Topological and lattice structures of $\mathcal{L}$-fuzzy rough sets determined by lower and upper sets

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Abstract
This paper builds the topological and lattice structures of $\mathcal{L}$-fuzzy rough sets by introducing lower and upper sets. In particular, it is shown that when the $\mathcal{L}$-relation is reflexive, the upper (resp. lower) set is equivalent to the lower (resp. upper) $\mathcal{L}$-fuzzy approximation set. Then by the upper (resp. lower) set, it is indicated that an $\mathcal{L}$-preorder is the equivalence condition under which the set of all the lower (resp. upper) $\mathcal{L}$-fuzzy approximation sets and the Alexandrov $\mathcal{L}$-topology are identical. However, associating with an $\mathcal{L}$-preorder, the equivalence condition that $\mathcal{L}$-interior (resp. closure) operator accords with the lower (resp. upper) $\mathcal{L}$-fuzzy approximation operator is investigated. At last, it is proven that the set of all the lower (resp. upper) $\mathcal{L}$-fuzzy approximation sets forms a complete lattice when the $\mathcal{L}$-relation is reflexive.

1. Introduction

The concept of rough set was originally proposed by Pawlak $^{[25,26]}$ as a mathematical approach to handle imprecision and uncertainty in data analysis. Usefulness and versatility of this theory have amply been demonstrated by successful applications in a variety of problems $^{[33,34]}$.

At present, there are two different basic approaches in the rough set theory, i.e. the axiomatic approach and constructive approach. In the axiomatic approach, the lower and upper approximation operators are the primitive notions. A set of axioms is used to characterize approximation operators $^{[2,19,22,23,32,35,37,40]}$. In contrast to axiomatic approach, the constructive approach takes binary relations on the universe as the primary notions by which the lower and upper approximation operators are constructed. Diverse forms of the constructive approach have been started from the properties of binary relations, for example, reflexivity, (fuzzy) preorder, (fuzzy) equivalence relation, and (fuzzy) coverings, to investigate the essential properties of the lower and upper approximation operators generated by such relations $^{[6,7,16,18,21,24,25,30,31,39,42–44]}$.

However, these forms in conjunction with additional topological or algebraic structures on a universe are considered. In $^{[15,38]}$, it was proven that, under a crisp preorder, the pair of lower and upper approximation operators is just a pair of interior and closure operators of an Alexandrov topology. In $^{[27]}$, it was pointed that inverse serial relations are the weakest relations which can induce topological spaces, and different relations based generalized rough set models can also induce different topological spaces. As the generalizations of rough sets from fuzzy sets point of view, in $^{[29]}$, it was verified that there exists a one-to-one correspondence between the set of all the lower approximation sets based on fuzzy preorder and
the set of all fuzzy topologies satisfying so-called (TC) axiom. In [31], \( L \)-fuzzy rough sets based on residuated lattices were proposed. In [32], the axiomatic characterizations of various \( L \)-fuzzy rough sets were investigated. Moreover, it was examined that an \( L \)-interior (resp. closure) operator of an \( L \)-topological space could associate with an \( L \)-preorder such that the corresponding lower (resp. upper) \( L \)-fuzzy approximation operator was the \( L \)-interior (resp. closure) operator. In [10,17], under the context of left continuous \( t \)-norms and residuated lattices, a one-to-one correspondence between the set of all fuzzy preorders and the set of all Alexandrov fuzzy topologies was obtained, where the fuzzy preorders and fuzzy topologies were connected by the upper sets and lower \( L \)-fuzzy approximation operators, respectively. However, it has not been studied on the relationships between upper sets and lower \( L \)-fuzzy approximation operators and applying the upper set to study \( L \)-fuzzy rough set theory, and the aim of the present paper is to investigate and solve these questions.

In this paper, we focus on a complete study on the topological and lattice structures in \( L \)-fuzzy approximation spaces by lower and upper sets. In Section 2, we recall some fundamental concepts and related properties. In Section 3, we propose some new properties of the \( L \)-fuzzy approximation operators. In Section 4, the main part of this paper, we prove that the upper (resp. lower) set is equivalent to the lower (resp. upper) \( L \)-fuzzy approximation set under a reflexive \( L \)-relation. Then by the upper (resp. lower) set, we point out that an \( L \)-preorder is the equivalence condition under which the set of all the lower (resp. upper) \( L \)-fuzzy approximation sets and the Alexandrov \( L \)-topology coincide. At the same time, associating with an \( L \)-preorder, we verify that the equivalence condition that \( L \)-interior (resp. closure) operator accords with the lower (resp. upper) \( L \)-fuzzy approximation operator. At last, when the \( L \)-relation is reflexive, we construct a complete lattice by the set of all the lower (resp. upper) \( L \)-fuzzy approximation sets.

2. Preliminaries

2.1. Residuated lattices

A residuated lattice is an algebra \( \mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1) \) such that \((L, \wedge, \vee, 0, 1)\) is a bounded lattice with the least element 0 and the greatest element 1, \((L, \otimes, 1)\) is a communicative monoid, and \(\otimes, \to\) form an adjoint pair, i.e. \(x \otimes y \leq z\) if and only if \(x \leq y \to z\) for each \(x, y, z \in L\). A residuated lattice \( \mathcal{L} \) is complete if the underlying lattice \((L, \wedge, \vee, 0, 1)\) is complete. \(\forall\) will be reserved as \(x \to 0\), for all \(x \in \mathcal{L}\). Throughout this paper, \( \mathcal{L} \) always denotes a complete residuated lattice.

Some basic properties of residuated lattices are collected in the following lemma, and the formulas except (1) and (9) are folklore in the literature, see, e.g. [11,36]. We do not find the proofs of (1) and (9) although it is highly likely that they have already appeared somewhere else.

**Lemma 2.1** [11,36]. Let \( \mathcal{L} \) be a residuated lattice. Then

\[
\begin{align*}
(1) \quad x \otimes (y \to z) & \leq (x \to y) \to z, \\
(2) \quad \forall_{\mathcal{L}}(x_0 \to y) & \leq \wedge_{\mathcal{L}}(x_1 \to y) , \\
(3) \quad x_1 \leq x_2 \text{ implies } x_2 \to y \leq x_1 \to y \text{ and } y \to x_1 \leq y \to x_2, \\
(4) \quad y \otimes (\wedge_{\mathcal{L}}(x_1)) & \leq \wedge_{\mathcal{L}}(y \otimes x_1), \\
(5) \quad y \otimes (\forall_{\mathcal{L}}(x_1)) & = \forall_{\mathcal{L}}(y \otimes x_1), \\
(6) \quad x \to (y \to z) & = (x \otimes y) \to z = y \to (x \to z), \\
(7) \quad \forall_{\mathcal{L}}(x_1) \to y & = \wedge_{\mathcal{L}}(x_1) \to y, \\
(8) \quad y \to \wedge_{\mathcal{L}}(x_1) & = \forall_{\mathcal{L}}(y \to x_1), \\
(9) \quad \forall_{\mathcal{L}}(x \to y) & = x, \\
(10) \quad y \to \forall_{\mathcal{L}}(x_1) & \geq \forall_{\mathcal{L}}(y \to x_1).
\end{align*}
\]

**Proof.** (1) Since \(x \otimes (x \to y) \leq y\), it follows that \(x \otimes (x \to y) \otimes (y \to z) \leq z\), that is, \(x \otimes (y \to z) \leq (x \to y) \to z\). Hence (1) holds.

(9) \(\forall_{\mathcal{L}}((x \to y) \to y) \leq (x \to y) \to x = x\). Consider the following instance of (1): \(x \leq (x \to y) \to y\), and hence \(x \leq \forall_{\mathcal{L}}((x \to y) \to y)\). Therefore (9) holds. \(\Box\)

2.2. \( \mathcal{L} \)-sets, \( \mathcal{L} \)-relations and \( \mathcal{L} \)-topologies

A fuzzy set \( \mu \) with truth degrees from a residuated lattice \( \mathcal{L} \) (\( \mathcal{L} \)-set for short) in a universe \( X [1] \) is a mapping \( \mu : X \to \mathcal{L}, \mu(x) \in \mathcal{L} \) which is a generalization of the notion of Zadeh's fuzzy sets [41] and Goguen's lattice-based fuzzy sets [9] (the set of all \( \mathcal{L} \)-sets will be denoted by \( \mathcal{L}^X \)). An \( \mathcal{L} \)-set \( \mu \) is said to be constant if \( \mu(x) = x_c \) for all \( x \in X \), noted by \( x_c \). And \( \mu \) is called a fuzzy point if

\[
\mu(x) = \begin{cases} 
1, & \text{if } x = y, \\
0, & \text{if } x \neq y,
\end{cases}
\]

noted by \( y_z \).
For all \( \mu, v \in \mathcal{L}^X \) and \( x \in X \),
\[
(\mu \otimes v)(x) = \mu(x) \otimes v(x),
(\mu \land v)(x) = \mu(x) \land v(x),
(\mu \lor v)(x) = \mu(x) \lor v(x),
(\mu \rightarrow v)(x) = \mu(x) \rightarrow v(x),
\neg(\mu)(x) = \neg\mu(x).
\]

We also write \( \mu \leq v \) to denote \( \mu(x) \leq v(x) \) for all \( x \in X \).

For a given fuzzy set \( \mu \), its height and plinth are defined as:
\[
h(\mu) = \forall_{x \in X} \mu(x),
p(\mu) = \land_{x \in X} \mu(x).
\]

Subsethood measure \([12]\) closely related to residuation was extensively studied in fuzzy set theory. It describes the degree to which a fuzzy set is a subset of another fuzzy set, and can be defined as:
\[
\mu \mapsto v = \land_{x \in X} (\mu(x) \rightarrow v(x)).
\]

By subsethood measure, \( \mathcal{L} \)-isotone Galois connection can be defined.

**Definition 2.2** (8.28). Let \( X, Y \) be two universes. A pair of functions \((\mathcal{L}^X, \mathcal{L}^Y)\), \( \mathcal{L}^X \rightarrow \mathcal{L}^Y \) and \( \mathcal{L}^Y \rightarrow \mathcal{L}^X \), is called an \( \mathcal{L} \)-isotone Galois connection between \( X \) and \( Y \) if
\[
\mu \mapsto v^\dagger = \mu^\dagger \mapsto v,
\]
for all \( \mu \in \mathcal{L}^X \), \( v \in \mathcal{L}^Y \).

An \( \mathcal{L} \)-set \( \theta \) on \( X^2 \) is called an \( \mathcal{L} \)-relation on \( X \). For an \( \mathcal{L} \)-relation \( \theta \), we define an \( \mathcal{L} \)-relation \( \theta^{-1} \) by \( \theta^{-1}(x, y) = \theta(y, x) \).

**Definition 2.3.** Let \( \theta \) be an \( \mathcal{L} \)-relation on \( X \). Then \( \theta \) is said to be

1. **serial** if \( \forall_{x \in X} \theta(x, y) = 1 \) for all \( x \in X \).
2. **reflexive** if \( \theta(x, x) = 1 \) for all \( x \in X \).
3. **symmetric** if \( \theta(x, y) = \theta(y, x) \) for all \( x, y \in X \).
4. **Euclidean** if \( \theta(x, z) \land \theta(z, y) \leq \theta(x, y) \) for all \( x, y, z \in X \).
5. **transitive** if \( \theta(x, z) \land \theta(z, y) \leq \theta(x, y) \) for all \( x, y, z \in X \).

An \( \mathcal{L} \)-relation \( \theta \) on \( X \) is called an \( \mathcal{L} \)-preorder if it is reflexive and transitive. And an \( \mathcal{L} \)-equivalence relation if it is reflexive, symmetric and transitive.

**Definition 2.4** (13,20). Let \( X \) be a universe and \( \tau \subseteq \mathcal{L}^X \). Then \( \tau \) is called an Alexandrov \( \mathcal{L} \)-topology on \( X \) if it satisfies

1. \( 0_X, 1_X \in \tau \),
2. \( \tau^1 \subseteq \tau \) implies \( \land_{\mu \in \tau^1} \mu \in \tau \) and \( \lor_{\mu \in \tau^1} \mu \in \tau \),
3. \( x \in \mathcal{L} \) and \( \mu \in \tau \) imply \( x \land \mu \in \tau \),
4. \( x \in \mathcal{L} \) and \( \mu \in \tau \) imply \( x \lor \mu \in \tau \).

**Definition 2.5** (20). A mapping \( \psi : \mathcal{L}^X \rightarrow \mathcal{L}^X \) is called an \( \mathcal{L} \)-interior operator if it satisfies

1. \( \psi(\mu) \leq \mu \),
2. \( \psi(x_X) = x_X \),
3. \( \psi(\mu \land v) = \psi(\mu) \land \psi(v) \).

**Definition 2.6** (20). A mapping \( \phi : \mathcal{L}^X \rightarrow \mathcal{L}^X \) is called an \( \mathcal{L} \)-closure operator if it satisfies

1. \( \mu \leq \phi(\mu) \),
2. \( \phi(x_X) = x_X \),
3. \( \phi(\phi(\mu)) = \phi(\mu) \),
4. \( \phi(\mu \lor v) = \phi(\mu) \lor \phi(v) \).
2.3. \(\mathcal{L}\)-fuzzy rough sets

The concept of \(\mathcal{L}\)-fuzzy rough sets was introduced by Radzikowska and Kerre in [31], and an axiomatic approach was proposed by She and Wang in [32]. It can be considered as a generalization of those in [6, 23, 30].

Assume that \(X\) is a nonempty universe and \(\vartheta\) is an arbitrary \(\mathcal{L}\)-relation on \(X\). A pair \((X, \vartheta)\) is called an \(\mathcal{L}\)-fuzzy approximation space.

**Definition 2.7** (31, 32). Let \((X, \vartheta)\) be an \(\mathcal{L}\)-fuzzy approximation space. Define the following two mappings \(\vartheta, \bar{\vartheta} : \mathcal{L}^X \rightarrow \mathcal{L}^X\), called lower and upper \(\mathcal{L}\)-fuzzy approximation operators, respectively, as follows: for all \(\mu \in \mathcal{L}^X\), and \(x \in X\),

\[
\vartheta(\mu)(x) = \land_{y \in X}(\vartheta(x, y) \rightarrow \mu(y)),
\]

\[
\bar{\vartheta}(\mu)(x) = \lor_{y \in X}(\vartheta(x, y) \rightarrow \mu(y)).
\]

\(\vartheta(\mu)\) (resp. \(\bar{\vartheta}(\mu)\)) is called a lower (resp. upper) \(\mathcal{L}\)-fuzzy approximation of \(\mu\). \(\vartheta(\mu) = (\vartheta(\mu), \bar{\vartheta}(\mu))\) is called an \(\mathcal{L}\)-fuzzy rough set with respect to \(\mu\) if \(\vartheta(\mu) \neq \bar{\vartheta}(\mu)\).

Some basic properties of the \(\mathcal{L}\)-fuzzy approximation operators are provided in the following two theorems.

**Theorem 2.8** (31). Let \((X, \vartheta)\) be an \(\mathcal{L}\)-fuzzy approximation space. Then for \(\mu, \upsilon, \mu_i \in \mathcal{L}^X\),

1. \(\vartheta(\text{0}_X) = 0_X\), \(\vartheta(1_X) = 1_X\).
2. \(\vartheta(\mu) \leq \vartheta(-\mu)\) and \(\vartheta(\mu) \leq \bar{\vartheta}(\mu)\).
3. \(-\vartheta(\mu) \geq \bar{\vartheta}(\mu)\) and \(-\vartheta(\mu) = \bar{\vartheta}(\mu)\).
4. If \(\mu \leq \upsilon\), then \(\vartheta(\mu) \leq \vartheta(\upsilon)\) and \(\bar{\vartheta}(\mu) \leq \bar{\vartheta}(\upsilon)\).
5. \(\vartheta(\land_{i \in I} \mu_i) = \land_{i \in I} \vartheta(\mu_i)\) and \(\vartheta(\lor_{i \in I} \mu_i) = \lor_{i \in I} \vartheta(\mu_i)\).
6. \(\bar{\vartheta}(\land_{i \in I} \mu_i) \geq \land_{i \in I} \bar{\vartheta}(\mu_i)\) and \(\bar{\vartheta}(\lor_{i \in I} \mu_i) = \lor_{i \in I} \bar{\vartheta}(\mu_i)\).

**Remark 2.9.** Theorem 2.8 (2) shows that \(\mathcal{L}\)-fuzzy approximation operators \(\vartheta\) and \(\bar{\vartheta}\) are not necessarily dual to each other. Moreover, She and Wang [32] proved that they were dual by taking regular residuated lattices as basic structures.

**Theorem 2.10** (32). Let \((X, \vartheta)\) be an \(\mathcal{L}\)-fuzzy approximation space. Then for all \(\mu \in \mathcal{L}^X\), \(x \in \mathcal{L}\) and \(x, y \in X\),

1. \(\vartheta(\text{1}_x)(y) = \vartheta(y, x)\).
2. \(\vartheta(\text{2}_X \otimes \mu) = \text{2}_X \otimes \bar{\vartheta}(\mu)\).
3. \(\vartheta(\text{3}_x \rightarrow \mu) = \text{3}_x \rightarrow \vartheta(\mu)\).
4. \(\vartheta(\text{4}_1 \rightarrow \text{2}_X)(y) = \vartheta(y, x) \rightarrow \text{4}_x\).

Furthermore, the relationships between special classes of \(\mathcal{L}\)-fuzzy relations and properties of \(\mathcal{L}\)-fuzzy approximation operators can be shown in the following theorems.

**Theorem 2.11** (31). Let \((X, \vartheta)\) be an \(\mathcal{L}\)-fuzzy approximation space and \(\vartheta\) be reflexive. Then \(\vartheta(\text{0}_X) = 0_X\) and \(\vartheta(1_X) = 1_X\).

**Note 2.12.** The theorem above can be proven when \(\vartheta\) is serial (see [32]).

**Theorem 2.13** (32). Let \((X, \vartheta)\) be an \(\mathcal{L}\)-fuzzy approximation space and \(\vartheta\) be serial. Then for all \(x \in \mathcal{L}\),

1. \(\vartheta(\text{5}_x) = \text{5}_x\).
2. \(\vartheta(\text{6}_x) = \text{6}_x\).

**Theorem 2.14** (31). Let \((X, \vartheta)\) be an \(\mathcal{L}\)-fuzzy approximation space. Then for all \(\mu \in \mathcal{L}^X\),

1. \(\vartheta\) is reflexive \(\iff\) \(\vartheta(\mu) \leq \mu \iff\) \(\mu \leq \bar{\vartheta}(\mu)\).
2. \(\vartheta\) is symmetric \(\iff\) \(\vartheta(\mu) \leq \mu \iff\) \(\mu \leq \bar{\vartheta}(\mu)\).
3. \(\vartheta\) is transitive \(\iff\) \(\vartheta(\mu) \leq \vartheta(\mu) \iff\) \(\vartheta(\mu) \leq \bar{\vartheta}(\mu)\).
4. \(\vartheta\) is Euclidean \(\iff\) \(\vartheta(\mu) \leq \bar{\vartheta}(\mu) \iff\) \(\vartheta(\mu) \leq \bar{\vartheta}(\mu)\).
5. \(\vartheta\) is reflexive and transitive \(\iff\) \(\vartheta(\mu) = \mu \iff\) \(\bar{\vartheta}(\mu) = \bar{\vartheta}(\mu)\).
6. \(\vartheta\) is an \(\mathcal{L}\)-equivalence relation \(\iff\) \(\vartheta(\mu) = \mu \iff\) \(\bar{\vartheta}(\mu) = \bar{\vartheta}(\mu)\).
3. New properties of $\mathcal{L}$-fuzzy approximation operators

In this section, associated with related operations of $\mathcal{L}$-sets, some new properties of $\mathcal{L}$-fuzzy approximation operators are obtained.

The theorem below generalizes the results in Theorem 2.8 (2) and (3) by replacing $\rightarrow$ with $\rightarrow$.

**Theorem 3.1.** Let $(X, \theta)$ be an $\mathcal{L}$-fuzzy approximation space. Then for all $\mu \in \mathcal{L}^X$ and $x \in \mathcal{L}$,

1. $\bar{\theta}(\mu) = \land_{x \in \mathcal{L}} (\bar{\theta}(\mu \rightarrow x) \rightarrow x)$.
2. $\bar{\theta}(\mu) = \land_{x \in \mathcal{L}} (\bar{\theta}(\mu \rightarrow x) \rightarrow x)$.
3. $\bar{\theta}(\mu \rightarrow x) = \bar{\theta}(\mu) \rightarrow x$.
4. $\bar{\theta}(\mu \rightarrow x) \subseteq \bar{\theta}(\mu) \rightarrow x$.

**Proof**

(1) By Lemma 2.1 (6)–(9), we get

$$\land_{x \in \mathcal{L}} (\bar{\theta}(\mu \rightarrow x) \rightarrow x) = \land_{x \in \mathcal{L}} \left\{ \left[ \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)) \right] \rightarrow x \right\} = \land_{x \in \mathcal{L}} \land_{y \in \mathcal{L}} \left\{ [\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)] \rightarrow x \right\}$$

$$= \land_{x \in \mathcal{L}} \land_{y \in \mathcal{L}} \left\{ \left[ \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)) \right] \rightarrow x \right\} = \land_{y \in \mathcal{L}} \left\{ \bar{\theta}(x \rightarrow y) \rightarrow (\mu \rightarrow x) \right\} = \bar{\theta}(\mu)(x).$$

(2) Using Lemma 2.1 (6), (7) and (9), we have

$$\land_{x \in \mathcal{L}} (\bar{\theta}(\mu \rightarrow x) \rightarrow x) = \land_{x \in \mathcal{L}} \left\{ \left[ \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)) \right] \rightarrow x \right\} = \land_{x \in \mathcal{L}} \left\{ [\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)] \rightarrow x \right\}$$

$$= \land_{x \in \mathcal{L}} \left\{ \left[ \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)) \right] \rightarrow x \right\} = \land_{y \in \mathcal{L}} \left\{ \bar{\theta}(x \rightarrow y) \rightarrow (\mu \rightarrow x) \right\} = \bar{\theta}(\mu)(x).$$

(3) Applying Lemma 2.1 (6) and (7), we find

$$\bar{\theta}(\mu \rightarrow x)(x) = \land_{y \in \mathcal{L}} [\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)] = \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \rightarrow x) = \bar{\theta}(\mu)(x).$$

(4) By Lemma 2.1 (1) and (2), we obtain

$$\bar{\theta}(\mu \rightarrow x)(x) = \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \land (\mu \rightarrow x)) \subseteq \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \rightarrow x) \subseteq \land_{y \in \mathcal{L}} (\bar{\theta}(x \rightarrow y) \rightarrow (\mu \rightarrow x)) \rightarrow x = \bar{\theta}(\mu)(x) \rightarrow x. \quad \square$$

In Pawlak rough set theory, both $(\overline{R}, \overline{R}^{-1})$ and $(\overline{R}^{1}, \overline{R})$ are Galois connections. In [8,28], the isotope form of $\mathcal{L}$-Galois connection was also studied. Here, we prove that the pair $(\overline{\theta}, \overline{\theta})$ is an $\mathcal{L}$-isotope Galois connection.

**Theorem 3.2.** Let $(X, \theta)$ be an $\mathcal{L}$-fuzzy approximation space. Then for all $\mu, v \in \mathcal{L}^X$,

$$\mu \mapsto \overline{\theta}(v) = \overline{\theta}^{-1}(\mu) \mapsto v.$$

**Proof.** By Lemma 2.1 (6)–(8), we have

$$\mu \mapsto \overline{\theta}(v) = \land_{x \in \mathcal{L}} \left\{ (\mu \rightarrow x) \rightarrow \left[ \land_{y \in \mathcal{L}} [\bar{\theta}(x \rightarrow y) \rightarrow v(y)] \right] \right\} = \land_{x \in \mathcal{L}} \land_{y \in \mathcal{L}} \left\{ [\bar{\theta}(x \rightarrow y) \rightarrow v(y)] \rightarrow (\mu \rightarrow x) \right\}$$

$$= \land_{x \in \mathcal{L}} \land_{y \in \mathcal{L}} \left\{ \left[ \land_{y \in \mathcal{L}} [\bar{\theta}(x \rightarrow y) \rightarrow v(y)] \right] \rightarrow (\mu \rightarrow x) \right\} = \bar{\theta}(\mu)(x).$$

Moreover, some new results can be obtained under special classes of $\mathcal{L}$-relations.

**Corollary 3.3.** Let $(X, \theta)$ be an $\mathcal{L}$-fuzzy approximation space and $\theta$ be symmetric. Then for all $\mu, v \in \mathcal{L}^X$,

$$\mu \mapsto \overline{\theta}(v) = \overline{\theta}(\mu) \mapsto v.$$

**Theorem 3.4.** Let $(X, \theta)$ be an $\mathcal{L}$-fuzzy approximation space and $\theta$ be an $\mathcal{L}$-equivalence relation. Then for all $\mu, v \in \mathcal{L}^X$,

1. $\overline{\theta}(\mu) \mapsto \overline{\theta}(v) = \overline{\theta}(\mu) \mapsto v$.
2. $\overline{\theta}(\mu) \mapsto \overline{\theta}(v) = \mu \mapsto \overline{\theta}(v)$.
3. $\overline{\theta}(\mu) \mapsto \overline{\theta}(v) = \mu \mapsto \overline{\theta}(v)$.

**Proof.** (1) By Corollary 3.3, we get $\overline{\theta}(\mu) \mapsto \overline{\theta}(v) = \overline{\theta}(\mu) \mapsto v$. It follows from Theorem 2.14(5) that $\overline{\theta}(\mu) \mapsto \overline{\theta}(v) = \overline{\theta}(\mu) \mapsto v$. (2) and (3) can be proven in a similar way as (1). \(\square\)
Theorem 3.5. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space and \(\theta\) be symmetric. Then for all \(\mu, v \in L^X\),
\[
\mu \leq \bar{\theta}(v) \iff \bar{\theta}(\mu) \leq v.
\]

Proof. For all \(x \in X\), \(\mu(x) = \bar{\theta}(v)(x) = \bigwedge_{y \in X}(\theta(x, y) \rightarrow v(y))\). That is for all \(x,y \in X\), \(\mu(x) \leq \theta(x,y) \rightarrow v(y)\), i.e. \(\theta(x,y) \otimes \mu(x) \leq v(y)\), and hence \(\bigwedge_{x \in X}(\theta(x,y) \otimes \mu(x)) \leq v(y)\), i.e. \(\bar{\theta}(\mu)(y) \leq v(y)\). Thus \(\bar{\theta}(\mu) \leq v\).

Conversely, for all \(x \in X\), \(\bar{\theta}(\mu)(x) = \bigvee_{y \in X}(\theta(x,y) \otimes \mu(y)) \leq v(x)\), i.e. for all \(x,y \in X\), \(\theta(x,y) \otimes \mu(y) \leq v(x)\). By symmetry of \(\theta\), \(\theta(y,x) \otimes \mu(y) \leq v(x)\), and hence \(\mu(y) \leq \bigwedge_{x \in X}(\theta(y,x) \rightarrow v(x)) = \bar{\theta}(v)(y)\). So \(\mu \leq \bar{\theta}(v)\). \(\Box\)

Corollary 3.6. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space and \(\theta\) be reflexive and symmetric. Then for all \(\mu \in L^X\),
\[
\bar{\theta}(\mu) = \mu \iff \bar{\theta}(\mu) = \mu.
\]

Proof. Consider the following instance of Theorem 3.5: \(\mu \leq \bar{\theta}(\mu) \iff \bar{\theta}(\mu) \leq \mu\) and Theorem 2.14 (1), the result holds. \(\Box\)

For a given fuzzy set \(\mu\), the following theorems show that the plinth (resp. height) of \(\mu\) and that of its lower (resp. upper) approximation are identical under special classes of \(L\)-relations.

Theorem 3.7. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space and \(\theta^{-1}\) be serial. Then for all \(\mu \in L^X\),
\[
p(\bar{\theta}(\mu)) = p(\mu).
\]

Proof. By Lemma 2.1 (7), \(p(\bar{\theta}(\mu)) = \bigwedge_{x \in X}(\bar{\theta}(\mu)(x) = \bigwedge_{y \in X}(\theta(x, y) \rightarrow \mu(y)) = \bigwedge_{y \in X}(\bigwedge_{x \in X}(\theta(x, y) \rightarrow \mu(y)) = \bigwedge_{y \in X}(\bigwedge_{x \in X}(\theta(x, y) \rightarrow \mu(y)) = \bar{\theta}(\mu)(y) = \bigwedge_{y \in X}(\mu(y) = p(\mu). \Box\)

Theorem 3.8. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space and \(\theta\) be reflexive. Then for all \(\mu \in L^X\),
\[
h(\bar{\theta}(\mu)) = h(\mu).
\]

Proof. Assume that \(x = h(\mu)\) holds. Then we have \(\mu \leq 2x\). By Theorem 2.8 (4), 2.13 (1) and 2.14 (1), we get \(\mu \leq \bar{\theta}(\mu) \leq \bar{\theta}(2x) = 2x\), and hence \(x = h(\mu) \leq h(\bar{\theta}(\mu)) \leq x\). \(\Box\)

4. Topological and lattice structures determined by lower and upper sets

In this section, the notions of lower and upper sets are introduced, by which the topological and lattice structures in an \(L\)-fuzzy approximation space are investigated.

4.1. Lower and upper sets in \(L\)-fuzzy approximation spaces

Here, lower and upper sets are introduced in an arbitrary \(L\)-fuzzy approximation space and the relationships between the upper (resp. lower) set and lower (resp. upper) \(L\)-fuzzy approximation operator are shown.

Definition 4.1. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space. An \(L\)-set \(\mu\) is said to be an upper set if \(\mu(x) \otimes \theta(x,y) \leq \mu(y)\) for all \(x,y \in X\). Dually, \(\mu\) is said to be a lower set if \(\mu(y) \otimes \theta(x,y) \leq \mu(x)\) for all \(x,y \in X\).

Particularly, for an arbitrary \(L\)-relation \(\theta\), we define \(L\)-sets \([x]^{\theta} : X \mapsto L, [x]^{\theta}(x) = \theta(x, x)\) and \([z]_{\theta} : X \mapsto L, [z]_{\theta}(x) = \theta(x, z)\).

Remark 4.2. The lower and upper sets were introduced to investigate the relationships between fuzzy preorder sets and fuzzy topology in [17] with respect to a fuzzy preorder. The upper set was also called extensional fuzzy set and generator in [3–5,14] with respect to an \(L\)-equivalence relation. Topological properties of the set of extensional fuzzy sets with respect to fuzzy preorder and fuzzy equivalence relation were obtained in [5,17] based on left continuous t-norm and its residuation.

The following lemmas are obvious.

Lemma 4.3. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space. Then for all \(x \in X\),

1. If \(\theta\) is symmetric, then \(\mu\) is an upper set \(\iff\ \mu\) is a lower set.
2. \(\theta\) is transitive \(\iff [x]^{\theta}\) is an upper set \(\iff [x]_{\theta}\) is a lower set.

Lemma 4.4. Let \((X, \theta)\) be an \(L\)-fuzzy approximation space. Then \(\mu\) is an upper set in \((X, \theta)\) if and only if \(\mu\) is a lower set in \((X, \theta^{-1})\).
Theorem 4.5. Let $(X, 0)$ be an $L$-fuzzy approximation space. Then for all $\mu \in L^X$,

(1) $\mu$ is an upper set $\iff \mu \leq \bar{\theta}(\mu)$.
(2) $\mu$ is a lower set $\iff \bar{\theta}(\mu) \leq \mu$.

Proof. (1) Assume that $\mu$ be an upper set. Then for all $x, y \in X$, we have $\mu(x) \otimes \theta(x, y) \leq \mu(y)$, and hence for all $x \in X$, $\mu(x) \leq \land_{\mu \in L^X}(\theta(x, y) \rightarrow \mu(y)) = \bar{\theta}(\mu)(x)$, i.e. $\mu \leq \bar{\theta}(\mu)$. Conversely, for all $x \in X$, $\mu(x) \leq \bar{\theta}(\mu)(x) = \land_{\mu \in L^X}(\theta(x, y) \rightarrow \mu(y))$, i.e. for all $x, y \in X$, $\mu(x) \leq \theta(x, y) \rightarrow \mu(y)$. And hence $\mu(x) \otimes \theta(x, y) \leq \mu(y)$, that is, $\mu$ is an upper set.

(2) It can be proven in a similar way as (1). □

Note 4.6. The theorem above confirms that the inequality $\bar{\theta}(\mu) \leq \mu \leq \bar{\theta}(\mu)$, which is true in Pawlak rough set theory, but not here. However, it holds if $\theta$ is reflexive.

Using the concepts of lower and upper sets, Theorem 2.14 (3) and (4) can be rewritten as:

Theorem 4.7. Let $(X, 0)$ be an $L$-fuzzy approximation space. Then for all $\mu \in L^X$,

(1) $\theta$ is transitive $\iff \bar{\theta}(\mu)$ is an upper set $\iff \bar{\theta}(\mu)$ is a lower set.
(2) $\theta$ is Euclidean $\iff \bar{\theta}(\mu)$ is a lower set $\iff \bar{\theta}(\mu)$ is an upper set.

When special classes of $L$-relations are considered, some new conclusions can be obtained.

Theorem 4.8. Let $(X, 0)$ be an $L$-fuzzy approximation space and $\theta$ be reflexive. Then for all $\mu \in L^X$,

(1) $\mu$ is an upper set $\iff \mu = \bar{\theta}(\mu)$.
(2) $\mu$ is a lower set $\iff \mu = \bar{\theta}(\mu)$.

Proof. It follows immediately from Theorems 4.5 and 2.14 (1). □

Theorem 2.14 (5) and (6) also can be rewritten as:

Theorem 4.9. Let $(X, 0)$ be an $L$-fuzzy approximation space and $\theta$ be reflexive. Then for all $\mu \in L^X$,

(1) $\theta$ is an $L$-preorder $\iff \bar{\theta}(\mu)$ is an upper set $\iff \bar{\theta}(\mu)$ is a lower set.
(2) $\theta$ is an $L$-equivalence relation $\iff \bar{\theta}(\mu)$ is an upper set $\iff \bar{\theta}(\mu)$ is a lower set.

The following lemma is obvious but useful in the sequel.

Lemma 4.10. Let $(X, 0)$ be an $L$-fuzzy approximation space and $\theta$ be reflexive. Then for all $x \in X$, the following statements are equivalent:

(1) $\theta$ is transitive.
(2) $\bar{\theta}(\mu)^\theta = [\mu]^\theta$.
(3) $\bar{\theta}(\mu)^\mu = [\mu]^\mu$.

Proof. It follows immediately from Lemmas 4.3 and 4.8. □

Some special classes of $L$-relations in an $L$-fuzzy approximation space can be represented by the lower (resp. upper) approximation operator and upper (resp. lower) set.

Theorem 4.11 (Representation of $L$-preorder). Let $(X, 0)$ be an $L$-fuzzy approximation space and $\theta$ be an $L$-preorder. Then

$$\theta(x, y) = \land_{\mu \in L^X}(\bar{\theta}(\mu)(y) \rightarrow \bar{\theta}(\mu)(x)) = \land_{\mu \in L^X}(\bar{\theta}(\mu)(x) \rightarrow \bar{\theta}(\mu)(y)) = \land_{\mu \in L^X}(\bar{\theta}(\mu)(x) \rightarrow \bar{\theta}(\mu)(y)) = \land_{\mu \in L^X}(\bar{\theta}(\mu)(x) \rightarrow \bar{\theta}(\mu)(y)).$$

Theorem 4.12 (Representation of $L$-equivalence relation). Let $(X, 0)$ be an $L$-fuzzy approximation space and $\theta$ be an $L$-equivalence relation. Then

$$\theta(x, y) = \land_{\mu \in L^X}(\bar{\theta}(\mu)(x) \rightarrow \bar{\theta}(\mu)(y)) = \land_{\mu \in L^X}(\bar{\theta}(\mu)(x) \rightarrow \bar{\theta}(\mu)(y)).$$
The proofs of the theorems above are similar to those of Theorem 3.13 and 3.14 in Lai and Zhang [17] and they are necessary in the next section, so we list them here.

4.2. Topological structures in \( L \)-fuzzy approximation spaces

In this part, the topological structures in \( L \)-fuzzy approximation spaces will be investigated by lower and upper sets in detail.

We put \( \text{Fix}(\emptyset) = \{ \mu \in L^X | \emptyset(\mu) = \mu \} \) and \( \text{Fix}(\emptyset) = \{ \mu \in L^X | \emptyset(\mu) = \mu \} \), that is, the set of all the lower and upper \( L \)-fuzzy approximation sets, respectively.

Remark 4.13. Generally, both \( \text{Fix}(\emptyset) \) and \( \text{Fix}(\emptyset) \) can not form an \( L \)-topology on \( X \), because Note 2.12 demonstrates that \( 0_X \in \text{Fix}(\emptyset) \) and \( 1_X \in \text{Fix}(\emptyset) \) do not always hold for an arbitrary \( L \)-fuzzy relation.

Even so, it holds for special class of \( L \)-fuzzy relations.

Theorem 4.14. Let \((X, \emptyset)\) be an \( L \)-fuzzy approximation space and \( \emptyset \) be reflexive. Then \( \text{Fix}(\emptyset) \) forms an Alexandrov \( L \)-topology on \( X \).

Proof. We verify the items in Definition 2.4.

1. It follows immediately from Theorem 2.8(1) and Note 2.12.
2. By Theorem 2.8(6) and 2.14(1), we have \( \vee_{\mu \in T} \mu = \vee_{\mu \in T} \emptyset(\mu) = \emptyset(\vee_{\mu \in T} \mu) \leq \vee_{\mu \in T} \emptyset \), and hence \( \vee_{\mu \in T} \mu \in \tau \). By Theorem 2.8(5), \( \wedge_{\mu \in T} \mu \in \tau \).
3. For all \( \mu \in \tau \), by Theorem 4.8(1), \( \mu \) is an upper set, i.e. for all \( x, y \in X \), we have \( \mu(x) \otimes \emptyset(x, y) \leq \mu(y) \), and hence for all \( x \in L \), \( \Delta \otimes \mu(x) \otimes \emptyset(x, y) \leq \Delta \otimes \mu \). Applying Theorem 4.8(1) again, we have \( \Delta \otimes \mu \in \tau \).
4. It follows immediately from Theorem 2.10(3).

Theorem 4.15. Let \((X, \emptyset)\) be an \( L \)-fuzzy approximation space and \( \emptyset \) be reflexive. Then \( \text{Fix}(\emptyset) \) forms an Alexandrov \( L \)-topology on \( X \).

Proof

1. It follows immediately from Theorem 2.8(1) and Note 2.12.
2. Assume that \( \tau_1 \subseteq \tau \). By Theorem 2.8(5) and 2.14(1), \( \wedge_{\mu \in T} \mu \subseteq \emptyset(\wedge_{\mu \in T} \mu) \leq \emptyset(\vee_{\mu \in T} \mu) = \vee_{\mu \in T} \emptyset(\mu) = \wedge_{\mu \in T} \emptyset(\mu) = \wedge_{\mu \in T} \emptyset \mu \in \tau \).
3. By Theorem 2.10(2), \( \emptyset(\Delta \otimes \mu) = \Delta \otimes \emptyset(\mu) = \Delta \otimes \mu \), hence \( \Delta \otimes \mu \in \tau \).
4. Assume that \( \mu \in \tau \). By Theorem 4.8(2), then \( \mu(y) \otimes \emptyset(x, y) \leq \mu(x) \), i.e. \( \emptyset(x, y) \leq \mu(y) \rightarrow \mu(x) \), and hence for all \( x \in L \), \( \emptyset(x, y) \leq \mu(y) \rightarrow \mu(x) \). Hence \( \Delta \rightarrow \mu \) is a lower set. Applying Theorem 4.8(2) again, \( \Delta \rightarrow \mu \in \tau \).

Remark 4.16

1. Reflexivity is the weakest condition under which \( \text{Fix}(\emptyset) \) and \( \text{Fix}(\emptyset) \) form Alexandrov \( L \)-topologies on \( X \).
2. By Theorem 2.14(5), \( \emptyset \) and \( \emptyset \) are not the \( L \)-interior and \( L \)-closure operators in the theorems above when \( \emptyset \) is reflexive.

But it holds if \( \emptyset \) is restricted to an \( L \)-preorder.

Theorem 4.17. Let \((X, \emptyset)\) be an \( L \)-fuzzy approximation space. Then the following statements are equivalent:

1. \( \emptyset \) is an \( L \)-preorder.
2. \( \emptyset \) is an \( L \)-interior operator of \( L \)-topological space \((X, \text{Fix}(\emptyset))\).
3. \( \emptyset \) is an \( L \)-closure operator of \( L \)-topological space \((X, \text{Fix}(\emptyset))\).

Proof. It follows immediately from Theorem 2.14(5).

Remark 4.18. Lemma 4.3(1) indicates that \( \text{Fix}(\emptyset) \) and \( \text{Fix}(\emptyset) \) coincide when \( \emptyset \) is reflexive and symmetric. Thus associated with the theorem above, we can conclude that \( \emptyset \) (resp. \( \emptyset \)) is not an \( L \)-closure (resp. \( L \)-interior) operator of \((X, \text{Fix}(\emptyset)) \) (resp. \((X, \text{Fix}(\emptyset)) \)) unless \( \emptyset \) is an \( L \)-equivalence relation.

A natural problem is that for a given \( L \)-topology, under what conditions it can be induced by the \( L \)-fuzzy approximation operators.
Theorem 4.19. Let $\tau$ be an Alexandrov $\mathcal{L}$-topology on $X$. Then there exists an $\mathcal{L}$-preorder $\vartheta$ such that $\tau = \text{Fix}(\vartheta)$.

Proof. Assume that $\tau$ is an Alexandrov $\mathcal{L}$-topology on $X$. We define an $\mathcal{L}$-relation $\vartheta$ as $\vartheta(x, y) = \wedge_{\mathcal{L}}(\mu(x) \rightarrow \mu(y))$. Then $\vartheta$ is an $\mathcal{L}$-preorder and $\tau \subseteq \text{Fix}(\vartheta)$. Next, we show that $\text{Fix}(\vartheta) \subseteq \tau$. For all $v \in \text{Fix}(\vartheta)$, by Lemma 4.4 and Theorem 4.8(2), we get $\vartheta^{-1}(v) = v$. For all $x \in X$,

$$v(x) = \vartheta^{-1}(v)(x) = \bigvee_{y \in X} \left( \vartheta^{-1}(x, y) \otimes v(y) \right) = \bigvee_{y \in X} \left( \left[ \wedge_{\mathcal{L}}(\mu(y) \rightarrow \mu(x)) \right] \otimes v(y) \right)$$

$$= \left\{ \bigvee_{y \in X} \left[ \left( \wedge_{\mathcal{L}}(\mu(y) \rightarrow \mu(x)) \otimes v(y) \right) \right] \right\}(x).$$

Since $\mu \in \tau$, by Definition 2.4, we have $v \in \tau$, and hence $\text{Fix}(\vartheta) = \tau$. \qed

Theorem 4.20. Let $\tau$ be an Alexandrov $\mathcal{L}$-topology on $X$. Then there exists an $\mathcal{L}$-preorder $\vartheta$ such that $\tau = \text{Fix}(\vartheta)$.

Proof. Assume that $\tau$ is an Alexandrov $\mathcal{L}$-topology on $X$. We define an $\mathcal{L}$-relation $\vartheta$ as $\vartheta(x, y) = \wedge_{\mathcal{L}}(\mu(y) \rightarrow \mu(x))$. Then $\vartheta$ is an $\mathcal{L}$-preorder and $\tau \subseteq \text{Fix}(\vartheta)$. Next, we show that $\text{Fix}(\vartheta) \subseteq \tau$. For all $v \in \text{Fix}(\vartheta)$, we have $\vartheta(v) = v$. For all $x \in X$,

$$v(x) = \vartheta(v)(x) = \bigvee_{y \in X} \left( \vartheta(x, y) \otimes v(y) \right) = \bigvee_{y \in X} \left( \left[ \wedge_{\mathcal{L}}(\mu(y) \rightarrow \mu(x)) \right] \otimes v(y) \right) = \left\{ \bigvee_{y \in X} \left[ \left( \wedge_{\mathcal{L}}(\mu(y) \rightarrow \mu(x)) \otimes v(y) \right) \right] \right\}(x).$$

By Definition 2.4, then $v \in \tau$. Hence $\text{Fix}(\vartheta) = \tau$. \qed

Remark 4.21. Theorem 4.14 and 4.19 (resp. Theorem 4.15 and 4.20) shows that the set of all the lower (resp. upper) $\mathcal{L}$-fuzzy approximation sets and the Alexandrov $\mathcal{L}$-topology can be induced each other if and only if the $\mathcal{L}$-relation associated with is an $\mathcal{L}$-preorder. Although Theorem 4.14 and 4.19 were also proven in [10], a brief proof by the notion of upper set is given here.

Since the $\mathcal{L}$-interior operators and $\mathcal{L}$-closure operators are basic concepts in $\mathcal{L}$-topological spaces, we also can consider the problem above as follows.

Theorem 4.22. Let $\psi$ be an $\mathcal{L}$-interior operator of $X$. Then there exists an $\mathcal{L}$-preorder $\vartheta$ such that $\psi = \vartheta$ if and only if $\psi$ satisfies:

1. $\psi(\wedge_{\mathcal{L}} \mu) = \wedge_{\mathcal{L}} \psi(\mu)$.
2. $\psi(x_1 \rightarrow x_2) = \left\{ \wedge_{\mathcal{L}}(\psi(x_1 \rightarrow \beta_2) \rightarrow \beta_1) \right\} \rightarrow x_2$.

Proof. Assume that $\vartheta$ is an $\mathcal{L}$-preorder. Then it follows from Theorem 2.8 (5) that (1) holds. Applying Theorem 2.10 (4), we get $\vartheta(\psi(x_1 \rightarrow \beta_2)(x) = \vartheta(x, y) \rightarrow \beta$. Lemma 2.1 (9) leads to $\left\{ \left[ \wedge_{\mathcal{L}}(\vartheta(x, y) \rightarrow \beta) \right] \rightarrow \alpha \right\} \rightarrow x_2 \rightarrow \alpha$, and hence (2) holds. Conversely, assume that (1) and (2) hold. By (2), an $\mathcal{L}$-relation $\vartheta$ is defined as $\vartheta(x, y) = \wedge_{\mathcal{L}}(\psi(y \rightarrow \beta_2)(x) \rightarrow \beta)$ for all $x, y \in X$. It is easy to prove that $\mu(x) = \wedge_{\mathcal{L}}(\mu(x) \rightarrow \psi(\mu))(x)$. By (1), we get $\vartheta(\mu(x)) = \wedge_{\mathcal{L}}(\vartheta(x, y) \rightarrow \mu(y))(x) = \psi(\wedge_{\mathcal{L}}(\psi(x_1 \rightarrow \beta_2) \rightarrow \beta_1)) \rightarrow x_2 \rightarrow \alpha$. Hence $\psi = \vartheta$. It is trivial to prove that $\vartheta$ is an $\mathcal{L}$-preorder. \qed

Theorem 4.23. Let $\varphi$ be an $\mathcal{L}$-closure operator of $X$. Then there exists an $\mathcal{L}$-preorder $\vartheta$ such that $\varphi = \vartheta$ if and only if $\varphi$ satisfies:

1. $\varphi(\bigvee_{\mathcal{L}} \mu) = \bigvee_{\mathcal{L}} \varphi(\mu)$.
2. $\varphi(\mu \otimes x) = \varphi(\mu) \otimes x$.

Proof. Assume that $\vartheta$ is an $\mathcal{L}$-preorder. It follows that (1) holds from Theorem 2.8 (6). Theorem 2.10 (2) yields that (2) holds. Conversely, it is easy to prove that $\mu = \bigvee_{\mathcal{L}}(\mu(y_1) \otimes \mu(y_2))$. By (2), an $\mathcal{L}$-relation $\vartheta$ is defined as $\vartheta(x, y) = \varphi(y_1)(x)$. For all $x \in X$, $\vartheta(\psi(x))(x) = \bigvee_{\mathcal{L}}(\vartheta(x, y) \rightarrow \mu(y))(x) = \bigvee_{\mathcal{L}}(\varphi(y_1)(x) \otimes \mu(y))(x) = \bigvee_{\mathcal{L}}(\varphi(y_1) \otimes \mu(y))(x) = \varphi(\bigvee_{\mathcal{L}}(\psi(x_1 \otimes \mu(y))(x))$. It is trivial to prove that $\vartheta$ is an $\mathcal{L}$-preorder. \qed

Remark 4.24. Although the two theorems above were also obtained in [32] from an axiomatic approach to $\mathcal{L}$-fuzzy rough sets, here they are proven from a constructive one.

4.3. Lattice structures in $\mathcal{L}$-fuzzy approximation spaces

As it is well-known that $(\mathcal{L}^X, \leq)$ forms a complete distributive lattice. In [7], Estaji defined the concept of lower fixed-point (i.e. lower $\mathcal{L}$-fuzzy approximation set), and proved that the set of all lower fixed-points was a sublattice of $(\mathcal{L}^X, \leq)$. But he cited incorrect results in his proof (See [7] Proposition 3.4) which even do not hold for any Pawlak rough set [25]. Here, we reconsider the fixed-points in $\mathcal{L}$-fuzzy rough sets.
An $\mathcal{L}$-set $\mu$ is called a lower (resp. upper) fixed-point if $\mathcal{L}(\mu) = \mu$ (resp. $\mathcal{R}(\mu) = \mu$).

**Theorem 4.25.** Let $(X, \theta)$ be an $\mathcal{L}$-fuzzy approximation space and $\theta$ be reflexive. Then for all $\mu \in \mathcal{L}^X$,

1. $\mu$ is an upper set if and only if $\mu$ is a lower fixed-point.
2. $\mu$ is a lower set if and only if $\mu$ is an upper fixed-point.

**Proof.** It follows immediately from Theorem 4.8. □

**Note 4.26.** Theorem 2.8(1) shows that $\text{Fix}(\mathcal{L})$ and $\text{Fix}(\mathcal{R})$ are always nonempty.

**Theorem 4.27.** Let $(X, \theta)$ be an $\mathcal{L}$-fuzzy approximation space and $\theta$ be reflexive. Then

1. $(\text{Fix}(\mathcal{L}), \wedge, \vee)$ forms a complete sublattice of $\mathcal{L}^X$.
2. $(\text{Fix}(\mathcal{R}), \wedge, \vee)$ forms a complete sublattice of $\mathcal{L}^X$.

**Proof.** Since $(\text{Fix}(\mathcal{L}), \wedge, \vee)$ and $(\text{Fix}(\mathcal{R}), \wedge, \vee)$ are subsets of $\mathcal{L}^X$ and $\mathcal{L}^X$ is a lattice, to prove $(\text{Fix}(\mathcal{L}), \wedge, \vee)$ and $(\text{Fix}(\mathcal{R}), \wedge, \vee)$ are lattices we only need to prove they are complete. These hold by Theorem 4.14. □

**Note 4.28.** When $\theta$ is reflexive, Theorem 2.14 (4) and (5) demonstrate that $\text{Fix}(\mathcal{L})$ and $\text{Fix}(\mathcal{R})$ are different. However, when $\theta$ is reflexive and symmetric, by Corollary 3.6, $\text{Fix}(\mathcal{L})$ and $\text{Fix}(\mathcal{R})$ possess the same lattice structure.

5. Conclusion

In this paper, the topological and lattice structures of $\mathcal{L}$-fuzzy rough sets by lower and upper sets were determined. Our main conclusions are list as follows:

1. The upper (resp. lower) set is equivalent to the lower (resp. upper) $\mathcal{L}$-fuzzy approximation set under a reflexive $\mathcal{L}$-relation.
2. An $\mathcal{L}$-preorder is the equivalence condition under which the set of all the lower (resp. upper) $\mathcal{L}$-fuzzy approximation sets and the Alexandrov $\mathcal{L}$-topology coincide.
3. Associating with an $\mathcal{L}$-preorder, the equivalence condition that $\mathcal{L}$-fuzzy interior (resp. closure) operator meets with the lower (resp. upper) $\mathcal{L}$-fuzzy approximation operator is provided.
4. The set of all the lower (resp. upper) $\mathcal{L}$-fuzzy approximation sets forms a complete lattice under a reflexive $\mathcal{L}$-relation.

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