An Inventory Model Perishable Products with Markovian Renewal Demands

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Abstract. In the inventory model, people usually assume that the inter-demand times are independent and identically distributed, but that may not be true in reality. This limitation motivated us to seek greater understanding of an important class of inventory models with dependent demand. In this paper we analyze an \((s,S)\) continuous review perishable inventory system with a Markovian renewal demand process. Using a Markovian renewal approach, we derive the transition matrix and some performance measures of the system. We then construct a closed-form expected cost function showing some managerial insights on controlling such inventory systems.

Keywords: Perishable inventory, Markovian renewal demand, optimization.

1. INTRODUCTION

In this paper, we study the replenishment problem of an inventory system for a single product with a fixed shelf lifetime. In the inventory literature, stored items with fixed finite lifetimes are usually referred to as perishable items. Examples of perishable products include fresh foods, medical products, whole-blood units, packaged chemical products, etc.

Classical inventory models have assumed the demand in each period to be a random variable independent of demands in other periods and of environmental factors other than time. As elaborated in Song and Zipkin (1993), many randomly changing environmental factors, such as fluctuating economic conditions and uncertain market conditions in different stages of a product life-cycle, can have a major effect on demand. This motivates us to consider a Markov process or a Markovian renewal process that provide a natural and flexible alternative for modeling the demand process.

The Earlier Markovian demand model was given by Karlin and Fabens (1959) and Iglehart and Karlin (1962) who formulate and characterize the form of the optimal policies. Many researchers studied the Markovian demand model later on. For example, Zipkin (1989), Aviv and Federgruen (1997) and Kapuscinski and Tayur (1998) studied the problem with cyclic demands; Song and Zipkin (1993) studied the optimal policy for continuous review Markov chain; Beyer and Sethi (1997) and Sethi and Cheng (1997) discussed the optimality of \((s,S)\) policies for models with Markovian demand; and Chen and Song (2001) studied a multi-stage serial inventory system. The assumption of Markovian demand may not be realistic because the inter-demand time is assumed to be exponential distributed. That motivated us to study a semi-Markov model in which the inter-demand time is generally distributed.

For a single item with a fixed lifetime, periodic inventory review seems ideal for inventory tracking and replenishing. A general \(m\) -period (lifetime equals \(m\) review periods) model with a zero replenishment lead time was separately studied by Fries (1975) and Nahmias (1975). While some properties for the optimal ordering policy were found through the dynamic programming approach, the exact format of the optimal ordering policy was not found. It has been shown (Nahmias 1975) that the critical number policy provides a good approximation of the true optimal policy. However, computing the optimal critical number policy \(S\) can still be very difficult for a large \(m\) (Nahmias, 1982).

Another approach to perishable inventory models is to adopt a continuous review policy. With a Poisson demand process, a zero lead time, and constant lifetime, Weiss (1980) showed that the optimal ordering policy is an \((s,S)\) policy. Schmidt and Nahmias (1985) considered an \((S-1,S)\) continuous review model with Poisson demands.
2. THE MODEL AND ANALYSIS

Consider an \((s, S)\) continuous review inventory system with unit demands. The customer demand follows a Markovian Renewal Process (MRP) with state space \(\{1, 2, \ldots, m\}\). The transition matrix is \(G(t) = (G_{i,j}(t))_{m \times m}\), which describes the type and the inter-arrival times of customer demands. The matrix \(G \triangleq \lim_{t \to \infty} G(t)\), is an irreducible stochastic matrix. \(\pi = (\pi_1, \ldots, \pi_m)\) is the stationary probability vector of the transition probability matrix \(G\).

The lifetime of a fresh item in storage is a constant \(T\). An order of size \(S-s\) is placed and is received without delay to bring the inventory level back to \(S\) whenever it drops to \(s\), where \(s \leq -1\). For convenience, we define \(x = -s\) that is the backorder quantity. Let \(I(t)\) be the inventory level, and \(D(t)\) be the demand state at time \(t\). We have \(I(t) = s + 1, \ldots, S\) and \(D(t) = 1, \ldots, m\). We also denote \(D_k\) be the inter-arrival time of the \(n^{th}\) demand. Though \(\{I(t), D(t), t \geq 0\}\) is not a Markov process, one can easily see that the epoch when a fresh order has just arrived is a Markov renewal point of the process \(\{I(t), D(t), t \geq 0\}\). For \(s < -1\), we have at least another Markovian renewal point at the epoch when the inventory level reduces from 0 to \(-1\).

Let \(Z_n\) be the epoch when the process \(\{I(t), D(t), t \geq 0\}\) makes the \(n^{th}\) transition into a Markov renewal point and let \(X_n\) identify the corresponding Markov point. Without loss of generality, we consider the case when \(s < -1\). Assuming that \(Z_0 = 0\), \(\{X_n, Z_n\}; n = 0, 1, 2, \ldots\) is a Markov renewal process with a state space \(J = \{(S,1), \ldots, (S,m), (-1,1), \ldots, (-1,m)\}\).

2.1 Transition Matrix

Let

\[
Q_{k,r,i,j}(t) = P[X_1 = (r, j), Z_1 = t | X_0 = (k, i)] \quad \text{and} \quad Q(t) = (Q_{k,r,i,j}(t), (k, i), (r, j)), t \geq 0 \}
\]

be the semi-Markov kernel of \((X, Z)\). For convenience, we define \(G_{i,j}^{(n)}(t) = 0\).

By the total probability formula, we have

\[
Q_{s-i,r,i,j}(t) = \sum_{k=0}^{s-1} P\left[\sum_{r=1}^{s} D_r \leq t \leq \sum_{r=1}^{s+1} D_r \right] +
\]

\[
P\left[\sum_{r=1}^{s} D_r \leq t, \sum_{r=1}^{s+1} D_r \leq t \right] \cdot G_{i,j}^{(s+1)}(t), \quad t < T,
\]

\[
G_{i,j}^{(s+1)}(t) + \sum_{u=0}^{S} \sum_{k=1}^{s} \int_0^t G_{i,s}^{(u)}(t-u) dG_{j,k}^{(u)}(u) - \sum_{u=0}^{S} G_{i,j}^{(u)}(T), \quad t \geq T,
\]

(1)

and

\[
Q_{-1,S,i,j}(t) = P\left[\sum_{k=1}^{s} D_k \leq t\right] = G_{i,j}^{(s+1)}(t), \quad (2)
\]

where \(G_{i,j}^{(n)}(t)\) is the element \((i, j)\) of the \(n^{th}\) fold convolution of the matrix \(G\) and \(G_{i,j}^{(0)}(t) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}\)

We break the transition matrix into blocks:

\[
Q(t) = \begin{pmatrix}
0 & Q_{S,-i}(t) \\
Q_{-1,S} & 0
\end{pmatrix}
\]

(3)

where \(Q_{S,-i}(t) = (Q_{S,-1,i,j}(t))_{m \times m}\) and \(Q_{-1,S}(t) = (Q_{-1,S,i,j}(t))_{m \times m}\).
\[
Q_{S,-1}(t) = \begin{cases} 
G^{(s+1)}(t), & t < T, \\
G(t) + \sum_{k=1}^{S} \int_{0}^{T} G(t-u)dG^{(k)}(u) - \sum_{k=0}^{S} G^{(k)}(T), & t \geq T, 
\end{cases}
\tag{4}
\]

\[
Q_{-1,S}(t) = P\left( \sum_{i=0}^{S} D_i \leq t \right) = G^{(s-1)}(t). 
\tag{5}
\]

Let \( \tilde{q}_{k,r,i,j}(v) = \int_{v}^{\infty} e^{-ivt}dQ_{k,r,i,j}(t), v > 0 \) be the Laplace-Stieltjes (LS) transform of \( Q_{k,r,i,j}(t) \), and \( q(v) = \begin{cases} 
0, & q_{S,-1}(v) \\
q_{-1,S}(v), & 0
\end{cases} \) be the corresponding matrix.

From (4) and (5), we have

\[
q_{S,-1}(v) = \int_{v}^{\infty} e^{-ivt}dG(t) = \frac{\pi_{S,-1}(v)}{\pi_{1,-1}(v)} = g^{-1}(v)
\]

Therefore,

\[
-\frac{q_{-1,S}^{-1}}{q_{S,-1}}(0) = -\left[ g^{-1}(v) \right] \bigg|_{v=0} = \sum_{i=0}^{S-2} G^{(i)} U G^{(i+2)} 
\tag{9}
\]

### 2.2 The Expected Cycle Time

With the transition probabilities mentioned above, we can derive the expected reorder cycle time \( E\tau \). We have

\[ E\tau = m_s + (x - 1)\mu^{-1} \tag{10} \]

where \( m_s = -\pi q_{S,-1}^{*}(0)e \) and \( \mu^{-1} = \pi U e \). Here \( e \) is an \( m \)-dimension column vector with all elements 1.

**Proof.** The expected reorder cycle time is the expected first arrival time to inventory level \( S \) starting from inventory level \( S \). In the steady state, it is equivalent to the expected first arrival time to inventory level \(-1\) starting from inventory level \(-1\). Let \( \tau_{ij} \) be the first arrival time to state \((1, j)\) starting from state \((-1, i)\) and \( F_{ij}(t) \) be the corresponding cumulative probability.

Denote \( F(t) = \left( F_{ij}(t) \right)_{n \times n} \) and \( f(v) = \int_{0}^{\infty} e^{-ivt}dF(t) \).

Then \( F(t) = Q_{-1,S} * Q_{S,-1}(t) \) and \( f(v) = q_{-1,S}^{-1}(v)q_{S,-1}(v) \). Therefore,

\[
-f(0) = -q_{-1,S}^{-1}(0)q_{S,-1}(0) - q_{-1,S}^{-1}(0)q_{S,-1}(0).
\]

From (9), the expected reorder cycle time

\[ E\tau = \pi f(0)e = \pi \sum_{i=0}^{S} G^{(i)} U G^{(i+2)} e - \pi G^{(s-1)} q_{S,-1}(0)e. \tag{12} \]

Since \( \pi \) is the stationary probability vector of the demand process, we have \( \pi G^{(i)} = \pi \) for any \( i > 0 \). Further more, \( G^{(s-2)} Q_{S,-1} e = G^{(s-2)} e = e \), we derive (10). \( \square \)

### 3. COST FUNCTION

Costs relevant to the choice of the optimal \( s \) and \( S \) values in this model are:

- \( C_h \) : the inventory carrying cost per unit per unit time;
- \( C_s \) : the disposal cost per unit decayed;
- \( C_r \) : the backorder penalty per unit backordered;
- \( C_b \) : the backorder penalty per unit per unit time;

...
C_o: the ordering cost per order.
Firstly, let’s calculate the disposed cost in a reorder cycle. If an item is not used by the time \( T \), the item will expire. Let \( \mathcal{N}(T) \) be the demands in \((0,T]\). Let \( R(T) \) be the number of perished units in a cycle. Then
\[
R(T) = \max\{S - \mathcal{N}(T), 0\},
\]
and
\[
ER(T) = \sum_{r=0}^{S} (S-r) \pi [G^{(r)}(T) - G^{(r+1)}(T)] e
\]
So the expected disposal cost per cycle is
\[
E(ER) = C_o \cdot ER(T) = C_o \left[ S - \sum_{r=1}^{S} \pi G^{(r)}(T) e \right] (13)
\]
To derive the holding cost and the shortage cost over a reorder cycle, we introduce the sojourn time in an inventory level.
Let
\[
V_{r,i}(t) = \begin{cases} P \left[ I(t) = r, Z_i > T \right] & I(0) = S, D(0) = i, r = 0, 1, \cdots, S, \\ P \left[ I(t) = r, Z_i > T \right] & I(0) = -1, D(0) = i, r = s + 1, \cdots, -1 \end{cases}
\]
where \( i = 1, \cdots, m \). Denote \( V_{r,i} = \int_0^T V_{r,i}(t) dt \) and we define a row vector \( V_r = [V_{r,1}, \cdots, V_{r,m}]^T \). For \( r = 0, 1, \cdots, S \), \( \pi V_r \) is the conditional expected sojourn time in inventory level \( r \) starting from the inventory level \( S \) over a reorder cycle. For \( r = -s + 1, \cdots, -1 \), \( \pi V_r \) is the conditional expected sojourn time in inventory level \( r \) starting from inventory level \( -1 \) over a reorder cycle.

**Theorem 2.** The expected holding cost over a reorder cycle can be given by
\[
E(HC) = C_h ST - C_h \sum_{r=1}^{S} \pi \left[ \int_0^T G^{(r)}(t) dt \right] e
\]
**Proof.** Here, we only prove the case when \( S > 0 \). By definition, we have
\[
V_{r,i}(t) = P \left[ I(t) = r, Z_i > T \right] I(0) = S, D(0) = i, \quad T > t \\
= \begin{cases} P \left[ \sum_{n=1}^{S-r} D_n \leq t < \sum_{n=1}^{S-r+1} D_n \right] D(0) = i, & T > t \\
0, & T \leq t.
\end{cases}
\]
Hence,
\[
V_{r,i} = \int_0^T V_{r,i}(t) dt = \sum_{j=1}^{m} \sum_{k=1}^{m} G_{r,i}^{(r)}(t) dG_{r,i}^{(r-1)}(u).
\]
Using matrix expression, we obtain
\[
V_r = \left[ \int_0^T G^{(r)}(t) dt \right] e.
\]
Since \( \pi V_r \) is the expected sojourn time in inventory level \( r \), so the expected holding cost in a reorder cycle can be given by
\[
E(HC) = C_h \sum_{r=1}^{S} \pi V_r
= C_h ST - C_h \sum_{r=1}^{S} \pi \left[ \int_0^T G^{(r)}(t) dt \right] e.
\]

**Theorem 3.** The expected shortage cost over a reorder cycle can be given by
\[
E(SC) = C_s (x-1) + \frac{1}{2} C_s x (x-1) \mu^{-1}.
\]
**Proof.** Similar to Theorem 2, we have
\[
V_{r,i}(t) = P \left[ \sum_{n=1}^{S-r} D_n \leq t < \sum_{n=1}^{S-r+1} D_n \right] D(0) = i
= \sum_{j=1}^{m} \sum_{k=1}^{m} G_{r,i}^{(r)}(t-u) dG_{r,i}^{(r-1)}(u).
\]
Rewrite it in matrix format,
\[ V_r(t) = \int_0^\infty G(t-u)dG^{(-1)}(u)e \quad (22) \]

So,
\[
V_r = \left[ \int_0^\infty G(t-u)dG^{(-1)}(u)dt \right] e
= \int_0^\infty G(t-u)dt dG^{(-1)}(u)e
= UG^{(-1)} e
= Ue
\quad (23) \]

Since \( \pi V_r \) is the expected sojourn time in inventory level \( r \), the expected shortage cost in a reorder cycle can be given by
\[
E(SC) = C_s(x-1) + C_r \sum_{i+1} (-r)\pi U e
= C_s(x-1) + \frac{1}{2} C_s(x-1)\mu^{-1}. \quad (24) \]

Summing up the three costs obtained above and the ordering cost and then dividing the total cost by the expected cycle length \( E\tau \), we obtain the expected total cost per unit time,
\[
C(x,S) = \frac{C_o + E(HC) + E(RC) + E(SC)}{m_s(x-1)/\mu^{-1}}. \quad (25) \]

We can see that the above formula is similar to (14) of Liu and Lian (1999).

4. NUMERICAL ILLUSTRATION

Section 3 provides formulae of the expected reorder cycle and the expected unit cost function. As a function of \( x \), \( C(x,S) \) has the similar properties analyzed in section 2.2 of Liu and Lian (1999).

Since the demand in this paper follows a Markovian renewal process rather than a renewal process, it is crucial to design a good algorithm in order to save the running time, especially when the number of states in the demand process is quite large.

For example, to avoid repeating the calculation, it is necessary to rewrite some formulae above as iterative functions.

Denoted by \( m_s(S) \) the expected cycle time when the order-up-to level is \( S \) and \( x = 1 \). Obviously, \( m_s(0) = \mu \), and when \( S \geq 1 \),
\[
m_s(S) = m_s(S-1) + \pi \left[ \int_0^T tG^{(S-1)}(t) - \int_0^T tG^{(S)}(t) dt \right] e
+ \pi \left[ \mu^{-1}G^{(S)}(T) + \int_0^T tG^{(S)}(T-t)G^{(S)}(t) dt \right] e, \quad (26) \]

where \( dG^{(S)}(t) = \int_0^\infty (t-v)dG^{(x-1)}(v)dt \). Matlab can help to do such symbol calculation.

Similarly, denoted by \( EHC(S) \) and \( ERC(S) \) the expected holding cost and expected replacement cost in a cycle respectively, we have \( EHC(0) = 0, ERC(0) = 0, \) and when \( S \geq 1 \),
\[
EHC(S) = EHC(S-1) + C_s T - C_r \pi \left[ \int_0^T G^{(S)}(t)dt \right] e, \quad (27) \]
\[
ERC(S) = ERC(S-1) + C_s \left[ 1 - \pi G^{(S)}(T) e \right]. \quad (28) \]

From the structure of \( C(x,S) \), one can see that the above formula save us much time to calculate \( C(x,S) \) for all \( x \) once we know \( EHC(S-1), ERC(S-1) \) and \( m_s(S-1) \).

Below we are going to explore how different is between the behaviors of the model with Markovian renewal demand and the model with the renewal demand in Liu and Lian [9]. Consider a Markovian renewal process with two states. The transition matrix is
\[
G(t) = \begin{pmatrix}
\frac{3}{5} & \frac{2}{5} \\
\frac{1}{5} & \frac{4}{5}
\end{pmatrix}. \quad (29) \]

So the stationary probability vector of the embedded Markov chain is
\[
\pi = \begin{pmatrix}
\frac{3}{5} \\
\frac{2}{5}
\end{pmatrix}. \quad (30) \]

And
\[
g(v) = \begin{pmatrix}
\frac{3v+1}{1+v^2} \\
\frac{1+v^2}{1+v^2}
\end{pmatrix}. \quad (31) \]

Hence the expected inter-demand time can be derived by
\[ \mu^{-1} = -\pi_0 e = \begin{pmatrix} 3 \frac{2}{5} \frac{3}{5} \frac{1}{5} \frac{1}{5} \end{pmatrix}^{T} = 0.617. \quad (32) \]

On the other hand, we consider a model in which the arrival demand follows a renewal process with exponential distributed inter-demand time, and the expected inter-demand time is \( \mu^{-1} = 0.617 \).

We assume that \( C_a = 5, \quad C_h = 0.005, \quad C_r = 8, \quad C_i = 0.05 \) and \( C_e = 1 \).

We first compare the expected reorder cycle time between the models with MRP and RP demands. Figure 1 and Figure 2 show that the expected order cycle time are very different. And the difference becomes larger when either the order-up-to level \( S \) or the item lifetime is increasing. While when the order-up-to level \( S \) or the item lifetime is quite large, the curve become flat, i.e., the reorder cycle time tend to be stable.

![Figure 1: Comparison of the order cycle time between RP and MRP (x = 1 and item lifetime = 4)](image1)

![Figure 2: Comparison of the order cycle time between RP and MRP (x = 1 and S = 4)](image2)

We then calculate the optimal \( x, S \) and minimum costs for when the item lifetime changes from 0.5 to 7. Table 1 and Figure 3 show the results. We can see, if the demand follows a Markovian renewal process but we consider it as a renewal process with the same arrival rate \( \mu \), then the derived \( x^* \) and \( S^* \) will be quite different from the optimal \( x \) and \( S \), so that the costs are also quite different when the item lifetime is not too short but not too long (between 2.0 to 5.0 in this example). While if the item lifetime is too short or too long, we can consider the renewal demand as an approximation of the Markovian renewal demand.

5. CONCLUSIONS

![Table 1: Comparison of minimum costs between RP model and MRP model](image3)

By constructing a Markovian renewal process, we derive the expected reorder cycle time and the expected total cost per unit time.

The iterative functions derived in section 4 are useful to effectively calculate the optimal values of \( x \) and \( S \), especially when the number of states in the demand process is quite large.

Numerical results show that the reorder cycle times and the optimal policies are different between MRP model and RP model, especially when the item lifetime is neither too big nor too small.
too small. When the item lifetime is very short, the optimal policy is basically a make-to-order policy \((S=0)\). When the lifetime is very long in comparison to the expected inter-demand time, one may ignore the perishability and we can consider a renewal demand with the same demand rate as an approximation of the Markovian renewal demand in order to simplify the calculation.

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