Global stability of an SEIR epidemic model with constant immigration

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Abstract

An SEIR epidemic model with the infectious force in the latent (exposed), infected and recovered period is studied. It is assumed that susceptible and exposed individuals have constant immigration rates. The model exhibits a unique endemic state if the fraction $p$ of infectious immigrants is positive. If the basic reproduction number $R_0$ is greater than 1, sufficient conditions for the global stability of the endemic equilibrium are obtained by the compound matrix theory.

1. Introduction

Many infectious diseases in nature incubate inside the hosts for a period of time before the hosts become infectious. We assume the population can be partitioned into four compartments: susceptible, latent or exposed, infectious and recovered, with sizes denoted by $S$, $E$, $I$ and $R$, respectively. The total population $N = S + E + I + R$. Using a compartmental approach, one may assume that a susceptible individual first goes through a latent period (in class $E$) after infection before becoming infectious. The resulting models are of SEI, SEIR or SEIRS type, respectively.


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infectious force in latent and infected period. Research on epidemic models of SEIR with infectious force in both latent, infected and recovered period are scarce in the literature. In the present paper, we consider an SEIR model with infectious force in latent, infected and recovered period and having constant immigration. To establish the global stability of the unique endemic equilibrium, we introduce a change of variables by which our four-dimensional SEIR model can be reduced to a three-dimensional asymptotical autonomous differential system. This paper is organized as follows. In Section 2, we formulate the model with infectious force in latent, infected and recovered periods and having constant immigration and discuss the existence of equilibrium. In Section 3, we consider the global stability of the endemic equilibrium when $p > 0$.

2. Model formulation

The total host population is partitioned into susceptibles, exposed (in the latent period), infectious, and recovered, with the densities, respectively, denoted by $S(t)$, $E(t)$, $I(t)$ and $R(t)$. The total population size at time $t$ is denoted by $N(t)$, with $N = S + E + I + R$. The transfer diagram is depicted in the following figure:

\[
\begin{array}{cccccc}
S & \xrightarrow{(1-p)A} & \beta_{10}SE & \xrightarrow{\beta_{20}SI} & \beta_{30}SR & \xrightarrow{\mu S} \\
& & \xrightarrow{\mu E} & \alpha_0 I & \xrightarrow{k_0 I} & R \\
& & \xrightarrow{\gamma_0 E} & & & \\
E & & & & & \\
I & & & & & \\
R & & & & & \\
\end{array}
\]

The model is described by the following system of differential equations:

\[
\begin{aligned}
S' &= (1-p)A - \beta_{10}SE - \beta_{20}SI - \beta_{30}SR - \mu S, \\
E' &= pA + \beta_{10}SE + \beta_{20}SI + \beta_{30}SR - \gamma_0 E - \mu E, \\
I' &= \gamma_0 E - k_0 I - \alpha_0 I - \mu I, \\
R' &= k_0 I - \mu R,
\end{aligned}
\]

where the positive parameters $\mu$ is the rate of natural death. $\alpha_0$ is a non-negative constant and denote the rate of disease-caused death. The parameter $\gamma_0$ denote the transfer rates between the exposed and the infectious. The constant $k_0$ is the rate at which the infectious individuals recover. $\beta_{10}$, $\beta_{20}$, $\beta_{30}$ are the rate of the efficient contact in the latent, infected and recovered period. $(1-p)A$, $pA$ are constant recruitment of susceptibles, exposed, respectively. Therefore $\frac{1}{\alpha_0}$ is the mean latent period and $\frac{1}{\gamma_0}$ is the mean infectious period.

Let

\[
\tau = \mu t, \quad \beta_1 = \frac{\beta_{10}}{\mu}, \quad \beta_2 = \frac{\beta_{20}}{\mu}, \quad \beta_3 = \frac{\beta_{30}}{\mu}, \quad \gamma = \frac{\gamma_0}{\mu}, \quad k = \frac{k_0}{\mu}, \quad c = \frac{A}{\mu}, \quad \alpha = \frac{\alpha_0}{\mu}.
\]

We obtain the following system analogous to (2.1):

\[
\begin{aligned}
\frac{dS}{dt} &= (1-p)c - \beta_1 SE - \beta_2 SI - \beta_3 SR - S, \\
\frac{dE}{dt} &= pc + \beta_1 SE + \beta_2 SI + \beta_3 SR - \gamma E - E, \\
\frac{dI}{dt} &= \gamma E - kI - \alpha I - I, \\
\frac{dR}{dt} &= kI - R.
\end{aligned}
\]

According to the fact, we assume $\beta_2 > \beta_3$. The total population size $N(t)$ can be determined by $N(t) = S(t) + E(t) + I(t) + R(t)$ or from the differential equation

\[
\frac{dN}{dt} = c - N - \alpha I.
\]
It is convenient to use $E$, $I$, $R$ and $N$ as variables and replace $S$ by $N - E - I - R$. This gives the model

$$\begin{align*}
\frac{dE}{dt} &= pc + (\beta_1 E + \beta_2 I + \beta_3 R)(N - E - I - R) - \delta E, \\
\frac{dI}{dt} &= \gamma E - \omega I, \\
\frac{dR}{dt} &= kI - R, \\
\frac{dN}{dt} &= c - N - \alpha I,
\end{align*}$$

(2.3)

where $\delta = 1 + \gamma$, $\omega = 1 + k + \alpha$. The system (2.3) is equivalent to the system (2.2). This allows us to attack (2.2) by studying the system (2.3). From biological considerations, we study the system (2.3) in the closed set

$$T = \left\{ (E, I, R, N) \in \mathbb{R}_+^4 : 0 \leq E + I + R \leq N \leq c = \frac{A}{\mu} \right\}.$$

It can be verified that $T$ is positively invariant with respect to the system (2.3).

If $\beta_1, \beta_2, \beta_3 = 0$, so that the exposed are those who have entered the population from outside, this reduces to the linear non-homogeneous system

$$\begin{align*}
\frac{dE}{dt} &= pc - \delta E, \\
\frac{dI}{dt} &= \gamma E - \omega I, \\
\frac{dR}{dt} &= kI - R, \\
\frac{dN}{dt} &= c - N - \alpha I.
\end{align*}$$

(2.3)

Every solution approaches the equilibrium

$$E = E_0 = \frac{pc}{\delta}, \quad I = I_0 = \frac{\gamma p c}{\delta \omega}, \quad R = R_0 = \frac{\gamma k p c}{\delta \omega}, \quad N = N_0 = \frac{c(\delta \omega - \gamma \alpha p)}{\delta \omega}.$$

We would expect an equilibrium with $\beta_2, \beta_3 > 0$ to satisfy $E \geq E_0, I \geq I_0, R \geq R_0, N \leq N_0$. So let the right side of each of the four differential equations equal to zero in system (2.3), obtaining the equation

$$\begin{align*}
pc + (\beta_1 E + \beta_2 I + \beta_3 R)(N - E - I - R) - \delta E &= 0, \\
\gamma E - \omega I &= 0, \\
kI - R &= 0, \\
c - N - \alpha I &= 0.
\end{align*}$$

(2.4)

In order to find equilibria for $\beta_1, \beta_2, \beta_3 > 0$, we substitute $N = c - \alpha I$ into

$$pA + (\beta_1 E + \beta_2 I + \beta_3 R)(N - E - I - R) - \delta E = 0$$

to obtain the quadratic equation

$$G(I) = (\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma) \delta I^2 + \gamma (\delta \omega - c(\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma)) I - \gamma^2 pc = 0.$$

If $p = 0$, one root is $I = 0$, and there is a second root

$$I^* = \frac{\gamma [c(\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma) - \delta \omega]}{\delta \omega (\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma)},$$

which is positive if and only if

$$\sigma = c(\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma) - \delta \omega > 0.$$

If $p > 0$, the quadratic equation $G(I) = 0$ has one positive and one negative root [12]. The disease-free equilibrium, $I = 0$, that occurs when $p = 0$ now become negative (not biologically feasible). The positive root is

$$I^* = \frac{\gamma \sigma + \gamma \sqrt{\sigma^2 + 4pc \delta \omega (\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma)}}{2 \delta \omega (\beta_1 \omega + \beta_2 \gamma + \beta_3 k \gamma)}.$$
The Jacobian matrix of the system (2.3) at a point $a$ is

$$J(a) = \begin{pmatrix} \gamma \beta_1 - n - \delta & (m - \frac{\gamma}{\omega}) \beta_2 - n & (m - \frac{\gamma}{\omega}) \beta_3 - n & n \\ \gamma & -\omega & 0 & 0 \\ 0 & k & -1 & 0 \\ 0 & -\gamma & 0 & -1 \end{pmatrix},$$

where $m = \frac{\delta \omega}{\omega \beta_1 + \beta_2 \beta_3}$, $n = \beta_1 E^* + \beta_2 I^* + \beta_3 R^*$. Its characteristic equation is $\det(\lambda I - J(a)) = 0$, where $I$ is the unit matrix and

$$N^* - E^* - I^* - R^* = \frac{\delta E^* - pc}{\beta_1 E^* + \beta_2 I^* + \beta_3 R^*}, \quad E^* = \frac{\omega \gamma}{\gamma}, \quad R^* = kI^*.$$

So the characteristic equation become to $(\lambda + 1)(\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3) = 0$, where

$$a_1 = 1 + \omega + n + \delta + \frac{pc \beta_1}{n} - m \beta_1 = 1 + \omega + n + \frac{pc \beta_1}{n} + \frac{\delta (\gamma \beta_2 + k \beta_3)}{\omega \beta_1 + \gamma \beta_2 + k \beta_3} > 0,$$

$$a_2 = \omega + \left(n + \delta + \frac{pc \beta_1}{n}\right) (1 + \omega) + \gamma \left(n + \frac{pc \beta_2}{n}\right) - m \beta_1 (1 + \omega) - \gamma m \beta_2 = \omega + n (\delta + \omega) + \frac{pc \beta_1}{n} (1 + \omega) + \frac{pc \beta_2}{n} + \frac{\delta \gamma (\beta_2 + (1 + \omega) \beta_3)}{\omega \beta_1 + \gamma \beta_2 + k \beta_3} > 0,$$

$$a_3 = (n + 1) \delta \omega + k \gamma \frac{pc \beta_3}{n} + \omega \frac{pc \beta_1}{n} + \gamma \frac{pc \beta_2}{n} - m (\omega \beta_1 + \gamma \beta_2 + k \beta_3) = n \delta \omega + k \gamma \frac{pc \beta_3}{n} + \omega \frac{pc \beta_1}{n} + \gamma \frac{pc \beta_2}{n} > 0.$$

We calculate easily $a_1 a_2 - a_3 > 0$ for $\beta_2 > \beta_3$. According to Hurwitz criterion, the epidemic equilibrium $P^*$ is local asymptotically stable. $\square$

To prove the global stability behavior of this equilibrium, we let $z = 0$ in the system (2.3) and define the new variables

$$X = \frac{\mu}{A} S, \quad Y = \frac{\mu}{A} E, \quad Z = \frac{\mu}{A} I, \quad R_1 = \frac{\mu}{A} R.$$
Using these change of variables, the system (2.3) becomes
\[
\begin{align*}
\frac{dX}{dt} &= (1 - p) - c\beta_1 XY - c\beta_2 XZ - c\beta_3 XR_1 - X, \\
\frac{dY}{dt} &= p + c\beta_1 XY + c\beta_2 XZ + c\beta_3 XR_1 - \gamma Y - Y, \\
\frac{dZ}{dt} &= \gamma Y - Z - kZ, \\
\frac{dR_1}{dt} &= kZ - R_1, \\
\end{align*}
\]
(3.1)
with \(N(t) = X(t) + Y(t) + Z(t) + R_1(t)\). The equation for the total population \(N_1\) is
\[
\frac{dN_1}{dt} = 1 - N_1.
\]
Since \(N \to 1\) as \(t \to +\infty\), substituting \(X = 1 - Y - Z - R_1\) into the system (3.1), we can obtain the limit system
\[
\begin{align*}
\frac{dY}{dt} &= p + c(\beta_1 Y + \beta_2 Z + \beta_3 R_1)(1 - Y - Z - R_1) - \gamma Y - Y, \\
\frac{dZ}{dt} &= \gamma Y - Z - kZ, \\
\frac{dR_1}{dt} &= kZ - R_1, \\
\end{align*}
\]
(3.2)
We make the change of variable \(x = 1 - Y - Z - R_1\), \(y = Y\), \(z = Z\), then the following system (3.3) is equivalent to (3.2):
\[
\begin{align*}
\frac{dx}{dt} &= 1 - p - x - c[\beta_1 y + \beta_2 z + \beta_3(1 - x - y - z)]x, \\
\frac{dy}{dt} &= p + c[\beta_1 y + \beta_2 z + \beta_3(1 - x - y - z)]x - \gamma y - y, \\
\frac{dz}{dt} &= \gamma y - z - kz, \\
\end{align*}
\]
(3.3)
with \(N_2(t) = x(t) + y(t) + z(t)\). The equation for the total population \(N_2\) is
\[
\frac{dN_2}{dt} = 1 - N_2 - kz.
\]
It is easy to see that the feasible region is
\[
\Gamma = \{ (x, y, z) \in R^3 : x, y, z \geq 0, x + y + z = N_2 \leq 1 \}.
\]
Let \(E^* = (x^*, y^*, z^*)\) is the unique positive equilibrium of the system (3.3). The Jacobian matrix \(J\) of the system (3.3) is
\[
J = \begin{bmatrix}
a_{11} & -c(\beta_1 - \beta_3)x & -c(\beta_2 - \beta_3)x \\
a_{21} & c(\beta_1 - \beta_3)x - \gamma - 1 & -c(\beta_2 - \beta_3)x \\
0 & \gamma & -1 - k
\end{bmatrix},
\]
(3.4)
where
\[
a_{11} = -1 + c\beta_3 x + c(\beta_1 y + \beta_2 z + \beta_3(1 - x - y - z)), \\
a_{21} = -c\beta_3 x + c(\beta_1 y + \beta_2 z + \beta_3(1 - x - y - z)).
\]
In the following, using the geometrical approach of Li and Muldowney in [13], we obtain simple sufficient conditions that the disease steady state \(E^*\) is globally asymptotically stable. At first, we give a brief outline of this geometrical approach.
Let \(x \mapsto f(x) \in R^e\) be a \(C^1\) function for \(x\) in an open set \(D \subset R^e\). Consider the differential equation
\[
x' = f(x).
\]
(3.5)
Denote by \(x(t, x_0)\) the solution to (3.5) such that \(x(0, x_0) = x_0\). We make the following two assumptions:

(H_1) There exists a compact absorbing set \(K \subset D\).

(H_2) Eq. (3.5) has a unique equilibrium \(\bar{x}\) in \(D\).
The equilibrium $\bar{x}$ is said to be globally stable in $D$ if it is locally stable and all trajectories in $D$ converge to $\bar{x}$. For $n \geq 2$, by a Bendixson criterion we mean a condition satisfied by $f$ which precludes the existence of non-constant periodic solutions of (3.5). The classical Bendixson’s condition $\text{div}(f(x)) < 0$ for $n = 2$ is robust under $C^1$ local perturbations of $f$. For higher dimensional systems, the $C^1$ robust properties are discussed in [13-15].

A point $x_0 \in D$ is wandering for (3.5) if there exists a neighborhood $U$ of $x_0$ and $T > 0$ such that $U \cap x(t, U)$ is empty for all $t > T$. Thus, for example, all equilibria and limit points are non-wandering. The following global-stability principle is established in Li and Muldowney [13] for autonomous systems in any finite dimension.

**Theorem 3.3.** Suppose that assumptions (H1) and (H2) hold. Assume that (3.5) satisfies a Bendixson criterion that is robust under $C^1$ local perturbations of $f$ at all non-equilibrium non-wandering points for (3.5). Then $\bar{x}$ is globally stable in $D$ provided it is stable.

The following Bendixson criterion is given in [13] and shown to have the robustness required by Theorem 3.3. Let $x \mapsto P(x)$ be an $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is $C^1$ for $x \in D$. Assume that $P^{-1}(x)$ exists and is continuous for $x \in K$, the compact absorbing set. A quantity $\bar{q}_2$ is defined as

$$\bar{q}_2 = \limsup_{t \to \infty} \sup_{x \in K} \frac{1}{t} \int_0^t \mu(B(x(s, x_0))) \, ds,$$

where

$$B = P_j P^{-1} + P_{ij}^{f[2]} P^{-1}$$

the matrix $P_j$ is obtained by replacing each entry $p$ of $P$ by its derivative in the direction of $f$, $p_{ij}$, and $\mu(B)$ is the Lozinskiĭ measure of $B$ with respect to a vector norm $|\cdot|$ in $\mathbb{R}^N$, $N = \binom{n}{2}$, defined by [18, p. 41]

$$\mu(B) = \lim_{h \to 0} \frac{|I + hB| - 1}{h}.$$

It is shown in [13] that, if $D$ is simply connected, the condition $\bar{q}_2 < 0$ rules out the presence of any orbit that gives rise to a simple closed rectifiable curve that is invariant for (3.5), such as periodic orbits, homoclinic orbits, and heteroclinic cycles. Moreover, it is robust under $C^1$ local perturbations of $f$ near any non-equilibrium point that is non-wandering. In particular, the following global-stability result is proved in Li and Muldowney [13].

**Theorem 3.4.** Assume that $D$ is simply connected and that the assumptions (H1) and (H2) hold. Then the unique equilibrium $\bar{x}$ of (3.5) is globally stable in $D$ if $\bar{q}_2 < 0$.

Now, we study the global stability of the disease steady state $E^*$, and obtain

**Theorem 3.5.** If $R_0 > 1$, $\beta_1 > \beta_3$ and $\gamma + c \beta_1(1 - p) < 1$, $c \beta_3(1 - p) < 1 + p$, then the disease steady state $E^*$ of the system (3.3) is globally asymptotically stable.

**Proof.** The Jacobian matrix $J$ associated with a general solution to (3.3) is (3.4), its second additive compound matrix $J^{[2]}$ is

$$J^{[2]} = \begin{bmatrix}
11 & c(\beta_2 - \beta_3)x & c(\beta_2 - \beta_3)x \\
\gamma & b_{22} & -c(\beta_1 - \beta_3)x \\
0 & b_{32} & b_{33}
\end{bmatrix},$$

where

$$b_{11} = -2 - \gamma + c \beta_1 x - c(\beta_1 y + \beta_3 z) - c \beta_1(1 - x - y - z),$$

$$b_{22} = -2 - k + c \beta_3 x - c(\beta_1 y + \beta_3 z) - c \beta_1(1 - x - y - z),$$

$$b_{32} = -c \beta_3 x + c(\beta_1 y + \beta_3 z) + c \beta_1(1 - x - y - z),$$

$$b_{33} = -2 - \gamma - k + c(\beta_1 - c \beta_3)x.$$

A comprehensive survey on compound matrices and their relations to differential equations is given in [16]. Set the function
\[
P(x, y, z) = \begin{bmatrix} z & 0 & 0 \\
0 & y & 0 \\
0 & y & y \end{bmatrix}.
\]

Then
\[
P_f P^{-1} = \text{diag}\left\{ \frac{x'}{x}, \frac{y'}{y}, \frac{z'}{z} \right\}
\]

and the matrix \( B = P_f P^{-1} + P^{(2)}_f P^{-1} \) in (3.7) can be written in block form
\[
B = \begin{bmatrix} B_{11} & B_{12} \\
B_{21} & B_{22} \end{bmatrix},
\]

where
\[
B_{11} = \frac{x'}{x} - 2 - \gamma + c\beta_1 x - c(\beta_1 y + \beta_2 z) - c\beta_3 (1 - x - y - z),
\]
\[
B_{12} = \begin{bmatrix} 0 & c(\beta_2 - \beta_3) x z \\
\gamma y & \gamma y z \end{bmatrix},
\]
\[
B_{21} = \begin{bmatrix} y' \gamma y + \gamma y z & -c(\beta_1 - \beta_3) x \\
\gamma y z & y' \gamma y + \gamma y z \end{bmatrix},
\]
\[
B_{22} = \begin{bmatrix} y' \gamma y + \gamma y z & -c(\beta_1 - \beta_3) x \\
\gamma y z & y' \gamma y + \gamma y z \end{bmatrix}.
\]

Let \((u, v, w)\) denotes the vectors in \(\mathbb{R}^3 \cong \mathbb{R}^{3(3)}\), we select a norm in \(\mathbb{R}^3\) as \(|(u, v, w)| = \max \{|u|, |v|, |w|\}\) and let \(\mu\) denotes the Lozinski measure with respect to this norm. Following the method in [17], we have the estimate \(\mu(B) \leq \sup \{g_1, g_2\}\), where
\[
g_1 = \mu_1(B_{11}) + |B_{12}|, \quad g_2 = |B_{21}| + \mu_1(B_{22}).
\]

\(|B_{12}|, |B_{21}|\) are matrix norms with respect to the \(l_1\) vector norm, and \(\mu_1\) denotes the Lozinski measure with respect to the \(l_1\) norm, see [18, p. 41]. More specifically, \(\mu_1(B_{11}) = \frac{\gamma}{3} - \frac{\gamma}{2} + c\beta_1 x - c(\beta_1 y + \beta_2 z) - c\beta_3 (1 - x - y - z),\)
\(|B_{12}| = \frac{c(\beta_2 - \beta_3) x z}{y}, |B_{22}| = \gamma y z\). To calculate \(\mu_1(B_{22})\), add the absolute value of the off-diagonal elements to the diagonal one in each column of \(B_{22}\), and then take the maximum of two sums, see [18, p. 41]. Since \(\beta_1 > \beta_3\), we have
\[
\mu_1(B_{22}) = \max \left\{ \frac{y'}{y} - 2 - k + \gamma + c\beta_1 x - c[\beta_1 y + \beta_2 z + \beta_3 (1 - x - y - z)], \frac{y'}{y} - 2 - k - \gamma + c(\beta_1 - \beta_3) x \right\}
\]
\[
\leq \frac{y'}{y} - 2 - k + \gamma + c\beta_1 x.
\]

From the system (3.3), we have
\[
x < 1 - p,
\]
\[
\frac{y'}{y} = \frac{p}{y} + c\beta_1 x - c\beta_3 x + \frac{c(\beta_2 - \beta_3) x z}{y} + \frac{c\beta_3 (1 - x) x}{y} - 1 - \gamma,
\]
\[
\frac{z'}{z} = \frac{\gamma y z}{z} - 1 - k.
\]

So we have
\[
g_1 = \frac{y'}{y} + \frac{z'}{z} - 1 - \frac{p}{y} + c\beta_3 x - c(\beta_1 y + \beta_2 z) - c\beta_3 (1 - x - y - z) - \frac{c\beta_3 (1 - x) x}{y}
\]
\[
< \frac{y'}{y} + \frac{z'}{z} - 1 - p + c\beta_3 (1 - p),
\]
\[
g_2 < \frac{y'}{y} + \frac{z'}{z} - 1 + \gamma + c\beta_1 x
\]
\[
< \frac{y'}{y} + \frac{z'}{z} - 1 + \gamma + c\beta_1 (1 - p).
\]
We can choose $t_1$ large enough such that
\begin{align*}
g_1 &< \frac{y'}{y} + \frac{z'}{z} - 1 - p + c\beta_2(1 - p), \\
g_2 &< \frac{y'}{y} + \frac{z'}{z} - 1 + \gamma + c\beta_1(1 - p)
\end{align*}
for $t \geq t_1$, where $\delta$ can be chosen arbitrarily small.
Therefore,
\[\mu(B) \leq \frac{y'}{y} + \frac{z'}{z} - \bar{b}\]
for $t \geq t_1$, where $\bar{b} = \min\{1 + p - c\beta_3(1 - p), 1 - \gamma - c\beta_1(1 - p)\} = 1 - \gamma - c\beta_1(1 - p) > 0$ is a constant. Along each solution $(x(t), y(t), z(t))$ of (3.3) with $(x(0), y(0), z(0)) \in \Gamma$, where $\Gamma$ is the compact absorbing set, we have
\[\frac{1}{t} \int_0^t \mu(B) \, ds = \frac{1}{t} \int_0^{t_1} \mu(B) \, ds + \frac{1}{t} \int_{t_1}^t \mu(B) \, ds \leq \frac{1}{t} \int_0^{t_1} \mu(B) \, ds + \frac{1}{t} \log \frac{y(t)}{y(t_1)} + \frac{1}{t} \log \frac{z(t)}{z(t_1)} - \bar{b},\]
which implies that $\bar{q}_2 \leq -\bar{b}/2 < 0$ from (3.6). This complete the proof. $\square$

4. Conclusion

In the paper, we discuss the SEIR model with constant immigration and infectious force in the latent and recovered period, too. We derive a basic reproduction number $R_0$ and that it determines the global dynamics of (2.1); if $R_0 > 1$, a unique endemic equilibrium $P^*$ is globally asymptotically stable in the interior of the feasible region so that the disease persists at the endemic equilibrium level if it is initially present. In the limiting case when $\gamma_0 \to \infty$, the latent period is negligible, and the model (2.1) reduces to an SIR model with bilinear incidence. When $\gamma_0 = 0$, there is no recovery from the disease, and (2.1) reduces to an SEI model. When $p = 0$, (2.1) becomes an SEIR with constant immigration only susceptibles.

References