Short Communication

On consistency of the weighted geometric mean complex judgement matrix in AHP

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Abstract

The weighted geometric mean method (WGMM) is the most common group preference aggregation method in the Analytic Hierarchy Process. This paper reports on research concerning the consistency of WGMM and proves that the weighted geometric mean complex judgement matrix (WGMCJM) is of acceptable consistency. In Saaty’s (T.L. Saaty, The Analytic Hierarchy Process, McGraw-Hill, New York, 1980) opinion, a consistency ratio (CR) of 0.1 or less is acceptable under the condition that all judgement matrices given by experts for the same problem of decision-making are of acceptable consistency. Accordingly, a theoretic basis has been developed for the application of the WGMM in group decision making. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Analytic Hierarchy Process (AHP) introduced by Saaty [3] is one of the popular and powerful methods for decision-making, which has also been applied to various decision areas such as economic analysis, strategic planning, forecasting, etc. (see Ref. [6]). Saaty [4] has discussed several practical and theoretical aspects of group decision-making using AHP. There are several methods employed in AHP for aggregating group opinions, one of which is the weighted geometric mean method (WGMM). The WGMM is the most common group preference aggregation method in AHP literature ([1,2,5,7]). As it is well known that if judgement matrices \( A_1, A_2, \ldots, A_s \) given by experts or decision-makers are of perfect consistency, then their WGMCJM \( \bar{A} \) is of perfect consistency. But if some of them are not of perfect consistency, then the above conclusion does not surely hold. Focusing on the problem, this paper proves that the WGMCJM is of acceptable consistency (i.e., \( CR \leq 0.1 \)) under the condition that each \( A_k (k = 1, 2, \ldots, s) \) is of acceptable consistency. A numerical example is also given.

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2. Basic concepts and properties

Definition 2.1. Let \( A = (a_{ij}) \) be an \( n \times n \) judgement matrix, \( \Omega = \{1, 2, \ldots, n\} \), if \( a_{ij} = a_{ik}a_{kj} \), for each \( i, j, k \in \Omega \), then judgement matrix \( A \) is perfectly consistent.

Definition 2.2. Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( n \times n \) judgement matrices, the Hardmard product of \( A, B \) can be denoted by \( C = A \circ B = (c_{ij}) \), where \( c_{ij} = a_{ij}b_{ij} \), for each \( i, j \in \Omega \).

Definition 2.3. Let \( A = (a_{ij}) \) be an \( n \times n \) judgement matrix, and let \( R \) be the set of real numbers, we denote by \( A^\dagger = (a_{ij}^\dagger) \), where \( \dagger \in R \).

Definition 2.4. Let \( A_1, A_2, \ldots, A_s \) be judgement matrices for the same decision problem, then the WGMJCM is \( A \), where

\[
\tilde{A} = A_1^{\dagger_1} \circ A_2^{\dagger_2} \circ \cdots \circ A_s^{\dagger_s},
\]

\[
\sum_{k=1}^{s} \lambda_k = 1, \quad \lambda_k > 0 \quad (k = 1, 2, \ldots, s).
\]

Since people have perceptions and judgements which are subject to change, due to their psychological states and information inputs, in general case, the perfect consistency condition is rarely satisfied. Hence, we assume that there is an underlying vector \( w = (w_1, w_2, \ldots, w_n)^T \), whose components are perturbed (by inconsistent human judgement) to give the elements of judgement matrix \( A \), namely, \( a_{ij} = w_i/w_j \hat{e}_{ij}, \quad i, j \in \Omega \), where \( \hat{e}_{ij} > 0 \) and \( e_{ij} = 1/\hat{e}_{ij} \).

Ref. [3] gave a consistency index

\[
\text{CI} = \frac{\lambda_{\text{max}} - n}{n - 1},
\]

where CI is related to the perturbation matrix \( E = (\hat{e}_{ij}) \), \( \lambda_{\text{max}} \) is the largest eigenvalue of \( A \).

We have \( A^\dagger w = \lambda_{\text{max}} w \), i.e.,

\[
\lambda_{\text{max}} w_i = \sum_{j=1}^{n} a_{ij} w_j, \quad i = 1, 2, \ldots, n.
\]

Since \( a_{ii} = 1 \) and \( a_{ij} = (1/(a_{ij})) \), hence

\[
n\lambda_{\text{max}} - n = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij} w_j w_i
\]

\[
= \sum_{1 \leq i < j \leq n} \left( a_{ij} w_j + a_{ji} w_i \frac{w_j}{w_i} \right).
\]

According to \( a_{ij} = (w_i/w_j) \hat{e}_{ij} \), we have

\[
n\lambda_{\text{max}} - n = \sum_{1 \leq i < j \leq n} \left( \hat{e}_{ij} + \frac{1}{\hat{e}_{ij}} \right),
\]

\[
\lambda_{\text{max}} - 1 = \frac{1}{n} \sum_{1 \leq i < j \leq n} \left( \hat{e}_{ij} + \frac{1}{\hat{e}_{ij}} - 2 \right),
\]

thus

\[
\text{CI} = \frac{\lambda_{\text{max}} - n}{n - 1}
\]

\[
= -1 + \frac{\lambda_{\text{max}} - 1}{n - 1}
\]

\[
= -1 + \frac{1}{n(n - 1)} \sum_{1 \leq i < j \leq n} \left( \hat{e}_{ij} + \frac{1}{\hat{e}_{ij}} - 2 \right).
\]

Definition 2.5. Let \( A \) be an \( n \times n \) judgement matrix, then \( A \) can be denoted by \( A = A^\dagger \circ E \), where \( A^\dagger \) is a perfectly consistent matrix, \( E \) is a perturbation matrix,

\[
E = (\hat{e}_{ij}), \quad \hat{e}_{ij} = \frac{1}{\hat{e}_{ij}}, \quad \hat{e}_{ij} > 0.
\]

We say that \( A \) is of acceptable consistency, if

\[
\text{CI} = \frac{1}{n(n - 1)} \sum_{1 \leq i < j \leq n} \left( \hat{e}_{ij} + \hat{e}_{ij} - 2 \right) \leq \varepsilon,
\]

where \( \varepsilon \) is the dead line of acceptable consistent judgement. In general, \( \varepsilon \) is equal to one-tenth of the mean consistency index of randomly generated matrices (for short, RI), which is given in Table 1.

Saaty pointed out in Ref. [3] that the consistency ratio \( \text{CR} = \text{CI}/\text{RI} \), thus, Definition 2.5 can be equally expressed as follows.

Judgement matrix \( A \) is of acceptable consistency, if \( \text{CR} \leq 0.1 \).

By Definitions 2.2 and 2.3, we have the following properties.
Let judgement matrices

Theorem 3.1. 

Proof. Since a judgement matrix can be regarded as the matrix obtained by perturbing a consistent matrix, we express the matrices $A_i (i = 1, 2, \ldots, s)$ as $A_1 = A \circ E_1, A_2 = A \circ E_2, \ldots, A_s = A \circ E_s$, where $A$ is a perfectly consistent matrix, $E_k = \left( \frac{e_{ij}^{(k)}}{e_{ij}} \right)$ is the perturbation matrix corresponding to $A_k (k = 1, 2, \ldots, s), \ e_{ij}^{(k)} > 0, \ e_{ij} = \frac{1}{e_{ij}^{(k)}}$.

According to the properties in Section 2, we can obtain

\[
A_i^{\hat{\lambda}} \circ A_2^{\hat{\lambda}} \circ \cdots \circ A_s^{\hat{\lambda}} = (A \circ E_1)^{\hat{\lambda}} \circ (A \circ E_2)^{\hat{\lambda}} \circ \cdots \circ (A \circ E_s)^{\hat{\lambda}}
\]

Then we can express the matrices $A_i$ as the matrix obtained by perturbing a consistent matrix, $E_k$.

Property 2.3.

Let $X$ and $\hat{X}$ be two consistent matrices. Then

$$
A \circ B \circ C = A \circ (B \circ C) = (A \circ B) \circ C.
$$

3. Lemma and theorem

Lemma 3.1. Let

$$
x_i > 0, \quad \hat{\lambda}_i > 0 \quad (i \in \Omega)
$$

and

$$
\sum_{i=1}^{n} \hat{\lambda}_i = 1,
$$

then

$$
\prod_{i=1}^{n} x_i^{\hat{\lambda}_i} \leq \sum_{i=1}^{n} \hat{\lambda}_i x_i
$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. This inequality follows the strict convexity of the function $f(x) = \exp(x)$ and by induction on $n$:

$$
\exp \left( \sum_{i=1}^{n} \hat{\lambda}_i \log(x_i) \right) \leq \sum_{i=1}^{n} \hat{\lambda}_i \exp(\log(x_i)),
$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3.1. Let judgement matrices $A_1, A_2, \ldots, A_s$ be of acceptable consistency, $\hat{\lambda}_k \in (0, 1)$, $\sum_{k=1}^{s} \hat{\lambda}_k = 1$, then the WGMCMJ $A_1^{\hat{\lambda}_1} \circ A_2^{\hat{\lambda}_2} \circ \cdots \circ A_s^{\hat{\lambda}_s}$ is of acceptable consistency.

Multiplying Eq. (1) by $\hat{\lambda}_k \in (0, 1)$, then

$$
\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left( \hat{\lambda}_k e_{ij}^{(k)} + \hat{\lambda}_k e_{ji}^{(k)} - 2 \hat{\lambda}_k \right) \leq \alpha \hat{\lambda}_k,
$$

By Eq. (2), and noting that $\sum_{k=1}^{s} \hat{\lambda}_k = 1$, it follows that

$$
\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{s} \hat{\lambda}_k e_{ij}^{(k)} + \sum_{k=1}^{s} \hat{\lambda}_k e_{ji}^{(k)} - 2 \right) \leq \alpha.
$$

From Lemma 3.1, we have

$$
\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[ \prod_{k=1}^{s} e_{ij}^{(k)} \hat{\lambda}_k + \prod_{k=1}^{s} e_{ji}^{(k)} \hat{\lambda}_k - 2 \right] \leq \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{s} \hat{\lambda}_k e_{ij}^{(k)} + \sum_{k=1}^{s} \hat{\lambda}_k e_{ji}^{(k)} - 2 \right).
$$
According to Eqs. (3) and (4), it can be obtained that
\[
\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[ \prod_{k=1}^{n} (e_{ij}^{(k)})^{\hat{w}_k} + \prod_{k=1}^{n} (e_{ji}^{(k)})^{\hat{w}_k} - 2 \right] \leq \lambda.
\]
(5)

From Eq. (5) and Definition 2.5, we can obtain that \( A_1^{\hat{w}_1} \circ A_2^{\hat{w}_2} \circ \cdots \circ A_r^{\hat{w}_r} \) is of acceptable consistency.

This completes the proof of Theorem 3.1. \( \square \)

4. Numerical example

The following four judgement matrices are given by four experts for a decision problem,
\[
A_1 = \begin{bmatrix}
1 & 4 & 6 & 7 \\
1/4 & 1 & 3 & 4 \\
1/6 & 1/3 & 1 & 2 \\
1/7 & 1/4 & 1/2 & 1 \\
\end{bmatrix},
\]
\[
w = (0.617, 0.224, 0.097, 0.062)^T, \quad \hat{\lambda}_{\text{max}} = 4.102, \quad \text{CR} = 0.038 < 0.1,
\]
\[
A_2 = \begin{bmatrix}
1 & 5 & 7 & 9 \\
1/5 & 1 & 4 & 6 \\
1/7 & 1/4 & 1 & 2 \\
1/9 & 1/6 & 1/2 & 1 \\
\end{bmatrix},
\]
\[
w = (0.653, 0.225, 0.076, 0.047)^T, \quad \hat{\lambda}_{\text{max}} = 4.181, \quad \text{CR} = 0.067 < 0.1,
\]
\[
A_3 = \begin{bmatrix}
1 & 3 & 5 & 8 \\
1/3 & 1 & 4 & 5 \\
1/5 & 1/4 & 1 & 2 \\
1/8 & 1/5 & 1/2 & 1 \\
\end{bmatrix},
\]
\[
w = (0.570, 0.277, 0.096, 0.057)^T, \quad \hat{\lambda}_{\text{max}} = 4.091, \quad \text{CR} = 0.034 < 0.1,
\]
\[
A_4 = \begin{bmatrix}
1 & 4 & 5 & 6 \\
1/4 & 1 & 3 & 3 \\
1/5 & 1/3 & 1 & 2 \\
1/6 & 1/3 & 1/2 & 1 \\
\end{bmatrix},
\]
\[
w = (0.597, 0.222, 0.109, 0.073)^T, \quad \hat{\lambda}_{\text{max}} = 4.126, \quad \text{CR} = 0.047 < 0.1.
\]

We first assume that the experts have equal weights in forming group opinion, i.e., \( \hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 = 1/4 \); then the WGMCJM is
\[
\hat{A} = \begin{bmatrix}
1 & 3.936 & 5.692 & 7.417 \\
0.254 & 1 & 3.464 & 4.356 \\
0.176 & 0.289 & 1 & 2 \\
0.135 & 0.230 & 1/2 & 1 \\
\end{bmatrix},
\]
\[
w = (0.610, 0.237, 0.094, 0.059)^T, \quad \hat{\lambda}_{\text{max}} = 4.119, \quad \text{CR} = 0.044 < 0.1.
\]

In case the weights to different experts are not the same, we suppose \( \hat{\lambda}_1 = 1/10 \), \( \hat{\lambda}_2 = 2/10 \), \( \hat{\lambda}_3 = 3/10 \), \( \hat{\lambda}_4 = 4/10 \); then the WGMCJM is
\[
\hat{A} = \begin{bmatrix}
1 & 3.837 & 5.446 & 7.204 \\
0.261 & 1 & 3.464 & 4.134 \\
0.184 & 0.287 & 1 & 2 \\
0.139 & 0.242 & 1/2 & 1 \\
\end{bmatrix},
\]
\[
w = (0.604, 0.239, 0.097, 0.061)^T, \quad \hat{\lambda}_{\text{max}} = 4.117, \quad \text{CR} = 0.043 < 0.1.
\]

5. Summary

This paper has proven that the WGMCJM is of acceptable consistency under the condition that all judgement matrices for the same problem are of acceptable consistency. This result is very convenient for dealing with group decision making problems. A theoretic basis has hence been developed for the application of the WGMM.

References
