ and $F = \{\wedge, \otimes, \Rightarrow, \iff\}$. Then $(L, \sim, F)$ is a GLE-algebra, but not an LE-algebra because $\sim \notin F$. A similar result can be obtained as above if expanding the residuated imso-monoid to a residuated lattice.

In the following part, we suppose that for all $\diamond \in F$, $\sigma(\diamond) = 2$. Let $\mathcal{H} = (A, \sim, F)$ be a GLE-algebra. Then for all $a, b, c, d \in A$ it holds that

$$(a \sim b) \otimes (c \sim d) \leq (a \circ c) \sim (b \circ d).$$

(\star)

Theorem 3.10. Let $\mathcal{H} = (A, \sim, F)$ be a GLE-algebra. Then for all $a, b, c \in A$ it holds that

$$a \sim b \leq (a \circ c) \sim (b \circ c).$$

Proof. Considering the following instance of (\star): $(a \sim b) \otimes (c \sim c) \leq (a \circ c) \sim (b \circ c)$, and hence $a \sim b \leq (a \circ c) \sim (b \circ c)$. \qed

Theorem 3.11. Let $\mathcal{H} = (A, \sim, F)$ be a GLE-algebra. Then for all $a, b, c, d \in A$ it holds that

$$(a \circ b) \sim c \otimes (a \sim d) \leq c \sim (d \circ b).$$

Proof. Using Theorem 3.10 and the transitivity of $\sim$, it yields that $((a \circ b) \sim c) \otimes (a \sim d) \leq ((a \circ b) \sim c) \otimes ((a \circ b) \sim (d \circ b)) \leq c \sim (d \circ b)$. \qed

Here, let $\mathcal{H} = \mathcal{L}$, then $\sim$ is both a fuzzy relation and an operation on $\mathcal{L}$. By Definition 3.8, $\sim$ is compatible w.r.t. itself. What is more, as an operation on $\mathcal{L}$, it may have an identity.

Definition 3.12. A GLE-algebra $\mathcal{L}$ is said to be good if for all $a \in L$ it holds that $a \sim 1 = a$.

Theorem 3.13. Let $\mathcal{L}$ be a good GLE-algebra. Then for all $a, b \in L$ it holds that

$$a \otimes b \leq a \sim b.$$

Proof. By Definition 3.12 and the transitivity of $\sim$, it holds that $a \otimes b = (a \sim 1) \otimes (b \sim 1) \leq a \sim b$. \qed

Theorem 3.14. Let $\mathcal{L}$ be a good GLE-algebra. Then for all $a, b \in L$ it holds that

$$a \leq b \rightarrow (a \otimes b).$$

Proof. Consider the following instance of Theorem 3.10: $a \sim 1 \leq (a \otimes b) \sim (1 \otimes b)$. Since $\mathcal{L}$ is good, it yields that $a \leq (a \otimes b) \sim b$. The isotonicity of $\otimes$ leads that $a \leq [(a \otimes b) \wedge b] \sim b \rightarrow (a \otimes b)$. \qed

Theorems 3.10 and 3.11 indicate that axiom (E4) is a direct result of the compatibility axiom of $\wedge$, and axiom (E5) itself is a compatibility axiom because $\sim$ is both an operation and a fuzzy equality. If we expand $\mathcal{L}$ to a residuated lattice, then axiom (E6) is a result of the compatibility axiom of $\vee$. These above inspire us to characterize EQ-algebras from the point of view of GLE-algebras.

4. EQ-algebras from the point of view of GLE-algebras

4.1. EQ-algebras and compatibility axiom

Examples are given to show the connections between EQ-algebras and the compatibility axiom.
Example 4.1. The EQ-algebra $E$ in [12] is defined as follows: $E = \{0, a, b, c, d, 1\}$ is a set with its partial order defined as:

$$0 \leq a \leq b \leq d \leq 1, \quad 0 \leq a \leq c \leq d \leq 1,$$

and its operations $\otimes$ and $\sim$ defined as:

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We obtain

$$a = (a \sim b) \otimes (c \sim d) \not\leq (a \otimes c) \sim (b \otimes d) = 0.$$ 

It shows that the operation $\otimes$ is not compatible w.r.t. $\sim$.

As it is pointed out in [2], the compatibility axiom, which is verbally formulated, maps similar elements to similar ones and plays an essential role from a logical point of view. So we propose a class of EQ-algebras which satisfies the compatibility axiom, and give some typical examples of them as follows:

Example 4.2. Let $L = \langle L, \wedge, \vee, \otimes, \Rightarrow, 0, 1 \rangle$ be a residuated lattice. In [12], it is pointed out that $L' = \langle L, \otimes, \Leftrightarrow, 1 \rangle$ is an EQ-algebra. Furthermore, we have the compatibility axiom (Theorem 3.6) in $L$. Thus $L$ can be seen as both a GLE-algebra and a lattice EQ-algebra. A similar conclusion holds if the residuated lattice is reduced to a residuated imso-monoid.

An EQ-algebra is called a compatible EQ-algebra, CEQ-algebra for short, if $\otimes$ is compatible w.r.t. $\sim$.

It is obvious that in a CEQ-algebra the operations $\wedge, \otimes$ and $\sim$ are compatible w.r.t. $\sim$. That is, CEQ-algebras are a special class of GLE-algebras, which leads us to characterize CEQ-algebras by GLE-algebras.

4.2. Characterizations of EQ-algebras by GLE-algebras

In this subsection, by GLE-algebras we mainly characterize EQ-algebras which satisfy the compatibility axiom.

Theorem 4.3. An algebra $\langle E, \wedge, \otimes, \sim, 1 \rangle$ of type $(2, 2, 2, 0)$ is a CEQ-algebra if and only if a GLE-algebra $\langle E, \sim, \{\wedge, \otimes, \sim\}, 1 \rangle$ satisfies the following axioms:

(L1) $\langle E, \wedge, 1 \rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with a top element 1),
(L2) $\langle E, \otimes, 1 \rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. $\leq$ (with $a \leq b$ iff $a \wedge b = a$),
(L3) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$,
(L4) $a \otimes b \leq a \sim b$.

Proof. Assume that $\langle E, \wedge, \otimes, \sim, 1 \rangle$ is a CEQ-algebra. Then (L1), (L2), (L3) and (L4) are obvious. By the definition of CEQ-algebras, $\otimes$ and $\sim$ are compatible w.r.t. $\sim$. It follows from Theorem 2.6(2) that $\wedge$ is compatible w.r.t. $\sim$.

Conversely, (E1), (E2), (E3), (E5), (E6) and (E7) are obvious. By replacing $\Diamond$ with $\wedge$ in Theorem 3.11, (E4) holds.

Corollary 4.4. An algebra $\langle E, \wedge, \otimes, \sim, 1 \rangle$ of type $(2, 2, 2, 0)$ is a good CEQ-algebra if and only if a good GLE-algebra $\langle E, \sim, \{\wedge, \otimes, \sim\}, 1 \rangle$ satisfies the axioms (L1), (L2) and (L3).

Proof. It follows immediately from Theorem 3.13.
Corollary 4.5. Each good CEQ-algebra is residuated.

Proof. It follows immediately from Theorems 2.11, 3.14 and 4.3. □

Remark 4.6.

(1) Corollary 4.4 indicates that in an arbitrary good EQ-algebra, the boundedness axiom (E7) follows from the “goodness” property and the transitivity of ~ (also see Theorem 3.13).

(2) Corollary 4.5 shows that good CEQ-algebras turn into residuated imso-monoids.

Similarly, for lattice EQ-algebras, we obtain the following results:

Theorem 4.7. An algebra \( \langle E, \lor, \land, \otimes, \sim, 1 \rangle \) of type \((2, 2, 2, 2, 0)\) is a lattice CEQ-algebra if and only if a GLE-algebra \( \langle E, \sim, \{\lor, \land, \otimes, \sim\}, 1 \rangle \) satisfies the axioms (L2), (L4) and the following axiom:

\( (L1') \langle E, \lor, \land, 1 \rangle \) is a lattice with a top element 1.

Proof. Assume that \( \langle E, \lor, \land, \otimes, \sim, 1 \rangle \) is a lattice CEQ-algebra. Then (L1'), (L2) and (L4) are obvious. By the definition of CEQ-algebras, \( \otimes \) and \( \sim \) are compatible w.r.t. \( \sim \). It follows from Theorems 2.6(2) and 2.12(2) that \( \land \) and \( \lor \) are compatible w.r.t. \( \sim \).

Conversely, (E1), (E2), (E3), (E5), (E6) and (E7) are obvious. By replacing \( \heartsuit \) with \( \land \) and \( \lor \), respectively, in Theorem 3.11, (E4) and (E9) hold. □

Corollary 4.8. An algebra \( \langle E, \lor, \land, \otimes, \sim, 1 \rangle \) of type \((2, 2, 2, 2, 0)\) is a good lattice CEQ-algebra if and only if a good GLE-algebra \( \langle E, \sim, \{\lor, \land, \otimes, \sim\}, 1 \rangle \) satisfies the axioms (L1') and (L2).

Remark 4.9. For an arbitrary good lattice CEQ-algebra \( \langle E, \lor, \land, \otimes, \sim, 1 \rangle \), a residuated lattice \( \langle E, \lor, \land, \otimes, \rightarrow, 1 \rangle \) can be induced. That is, good lattice CEQ-algebras turn into residuated lattices.

5. Conclusion

In this paper, we have examined the compatibility of multiplication w.r.t. the fuzzy equality in an arbitrary EQ-algebra. In particular, by introducing GLE-algebras we characterized a special class of EQ-algebras called CEQ-algebras. Our study also showed that associated with the “goodness” property, CEQ-algebras and lattice CEQ-algebras turn into residuated imso-monoids and residuated lattices, respectively. To study whether the non-CEQ-algebras are algebras of truth values in fuzzy type theory is a further topic.

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