An interactive method for fuzzy multiple attribute group decision making

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Abstract

In this paper, we develop an interactive method for multiple attribute group decision making under fuzzy environment. The method can be used in situations where the information about attribute weights is partly known, the weights of decision makers are expressed in exact numerical values or triangular fuzzy numbers, and the attribute values are triangular fuzzy numbers. The method transforms fuzzy decision matrices into their expected decision matrices, constructs the corresponding normalized expected decision matrices by two simple formulas, and then aggregates these normalized expected decision matrices into a complex decision matrix. Moreover, the decision makers are asked to provide their preferences gradually in the course of interactions. By solving linear programming models, the method diminishes the given alternative set gradually, and finally finds the most preferred alternative. By using the method, the decision makers can provide and modify their preference information gradually in the process of decision making so as to make the decision result more reasonable. The method can not only reflect the importance of the given arguments and the ordered positions of the arguments, but also relieve the influence of unfair arguments on the decision result. Finally, a practical problem is used to illustrate the developed method.

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1. Introduction

Multiple attribute decision making is a usual task in human activities. It consists of finding the most preferred alternative from a given alternative set. The increasing complexity of the socio-economic environment makes it less and less possible for a single decision maker to consider all relevant aspects of a problem...
In such situations, the preference information provided by the decision maker may be imprecise or incomplete. As a result, many decision making processes, in the real world, take place in group settings with incomplete information. There have been a few studies employing imprecise preference models in group settings so far [1,4,10,15–18,20,23,25,29]. Anandaligam [1] developed a methodology to use multiple attribute utility functions within a Nash bargaining model. Salo [29] developed an interactive method to aggregate the preferences of group members in the context of an evolving value representation. Kim and Ahn [15] suggested the possibility that individually optimized results can be used to build group consensus, and considered strict or weak dominance values as input for aggregation procedures. Park and Kim [25] proposed a dominance graph and also presented an algorithm to generate the dominance graph based on the information of pairwise dominance. Kim et al. [17] presented an interactive procedure for multiple attribute group decision making with incomplete information and described some theoretical models to establish group’s pairwise dominance relations with group’s utility ranges by using a separable linear programming technique. Kim and Ahn [16] suggested a procedure to rank alternatives by comparing the net strengths of alternatives. Chen [4] extended the TOPSIS of Hwang and Yoon [11] to fuzzy environment and developed a vertex procedure to calculate the distance between two triangular fuzzy numbers, and defined a closeness coefficient to determine the ranking order of all alternatives by calculating the distances to both the fuzzy positive-ideal solution (FPIS) and fuzzy negative-ideal solution (FNIS) simultaneously. Li and Yang [20] extended the classical linear programming technique for multidimensional analysis of preference (LINMAP) to develop a new methodology to solve multiple attribute group decision making problems under fuzzy environment. They constructed a fuzzy linear programming model to rank alternatives by using the pairwise comparisons between alternatives, which can be used in both crisp and fuzzy environments. Lahdelma et al. [18] developed a Ref-SMAA method to solve the problems where both attribute data and preference information are uncertain or inaccurate (or preference information is totally missing). Ölcər and Odabasi [23] introduced an attribute based aggregation technique to deal with fuzzy multiple attribute group decision making problems. Herrera et al. [10] presented an aggregation procedure to manage non-homogeneous information of different nature (numerical, interval-valued and linguistic).

However, in many real-life cases, such as negotiation processes, the high technology project investment of venture capital firms, etc., a decision maker cannot generally specify exact attribute weights but can provide value ranges [19,24,25,27], and the information about attribute values usually takes the form of linguistic variables or triangular fuzzy numbers [4,20] because that (1) a decision should be made under time pressure and lack of knowledge or data [16,26,32,35,43]; (2) many of the attributes are intangible or non-monetary because they reflect social and environmental impacts [17]; (3) the decision maker has limited attention and information processing capabilities [12]; (4) in group settings, all participants do not have equal expertise about problem domain [28]. Furthermore, in the process of decision making, a decision maker often needs to interact with group members (or analysts) by providing and modifying his/her incomplete preference information gradually. All the above methods are somewhat unsuitable for dealing with these situations, and thus, it is interesting and necessary to pay attention to this issue. In this paper, we will develop an interactive method for multiple attribute group decision making under fuzzy environment, where the information about attribute weights is partly known, the weights of decision makers are expressed in exact numerical values or triangular fuzzy numbers, and the attribute values are triangular fuzzy numbers. To do so, the rest of this paper is arranged as follows: Section 2 gives a simple representation of the fuzzy multiple attribute group decision making problem and reviews some aggregation operators. Section 3 develops an interactive method for fuzzy multiple attribute group decision making and gives a comparative analysis of the developed method and the extended TOPSIS of Chen [4]. In Section 4, a fuzzy multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library is used to illustrate the developed method and to demonstrate its feasibility and practicality. Section 5 concludes the paper.

2. Preliminaries

A fuzzy multiple attribute group decision making problem considered in this paper is represented as follows:
Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a discrete set of alternatives, \( D = \{d_1, d_2, \ldots, d_l\} \) be the set of decision makers, and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^\top \) be the weight vector of decision makers, where \( \lambda_k \geq 0 \), \( k = 1, 2, \ldots, l \), and \( \sum_{k=1}^{l} \lambda_k = 1 \). Let \( G = \{G_1, G_2, \ldots, G_t\} \) be the set of attributes, and \( w = (w_1, w_2, \ldots, w_t)^\top \) be the weight vector of attributes, where \( w_j \geq 0 \), \( i = 1, 2, \ldots, s \), \( \sum_{j=1}^{s} w_j = 1 \), and \( w \in H \). \( H \) is the set of the known information about attribute weights given by the decision makers, which can be constructed in the following forms [16,17,25], for \( i \neq j \):

1. A weak ranking: \( \{w_i \geq w_j\} \).
2. A strict ranking: \( \{w_i - w_j \geq \alpha_i (>0)\} \).
3. A ranking with multiples: \( \{w_i \geq \alpha_i w_j\}, \, 0 \leq \alpha_i \leq 1 \).
4. An interval form: \( \{\alpha_i \leq w_i \leq \alpha_i + e_i\}, \, 0 \leq \alpha_i < \alpha_i + e_i \leq 1 \).
5. A ranking of differences: \( \{w_i - w_j \geq w_k - w_l\} \) for \( i \neq k \neq l \).

(1)–(4) are well known types of imprecise information, and (5) is a ranking of differences of adjacent parameters obtained by weak rankings among the parameters, which can be subsequently constructed based on (1).

Let \( \tilde{A}^{(k)} = (\tilde{a}^{(k)}_{ij})_{s \times n} \) be a fuzzy decision matrix, where \( \tilde{a}^{(k)}_{ij} = [a^{(k)}_{ij}, a^{(k)}_{mij}, a^{(k)}_{sij}] \) is an attribute value, given by the decision maker \( d_k \in D \), for the alternative \( x_j \in X \) with respect to the attribute \( G_i \in G \).

For simplicity of calculation, we transform the fuzzy decision matrix \( \tilde{A}^{(k)} = (\tilde{a}^{(k)}_{ij})_{s \times n} \) into an expected decision matrix \( \bar{A}^{(k)} = (\bar{a}^{(k)}_{ij})_{s \times n} \) by using the following formula [21]:

\[
\bar{a}^{(k)}_{ij} = \frac{1}{2} [(1 - \eta_k) a^{(k)}_{ij} + a^{(k)}_{mij} + \eta_k a^{(k)}_{sij}], \quad \eta_k \in [0, 1], \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, t
\]

(1)

where \( \eta_k \) is an index that reflects the decision maker’s risk-bearing attitude. If \( \eta_k < 0.5 \), then the decision maker is a risk lover. If \( \eta_k = 0.5 \), then the attitude of the decision maker is neutral to the risk. If \( \eta_k > 0.5 \), then the decision maker is a risk avertor. In general, \( \eta_k \) can be given by the decision maker directly.

In general, there are benefit attributes and cost attributes in multiple attribute decision making problems. In order to measure all attributes in dimensionless units and to facilitate inter-attribute comparisons, we need to normalize each expected attribute value \( \bar{a}^{(k)}_{ij} \) in the matrix \( \bar{A}^{(k)} = (\bar{a}^{(k)}_{ij})_{s \times n} \) into a corresponding element in the matrix \( \bar{R}^{(k)} = (\bar{r}^{(k)}_{ij})_{s \times n} \), where

\[
\bar{r}^{(k)}_{ij} = \bar{a}^{(k)}_{ij} / \sum_{j=1}^{n} \bar{a}^{(k)}_{ij} \quad \text{for benefit attribute} \ G_i, \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, t
\]

(2)

\[
\bar{r}^{(k)}_{ij} = (1/\bar{a}^{(k)}_{ij}) / \sum_{j=1}^{n} (1/\bar{a}^{(k)}_{ij}) \quad \text{for cost attribute} \ G_i, \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, t
\]

(3)

From (2) and (3), we can obtain the following conclusions:

1. The normalized expected decision matrices can preserve all the information that the original expected decision matrices contain.
2. The transformation by using (2) or (3) is straightforward, and can be performed on computer easily.

In the following, we introduce some operators for aggregating decision information:

**Definition 1** [9]. Let WAA: \( R^n \rightarrow R \).

\[
\text{WAA}_\omega (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \omega_j a_j
\]

(4)

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^\top \) is the weight vector of the arguments \( a_i \) \( (i = 1, 2, \ldots, n) \), \( \omega_j \geq 0 \), \( j = 1, 2, \ldots, n \), \( \sum_{j=1}^{n} \omega_j = 1 \), and \( R \) is the set of all real numbers, then WAA is called a weighted arithmetic averaging operator.

Obviously, the WAA operator weights all the given arguments by a normalized weight vector, and then aggregates these weighted arguments.
Definition 2 [39]. An ordered weighted averaging (OWA) operator of dimension $n$ is a mapping OWA: $R^n \rightarrow R$ that has an associated vector $v = (v_1, v_2, \ldots, v_n)^T$ such that $v_j \geq 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} v_j = 1$. Furthermore

$$\text{OWA}_v(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} v_j b_j$$

where $b_j$ is the $j$th largest of the arguments $a_i$ ($i = 1, 2, \ldots, n$).

The fundamental aspect of the OWA operator is its reordering step. Several methods have been developed to obtain the OWA weights. Yager [39] suggested a way to compute the OWA weights using linguistic quantifiers. O’Hagan [22] developed a procedure to generate the OWA weights that have a predefined degree of orness and maximize the entropy of the OWA weights. Yager [40] introduced some families of the OWA weights. Filev and Yager [7] developed two procedures, based on the exponential smoothing, to obtain the OWA weights. Yager [39] suggested a way to compute the OWA weights using linguistic quantifiers. Filev and Yager [42] suggested an algorithm to obtain the OWA weights from a collection of samples with the relevant aggregated data. Füllér and Majlender [8] used the method of Lagrange multipliers to solve O’Hagan’s procedure analytically. Xu and Da [37] established a linear objective-programming model to obtain the OWA weights under partial weight information. Especially, based on the normal distribution (Gaussian distribution), Xu [36] developed a method to obtain the OWA weights, whose prominent characteristic is that it can relieve the influence of unfair arguments on the decision result by assigning low weights to those “false” or “biased” ones.

At present, the WAA and OWA operators are two of the most common operators for aggregating information [38,41]. From Definitions 1 and 2, we know that the WAA operator only weights the argument itself, but ignores the importance of the ordered position of the argument, while the OWA operator only weights the ordered position of the argument, but ignores the importance of the argument. To solve this drawback, Xu and Da [38] introduced a hybrid weighted averaging (HWA) operator, which weights both the given argument and its ordered position.

Definition 3 [38]. A hybrid weighted averaging (HWA) operator is a mapping HWA: $R^n \rightarrow R$ that has an associated vector $v = (v_1, v_2, \ldots, v_n)^T$ with $v_j \geq 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} v_j = 1$, such that

$$\text{HWA}_{v, \omega}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} v_j b_j$$

where $b_j$ is the $j$th largest of the weighted arguments $n \omega a_i$ ($i = 1, 2, \ldots, n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of the arguments $a_i$ ($i = 1, 2, \ldots, n$), $\omega_j \geq 0$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} \omega_j = 1$, and $n$ is the balancing coefficient, which plays a role of balance (in this case, if the vector $(\omega_1, \omega_2, \ldots, \omega_n)^T$ approaches $(1/n, 1/n, \ldots, 1/n)^T$, then the vector $(n \omega a_1, n \omega a_2, \ldots, n \omega a_n)^T$ approaches $(a_1, a_2, \ldots, a_n)^T$).

From Definition 3, we know that the HWA operator is carried out in the following three phases:

1. Multiply the arguments $a_i$ ($i = 1, 2, \ldots, n$) by the associated weights $\omega_i$ ($i = 1, 2, \ldots, n$) and the balancing coefficient $n$, and then get the weighted arguments $n \omega a_i$ ($i = 1, 2, \ldots, n$).
2. Reorder the weighted arguments $n \omega a_i$ ($i = 1, 2, \ldots, n$) in descending order $(b_1, b_2, \ldots, b_n)$, where $b_j$ is the $j$th largest of $n \omega a_i$ ($i = 1, 2, \ldots, n$).
3. Multiply these ordered arguments $b_j$ ($j = 1, 2, \ldots, n$) by the HWA weights $v_j$ ($j = 1, 2, \ldots, n$), and then aggregate all the weighted arguments $v_j b_j$ ($j = 1, 2, \ldots, n$).

It is clear that the HWA operator generalizes the WAA and OWA operators, and can reflect the importance of both the given argument and the ordered position of the argument.

In Section 3, we will adopt the HWA operator to aggregate decision information and develop an interactive method for multiple attribute group decision making under fuzzy environment.
3. An interactive method for fuzzy multiple attribute group decision making

We first utilize the HWA operator to aggregate the normalized expected decision matrices \( \bar{R}(k) = (\bar{r}_{ij}^{(k)})_{s \times n} \) \( (k = 1, 2, \ldots, t) \) into a complex decision matrix \( \bar{R} = (\bar{r}_{ij})_{s \times n} \), where

\[
\bar{r}_{ij} = \text{HWA}_{v_j}(\bar{r}_{ij}^{(1)}, \bar{r}_{ij}^{(2)}, \ldots, \bar{r}_{ij}^{(t)}), \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, n
\]

\( v = (v_1, v_2, \ldots, v_t)^T \) is the associated vector of the HWA operator, and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T \) is the weight vector of decision makers.

Based on the complex decision matrix \( \bar{R} = (\bar{r}_{ij})_{s \times n} \), the overall value of an alternative \( x_j \) can be expressed as follows:

\[
z_j(w) = \sum_{i=1}^{s} \bar{r}_{ij} w_i, \quad j = 1, 2, \ldots, n
\]

Obviously, the greater the value \( z_j(w) \), the better the alternative \( x_j \).

In general, the overall values of alternatives are used to rank alternatives \([5,6,11]\), and a decision maker chooses an alternative \( x_i \) such that \( z_i(w) \geq z_j(w), \) for all \( j \).

To get a preferred alternative, we give the following definition.

**Definition 4.** For an alternative \( x_p \in X \), if there exists an alternative \( x_q \in X \), such that \( z_q(w) > z_p(w) \), then \( x_p \) is called a dominated alternative; otherwise, the alternative \( x_p \) is called a non-dominated alternative.

Dominated alternatives should be eliminated because they are inferior to non-dominated alternatives. As a result, the given alternative set will get diminished.

The following theorem will provide a method to identify dominated alternatives.

**Theorem 1.** The alternative \( x_p \) is a dominated alternative if and only if \( F_p < 0 \), where

\[
F_p = \max_{w, \theta} \left( \sum_{i=1}^{s} \bar{r}_{ip} w_i + \theta \right)
\]

s.t.

\[
\sum_{i=1}^{s} \bar{r}_{ij} w_i + \theta \leq 0, \quad j \neq p
\]

\[
w = (w_1, w_2, \ldots, w_s)^T \in H, \quad w_i \geq 0, \quad \sum_{i=1}^{s} w_i = 1
\]

\( \theta \) is only an unconstrained auxiliary variable, which has no actual meaning.

**Proof.** SUFFICIENCY: If \( F_p < 0 \), then by the constraint conditions in Theorem 1, we have \( \sum_{i=1}^{s} \bar{r}_{ip} w_i \leq -\theta \), for any \( j \neq p \). When the optimum solution is taken, there exists at least an integer \( q \), such that \( j = q \), and the equality holds, i.e., \( \sum_{i=1}^{s} \bar{r}_{iq} w_i = -\theta \). From \( F_p < 0 \), we have

\[
F_p = \max_{w, \theta} \left( \sum_{i=1}^{s} \bar{r}_{ip} w_i + \theta \right) = \max_{w} \left( \sum_{i=1}^{s} \bar{r}_{ip} w_i - \sum_{i=1}^{s} \bar{r}_{iq} w_i \right) < 0
\]

and thus, \( \sum_{i=1}^{s} \bar{r}_{ip} w_i < \sum_{i=1}^{s} \bar{r}_{iq} w_i \), i.e., \( z_p(w) < z_q(w) \). Therefore, \( x_p \) is a dominated alternative.

NECESSITY: Since \( x_p \in X \) is a dominated alternative, there exists an alternative \( x_q \in X \), such that \( \sum_{i=1}^{s} \bar{r}_{iq} w_i \leq \sum_{i=1}^{s} \bar{r}_{ip} w_i \). By the constraint conditions in Theorem 1, we have \( \sum_{i=1}^{s} \bar{r}_{iq} w_i \leq -\theta \), and thus

\[
\sum_{i=1}^{s} \bar{r}_{ip} w_i - (-\theta) \leq \sum_{i=1}^{s} \bar{r}_{iq} w_i - \sum_{i=1}^{s} \bar{r}_{iq} w_i < 0
\]

i.e., \( F_p < 0 \). This completes the proof of Theorem 1. \( \Box \)
We only need to identify every alternative in $X$ and understand whether it is a dominated alternative or not. As a result, we can eliminate any dominated alternatives from the alternative set $X$, and then the set $\overline{X}$ whose elements are non-dominated alternatives can be obtained. Obviously, $\overline{X}$ is a subset of $X$, and thus the alternative set $X$ is diminished.

By Theorem 1, below we develop an interactive procedure to find out the most preferred alternative.

**Procedure 1**

Step 1. For a fuzzy multiple attribute group decision making problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives, $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision makers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$ be the weight vector of decision makers, where $\lambda_k \geq 0$, $k = 1, 2, \ldots, t$ and $\sum_{k=1}^{t} \lambda_k = 1$ (although all decision makers generally have equal weights in deciding group preferences, there are many applications, especially in situations involving policy specification, which necessitate different weights [28] because a decision maker cannot be expected to have sufficient expertise to comment on all aspects of the problem but on a part of the problem for which he/she is competent [33]. Up to now, many methods have been developed to determine the weights of decision makers. Theil [30] proposed a method based on symmetry when the utilities of the members are measurable. Keeney and Kirkwood [14], and Keeney [13] suggested the use of interpersonal comparisons to obtain the values of scaling constants in the weighted additive social choice function. Bodily [2] derived the member weight as a result of designation of voting weights from a member to a delegation subcommittee made up of other members of the group. Brock [3] used a Nash bargaining based approach to estimate the weights of group members intrinsically. Ramanathan and Ganesh [28] proposed a simple and intuitively appealing eigenvector based method to intrinsically determine the weights of group members using their own subjective opinions). Let $G = \{G_1, G_2, \ldots, G_n\}$ be the set of attributes, and $w = (w_1, w_2, \ldots, w_x)^T$ be the weight vector of attributes, where $w_i \geq 0$, $i = 1, 2, \ldots, s$, $\sum_{i=1}^{s} w_i = 1$, and $w \in H$. $H$ is the set of the known weight information, given by the decision makers, as described in Section 2. Let $A^{(k)} = (a^{(k)}_{ij})_{s \times n}$ be a fuzzy decision matrix, given by the decision maker $d_k \in D$, where $a^{(k)}_{ij} = (a^{(k)}_{ij}, a^{(k)}_{i}, a^{(k)}_{ij})$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, t$.

Step 2. Determine the index of rating attitude, $\eta_k$ (in general, it can be given by the decision maker $d_k$ directly) and then utilize the fuzzy decision matrix $A^{(k)}$ and (1) to get an expected decision matrix $\overline{A}^{(k)} = (\overline{a}^{(k)}_{ij})_{s \times n}$.

Step 3. Normalize the expected decision matrix $\overline{A}^{(k)} = (\overline{a}^{(k)}_{ij})_{s \times n}$ into a corresponding matrix $\overline{R}^{(k)} = (r^{(k)}_{ij})_{s \times n}$ by using (2) and (3).

Step 4. Utilize the HWA operator (7) to aggregate the normalized expected decision matrices $\overline{R}^{(k)} = (r^{(k)}_{ij})_{s \times n}$ ($k = 1, 2, \ldots, t$) into a complex decision matrix $\overline{R} = (\overline{r}_{ij})_{s \times n}$, where $\overline{r}_{ij} = \text{HWA}_{v, \lambda}(\overline{r}^{(1)}_{ij}, \overline{r}^{(2)}_{ij}, \ldots, \overline{r}^{(t)}_{ij})$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, n$, $v = (v_1, v_2, \ldots, v_x)^T$ is the associated vector of the HWA operator, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$ is the weight vector of decision makers.

Step 5. By Theorem 1, we identify whether the alternative $x_j$ ($j = 1, 2, \ldots, n$) is a dominated alternative or not, eliminate dominated alternatives, and then get a set $\overline{X}$, whose elements are non-dominated alternatives. If most of the decision makers suggest that an alternative $x \in \overline{X}$ be superior to any other alternatives in $\overline{X}$, or the alternative $x$ is the only one alternative left in $\overline{X}$, then the most preferred alternative is $x$, go to Step 7; otherwise, go to Step 6.

Step 6. Interact with the decision makers, and add the decision information (provided by the decision makers) as the weight information to the set $H$. If the added information given by a decision maker contradicts the information in $H$, then return it to the decision maker for reassessment, and go to Step 5.

Step 7. End.

**Theorem 2.** The above interactive procedure is convergent.

**Proof.** By Theorem 1, we can identify whether an alternative $x$ is a dominated alternative or not, eliminate dominated alternatives, and then get a set $\overline{X}$, whose elements are non-dominated alternatives. Interacting with
the decision makers, if most of the decision makers suggest that the alternative \( x \in X \) be superior to any other alternatives in \( X \), or the alternative \( x \) is the only one alternative left in \( X \), then the alternative \( x \) is the most preferred one; otherwise, we add the decision information (provided by the decision makers) as the weight information to the set \( H \). If the added information given by a decision maker contradicts the information in \( H \), then we return it to the decision maker for reassessment. With the increase of the weight information, the number of alternatives in \( X \) will be diminished gradually. Ultimately, either most of the decision makers suggest that a certain alternative in \( X \) be the most preferred one, or there is only one alternative left in the set \( X \), then this alternative is the most preferred one. This completes the proof of Theorem 2. \( \square \)

In the above procedure, the weight information of decision makers is assessed using exact numerical values. However, in some real-life situations, it is difficult, if not impossible, to use exact numerical values to capture imprecision and vagueness inherent in subjective assessments because of the lack of data or the inability of assessors to provide precise assessments. In such cases, the weight information of decision makers may take the form of triangular fuzzy numbers rather than exact numerical ones.

For convenience, in what follows, we introduce the operational laws and the distance measure of triangular fuzzy numbers.

**Definition 5** [31]. Let \( \tilde{a} = [a_l, a_m, a_u] \) and \( \tilde{b} = [b_l, b_m, b_u] \) be two triangular fuzzy numbers, where \( a_u \geq a_m \geq a_l \geq 0 \) and \( b_u \geq b_m \geq b_l \geq 0 \), then

1. \( \tilde{a} \oplus \tilde{b} = [a_l + b_l, a_m + b_m, a_u + b_u] \);
2. \( \tilde{a} \otimes \tilde{b} = [a_l b_l, a_m b_m, a_u b_u] \);
3. \( \mu \tilde{a} = [\mu a_l, \mu a_m, \mu a_u] \), \( \mu > 0 \).

**Definition 6** [4]. Let \( \tilde{a} = [a_l, a_m, a_u] \) and \( \tilde{b} = [b_l, b_m, b_u] \) be two triangular fuzzy numbers, then the distance between \( \tilde{a} \) and \( \tilde{b} \) is defined as follows:

\[
d(\tilde{a}, \tilde{b}) = \sqrt{\frac{1}{3} [(a_l - b_l)^2 + (a_m - b_m)^2 + (a_u - b_u)^2]}
\] (11)

Similar to Procedure 1, below we develop an interactive procedure, which can be used in situations where the weights of decision makers and the attribute values are triangular fuzzy numbers, and the information about attribute weights is partly known.

**Procedure 2**

**Step 1.** For a fuzzy multiple attribute group decision making problem, let \( \tilde{\lambda} = [\lambda_{ik}, \lambda_{mk}, \lambda_{uk}] \) be the weight of the decision maker \( d_k \in D \), where \( \lambda_{ik}, \lambda_{mk}, \lambda_{uk} \in [0, 1], k = 1, 2, \ldots, t, \sum_{k=1}^{t} \lambda_{ik} \leq 1 \) and \( \sum_{k=1}^{t} \lambda_{uk} \geq 1 \). Let \( w = (w_1, w_2, \ldots, w_s)^T \) be the weight vector of attributes, and \( \tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{s \times n} \) be a fuzzy decision matrix given by the decision maker \( d_k \in D \), where \( w_i \geq 0, i = 1, 2, \ldots, s, \sum_{i=1}^{s} w_i = 1, w \in H \), and \( \tilde{a}_{ij}^{(k)} = [a_{ij}^{(k)}, a_{mij}^{(k)}, a_{uij}^{(k)}], i = 1, 2, \ldots, s, j = 1, 2, \ldots, n, k = 1, 2, \ldots, t \).

**Step 2.** Construct the weighted fuzzy decision matrix \( \tilde{B}^{(k)} = (\tilde{b}_{ij}^{(k)})_{s \times n}, \) where

\[
\tilde{b}_{ij}^{(k)} = [b_{ij}^{(k)}, b_{mij}^{(k)}, b_{uij}^{(k)}] = \lambda_{ik} \otimes \tilde{a}_{ij}^{(k)} = [\lambda_{ik}, \lambda_{mk}, \lambda_{uk}] \otimes [a_{ij}^{(k)}, a_{mij}^{(k)}, a_{uij}^{(k)}] = [\lambda_{ik} a_{ij}^{(k)}, \lambda_{mk} a_{mij}^{(k)}, \lambda_{uk} a_{uij}^{(k)}],
\]

\( i = 1, 2, \ldots, s, \) \( j = 1, 2, \ldots, n, \) \( k = 1, 2, \ldots, t \) (12)

**Step 3.** Determine \( \eta_k \), and then utilize the weighted fuzzy decision matrix \( \tilde{B}^{(k)} \) and (1) to get an expected decision matrix \( \tilde{B}^{(k)} = (\tilde{b}_{ij}^{(k)})_{s \times n}, \) where

\[
\tilde{b}_{ij}^{(k)} = \frac{1}{2} [(1 - \eta_k) b_{ij}^{(k)} + b_{mij}^{(k)} + \eta_k b_{uij}^{(k)}], \quad i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, t \) (13)

**Step 4.** Normalize the expected decision matrix \( \tilde{B}^{(k)} = (\tilde{b}_{ij}^{(k)})_{s \times n} \) into a corresponding matrix \( \tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{s \times n} \) by using (2) and (3).
Step 5. Utilize the OWA operator to aggregate the normalized expected decision matrices \( \hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{s \times n} \) into a complex decision matrix \( \hat{R} = (\hat{r}_{ij})_{s \times n} \), where

\[
\hat{r}_{ij} = \text{OWA}_v(\hat{r}_{ij}^{(1)}, \hat{r}_{ij}^{(2)}, \ldots, \hat{r}_{ij}^{(t)}), \quad i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n
\]

and \( v = (v_1, v_2, \ldots, v_t)^T \) is the associated vector of the OWA operator.


Step 7. See Step 6 of Procedure 1.

Step 8. End.

Hwang and Yoon [11] developed a TOPSIS for multiple attribute decision making. The fundamental idea of the TOPSIS is that the chosen alternative should have the shortest distance from the positive-ideal solution and the farthest distance from the negative-ideal solution. The TOPSIS can only be used in situations where the attribute values and the attribute weights are expressed in exact numerical values. Chen [4] proposed an extended TOPSIS for multiple attribute group decision making under fuzzy environment, which can be described as follows.

Step 1. For a multiple attribute group decision making problem, suppose that all the decision makers \( d_k \in D \) \( (k = 1, 2, \ldots, t) \) have equal weights \( \lambda_i = \frac{1}{t} \) \( (i = 1, 2, \ldots, s) \). Let \( \hat{w}_i^{(k)} \) be the weight of the attribute \( G_i \in G \), and \( \hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{s \times n} \) be a fuzzy decision matrix, given by the decision maker \( d_k \), where \( \hat{w}_i^{(k)} \) and \( \hat{a}_{ij}^{(k)} \) \( (i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, t) \) are linguistic variables (linguistic variables can be described by triangular fuzzy numbers, as shown in Table 1) or triangular fuzzy numbers, \( \hat{w}_i^{(k)} = [w_{li}^{(k)}, w_{mi}^{(k)}, w_{ui}^{(k)}], \ w_{li}^{(k)}, w_{mi}^{(k)}, w_{ui}^{(k)} \in [0, 1] \), and \( \hat{a}_{ij}^{(k)} = [a_{li}^{(k)}, a_{mi}^{(k)}, a_{ui}^{(k)}], \ i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, t \).

Step 2. Aggregate the fuzzy decision matrices \( \hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{s \times n} \) \( (k = 1, 2, \ldots, t) \) into a complex fuzzy decision matrix \( \hat{A} = (\hat{a}_{ij})_{s \times n} \), where

\[
\hat{a}_{ij} = \frac{1}{t} (\hat{a}_{ij}^{(1)} \oplus \hat{a}_{ij}^{(2)} \oplus \cdots \oplus \hat{a}_{ij}^{(t)}) = [a_{ij}, a_{mi}, a_{ui}] = \left[ \frac{1}{t} \sum_{k=1}^{t} a_{ij}^{(k)}, \frac{1}{t} \sum_{k=1}^{t} a_{mi}^{(k)}, \frac{1}{t} \sum_{k=1}^{t} a_{ui}^{(k)} \right], \quad i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n
\]

and aggregate the attribute weights \( w_i^{(k)} = [w_{li}^{(k)}, w_{mi}^{(k)}, w_{ui}^{(k)}] \) \( (i = 1, 2, \ldots, s, \ k = 1, 2, \ldots, t) \) into the corresponding complex attribute weights \( \hat{w}_i = [\hat{w}_{li}, \hat{w}_{mi}, \hat{w}_{ui}] \) \( (i = 1, 2, \ldots, s) \), where

\[
\hat{w}_i = \frac{1}{t} (\hat{w}_i^{(1)} \oplus \hat{w}_i^{(2)} \oplus \cdots \oplus \hat{w}_i^{(t)}) = \left[ \frac{1}{t} \sum_{k=1}^{t} w_{li}^{(k)}, \frac{1}{t} \sum_{k=1}^{t} w_{mi}^{(k)}, \frac{1}{t} \sum_{k=1}^{t} w_{ui}^{(k)} \right], \quad i = 1, 2, \ldots, s
\]

Step 3. Normalize the complex fuzzy decision matrix \( \hat{A} = (\hat{a}_{ij})_{s \times n} \) into a corresponding matrix \( \hat{R} = (\hat{r}_{ij})_{s \times n} \), where

<table>
<thead>
<tr>
<th>Linguistic variables</th>
<th>Triangular fuzzy numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very low (VL)</td>
<td>[0.0, 0.1]</td>
</tr>
<tr>
<td>Low (L)</td>
<td>[0.0, 1, 0.3]</td>
</tr>
<tr>
<td>Medium low (ML)</td>
<td>[0.1, 0.3, 0.5]</td>
</tr>
<tr>
<td>Medium (M)</td>
<td>[0.3, 0.5, 0.7]</td>
</tr>
<tr>
<td>Medium high (MH)</td>
<td>[0.5, 0.7, 0.9]</td>
</tr>
<tr>
<td>High (H)</td>
<td>[0.7, 0.9, 1.0]</td>
</tr>
<tr>
<td>Very high (VH)</td>
<td>[0.9, 1.0, 1.0]</td>
</tr>
</tbody>
</table>
Step 4. Construct the weighted normalized fuzzy decision matrix \( \tilde{V} = (\tilde{v}_{ij})_{n \times n} \), where 
\[
\tilde{v}_{ij} = \tilde{w}_i \otimes \tilde{r}_{ij}, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n
\]
Step 5. Calculate the fuzzy positive-ideal solution and fuzzy negative-ideal solution, respectively:
\[
x^+ = (\hat{v}_{1}^+, \hat{v}_{2}^+, \ldots, \hat{v}_{n}^+), \quad x^- = (\hat{v}_{1}^-, \hat{v}_{2}^-, \ldots, \hat{v}_{n}^-)
\]
where \( \hat{v}_{i}^+ = [1, 1, 1] \) and \( \hat{v}_{i}^- = [0, 0, 0] \), \( i = 1, 2, \ldots, n \).
Step 6. Calculate the distance between each alternative and positive-ideal solution and the distance between each alternative and negative-ideal solution, respectively:
\[
d_j^+ = \sum_{i=1}^{j} d(\hat{v}_{ij}, \hat{v}_{i}^+), \quad d_j^- = \sum_{i=1}^{j} d(\hat{v}_{ij}, \hat{v}_{i}^-), \quad j = 1, 2, \ldots, n
\]
Step 7. Calculate the closeness coefficient of each alternative:
\[
cc_j = \frac{d_j^-}{d_j^+ + d_j^-}, \quad j = 1, 2, \ldots, n
\]
Step 8. Rank all the alternatives \( x_j (j = 1, 2, \ldots, n) \) according to the closeness coefficients \( cc_j (j = 1, 2, \ldots, n) \), the greater the value \( cc_j \), the better the alternative \( x_j \).
Step 9. End.

From the developed interactive method (Procedures 1 and 2) and the extended TOPSIS, we can conclude the following:

1. The interactive method can be used in situations where the information about attribute weights can be constructed in five different forms and the weights of decision makers can be expressed in exact numerical values (Procedure 1) or triangular fuzzy numbers (Procedure 2), and the attribute values are triangular fuzzy numbers, while the extended TOPSIS can only be used in situations where the weights of decision makers are expressed in exact numerical values [4] only considered the cases where all decision makers have equal weights, and all attribute weights and attribute values take the form of triangular fuzzy numbers.
2. The interactive method can not only reflect the importance of the given arguments and the ordered positions of the arguments, but also relieve the influence of unfair arguments on the decision result by using the normal distribution based method [36] to assign low weights to those “false” or “biased” ones. However, the extended TOPSIS only considers weighting the given arguments, and cannot relieve the influence of unfair arguments on the decision result.
3. The interactive method can actualize the process of interactive decision making by interactively providing or modifying decision makers’ preference information so as to make the decision result more reasonable.
4. In the interactive method, the number of linear programming models to be solved is no more than \( n \) at one interaction, and the number of linear programming models to be solved decreases as the interactive time steadily increases.
5. The interactive method can be used to many real-life applications under fuzzy environment, such as negotiation processes, the high technology project investment of venture capital firms, etc., in which...
attribute weights and attribute values are usually uncertain or inaccurate, and decision makers are asked
to give their preferences gradually in the course of decision making.

4. Illustrative example

In this section, a fuzzy multiple attribute group decision making problem of determining what kind of air-
conditioning systems should be installed in a library (adapted from [44]) is used to illustrate the proposed
approach.

A city is planning to build a municipal library. One of the problems facing the city development commis-
sioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor
offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives
\( x_j \) \((j = 1, 2, 3, 4, 5)\) are to be evaluated by three decision makers
\( d_k \) \((k = 1, 2, 3)\) under three major impacts: economic, functional and operational. Two monetary attributes and six non-monetary attributes (that is, \( G_1 \): owning cost ($/ft^2), G_2 \): operating cost ($/ft^2), G_3 \): performance (*), G_4 \): noise level (Db), G_5 \): maintainability (*), G_6 \): flexibility (*) , G_7 \): reliability (%), G_8 \): safety (*), where * unit is from 0 to 1 scale, three attributes \( G_1 \), \( G_2 \) and \( G_4 \) are cost attributes, and the other five attributes are benefit attributes) emerged from three impacts in
Tables 2–4.

Table 2
Decision matrix \( \tilde{A}^{(1)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>[3.7, 4.2, 4.7]</td>
<td>[1.5, 2.2, 2.5]</td>
<td>[3.4, 5]</td>
<td>[3.5, 4.2, 4.5]</td>
<td>[2.5, 3.2, 3.5]</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>[5.9, 6.3, 6.9]</td>
<td>[4.7, 5.1, 5.7]</td>
<td>[4.2, 4.5, 5.2]</td>
<td>[4.5, 5.1, 5.5]</td>
<td>[5.6, 7]</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>[0.8, 0.9, 1]</td>
<td>[0.4, 0.5, 0.6]</td>
<td>[0.4, 0.5, 0.7]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.6, 0.7, 0.8]</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>[30, 35, 40]</td>
<td>[65, 70, 75]</td>
<td>[60, 65, 70]</td>
<td>[35, 40, 45]</td>
<td>[50, 55, 60]</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>[0.3, 0.4, 0.5]</td>
<td>[0.2, 0.4, 0.5]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.8, 0.9, 1]</td>
<td>[0.5, 0.6, 0.7]</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>[90, 95, 100]</td>
<td>[70, 75, 80]</td>
<td>[80, 85, 90]</td>
<td>[85, 90, 95]</td>
<td>[83, 90, 92]</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>[0.3, 0.4, 0.5]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.6, 0.8, 1]</td>
<td>[0.6, 0.7, 0.8]</td>
<td>[0.4, 0.5, 0.6]</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>[0.6, 0.7, 0.8]</td>
<td>[0.4, 0.5, 0.6]</td>
<td>[0.5, 0.6, 0.7]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.8, 0.9, 1]</td>
</tr>
</tbody>
</table>

Table 3
Decision matrix \( \tilde{A}^{(2)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>[3.9, 4.4, 5.1]</td>
<td>[1.4, 2.1, 2.3]</td>
<td>[4.5, 6]</td>
<td>[3.4, 4.4, 4.7]</td>
<td>[2.3, 3.3, 3.6]</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>[6.3, 6.5, 7.2]</td>
<td>[4.8, 5.1, 5.5]</td>
<td>[4.4, 4.6, 5.1]</td>
<td>[4.2, 5.0, 5.6]</td>
<td>[6.7, 8]</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.3, 0.4, 0.6]</td>
<td>[0.3, 0.5, 0.7]</td>
<td>[0.6, 0.7, 0.8]</td>
<td>[0.6, 0.7, 0.8]</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>[32, 37, 39]</td>
<td>[62, 71, 74]</td>
<td>[62, 67, 72]</td>
<td>[37, 42, 44]</td>
<td>[52, 54, 62]</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>[0.2, 0.4, 0.5]</td>
<td>[0.4, 0.5, 0.6]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.7, 0.8, 1]</td>
<td>[0.5, 0.6, 0.7]</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>[92, 95, 98]</td>
<td>[69, 74, 81]</td>
<td>[81, 83, 92]</td>
<td>[84, 90, 96]</td>
<td>[83, 92, 94]</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>[0.3, 0.5, 0.6]</td>
<td>[0.6, 0.8, 0.9]</td>
<td>[0.6, 0.7, 0.9]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.3, 0.4, 0.5]</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.5, 0.6, 0.7]</td>
<td>[0.6, 0.7, 0.8]</td>
<td>[0.6, 0.7, 0.8]</td>
<td>[0.7, 0.8, 0.9]</td>
</tr>
</tbody>
</table>

Table 4
Decision matrix \( \tilde{A}^{(3)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>[3.8, 4.3, 4.9]</td>
<td>[1.7, 2.3, 2.7]</td>
<td>[3.5, 7]</td>
<td>[3.6, 4.7, 4.9]</td>
<td>[2.3, 3.1, 3.7]</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>[5.5, 6.4, 6.7]</td>
<td>[4.5, 5.0, 5.9]</td>
<td>[4.5, 4.7, 5.5]</td>
<td>[3.5, 5.5, 5.9]</td>
<td>[4.6, 8]</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>[0.8, 0.9, 1]</td>
<td>[0.3, 0.5, 0.8]</td>
<td>[0.5, 0.6, 0.8]</td>
<td>[0.6, 0.8, 1]</td>
<td>[0.5, 0.7, 0.9]</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>[32, 36, 41]</td>
<td>[67, 72, 77]</td>
<td>[62, 67, 75]</td>
<td>[37, 44, 48]</td>
<td>[53, 56, 64]</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>[0.3, 0.4, 0.6]</td>
<td>[0.3, 0.4, 0.5]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.7, 0.9, 1]</td>
<td>[0.4, 0.6, 0.8]</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>[92, 93, 98]</td>
<td>[74, 77, 85]</td>
<td>[85, 89, 95]</td>
<td>[86, 94, 96]</td>
<td>[84, 93, 97]</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>[0.3, 0.4, 0.7]</td>
<td>[0.7, 0.8, 0.9]</td>
<td>[0.8, 0.9, 1]</td>
<td>[0.4, 0.6, 0.8]</td>
<td>[0.3, 0.5, 0.7]</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>[0.5, 0.7, 0.8]</td>
<td>[0.5, 0.6, 0.8]</td>
<td>[0.4, 0.6, 0.8]</td>
<td>[0.6, 0.8, 0.9]</td>
<td>[0.7, 0.9, 1]</td>
</tr>
</tbody>
</table>
Suppose that the decision makers utilize linguistic variables to provide the weights of the attributes, as listed in Table 5.

In this case, the extended TOPSIS can be applied to the selection of air-conditioning systems, which involves the following steps:

Step 1. Utilize (15) to aggregate the decision matrices $\hat{A}^{(k)}$ ($k = 1, 2, 3$) into a complex decision matrix $\hat{A}$ (see Table 6).

Step 2. Utilize (17)–(20) to normalize the complex fuzzy decision matrix $\hat{A}$, and then get a normalized matrix $\hat{R}$ (see Table 7).

Step 3. Utilize (16) to derive the complex attribute weights as follows:

$\hat{w}_1 = [0.47, 0.60, 0.70]$, \ $\hat{w}_2 = [0.40, 0.57, 0.73]$, \ $\hat{w}_3 = [0.83, 0.97, 1.0]$, \ $\hat{w}_4 = [0.77, 0.90, 0.97]$, \ $\hat{w}_5 = [0.27, 0.40, 0.57]$, \ $\hat{w}_6 = [0.57, 0.77, 0.90]$, \ $\hat{w}_7 = [0.23, 0.43, 0.53]$, \ $\hat{w}_8 = [0.33, 0.50, 0.67]$

Step 4. Calculate the distance between each alternative and positive-ideal solution and the distance between each alternative and negative-ideal solution by using (21)–(23), respectively:

$d_1^+ = 4.495$, \ $d_1^- = 3.830$, \ $d_2^+ = 4.914$, \ $d_2^- = 3.391$, \ $d_3^+ = 4.779$, \ $d_3^- = 3.532$, \ $d_4^+ = 4.285$, \ $d_4^- = 4.082$, \ $d_5^+ = 4.697$, \ $d_5^- = 3.589$

Step 5. Calculate the closeness coefficient of each alternative by using (24):

$cc_1 = 0.460$, \ $cc_2 = 0.408$, \ $cc_3 = 0.425$, \ $cc_4 = 0.488$, \ $cc_5 = 0.433$

Table 5

<table>
<thead>
<tr>
<th></th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>$G_4$</th>
<th>$G_5$</th>
<th>$G_6$</th>
<th>$G_7$</th>
<th>$G_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>VL</td>
<td>MH</td>
<td>VH</td>
<td>VH</td>
<td>MH</td>
<td>H</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>$D_2$</td>
<td>H</td>
<td>L</td>
<td>H</td>
<td>VH</td>
<td>VL</td>
<td>H</td>
<td>H</td>
<td>L</td>
</tr>
<tr>
<td>$D_3$</td>
<td>H</td>
<td>H</td>
<td>VH</td>
<td>MH</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>H</td>
</tr>
</tbody>
</table>

Table 6

Complex decision matrix $\hat{A}$

<table>
<thead>
<tr>
<th>$G_1$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>[3.80,4.30,4.90]</td>
<td>[1.53,2.20,2.50]</td>
<td>[3.33,4.67,6.00]</td>
<td>[3.50,4.43,4.70]</td>
<td>[2.37,3.20,3.60]</td>
</tr>
<tr>
<td>$G_2$</td>
<td>[6.10,6.40,6.93]</td>
<td>[4.67,5.07,5.70]</td>
<td>[4.37,4.60,5.27]</td>
<td>[4.07,5.20,5.67]</td>
<td>[5.00,6.33,7.67]</td>
</tr>
<tr>
<td>$G_3$</td>
<td>[0.77,0.87,0.97]</td>
<td>[0.33,0.47,0.67]</td>
<td>[0.40,0.53,0.73]</td>
<td>[0.63,0.77,0.90]</td>
<td>[0.57,0.70,0.83]</td>
</tr>
<tr>
<td>$G_4$</td>
<td>[31.3,36.0,40.0]</td>
<td>[64.7,71.0,75.3]</td>
<td>[61.3,66.3,72.3]</td>
<td>[36.3,42.0,45.7]</td>
<td>[51.7,55.0,62.0]</td>
</tr>
<tr>
<td>$G_5$</td>
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<td>[0.30,0.43,0.53]</td>
<td>[0.70,0.80,0.90]</td>
<td>[0.73,0.87,1.00]</td>
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</tr>
<tr>
<td>$G_6$</td>
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<td>[71.0,75.3,82.0]</td>
<td>[82.0,85.7,92.3]</td>
<td>[85.0,91.3,95.7]</td>
<td>[83.3,91.7,94.3]</td>
</tr>
<tr>
<td>$G_7$</td>
<td>[0.30,0.43,0.60]</td>
<td>[0.67,0.80,0.90]</td>
<td>[0.67,0.80,0.97]</td>
<td>[0.57,0.70,0.83]</td>
<td>[0.33,0.47,0.60]</td>
</tr>
<tr>
<td>$G_8$</td>
<td>[0.60,0.73,0.83]</td>
<td>[0.47,0.57,0.70]</td>
<td>[0.50,0.63,0.77]</td>
<td>[0.63,0.77,0.87]</td>
<td>[0.73,0.87,0.97]</td>
</tr>
</tbody>
</table>

Table 7

Normalized decision matrix $\hat{R}$

<table>
<thead>
<tr>
<th>$G_1$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>[0.31,0.36,0.40]</td>
<td>[0.61,0.70,1.00]</td>
<td>[0.26,0.33,0.46]</td>
<td>[0.33,0.35,0.44]</td>
<td>[0.42,0.48,0.65]</td>
</tr>
<tr>
<td>$G_2$</td>
<td>[0.59,0.64,0.67]</td>
<td>[0.71,0.80,0.87]</td>
<td>[0.77,0.88,0.93]</td>
<td>[0.72,0.78,1.00]</td>
<td>[0.53,0.64,0.81]</td>
</tr>
<tr>
<td>$G_3$</td>
<td>[0.79,0.90,1.00]</td>
<td>[0.34,0.48,0.69]</td>
<td>[0.41,0.55,0.75]</td>
<td>[0.65,0.79,0.93]</td>
<td>[0.59,0.72,0.86]</td>
</tr>
<tr>
<td>$G_4$</td>
<td>[0.78,0.87,1.00]</td>
<td>[0.42,0.44,0.48]</td>
<td>[0.43,0.47,0.51]</td>
<td>[0.69,0.75,0.86]</td>
<td>[0.51,0.57,0.61]</td>
</tr>
<tr>
<td>$G_5$</td>
<td>[0.27,0.40,0.53]</td>
<td>[0.30,0.43,0.53]</td>
<td>[0.70,0.80,0.90]</td>
<td>[0.73,0.87,1.00]</td>
<td>[0.47,0.60,0.73]</td>
</tr>
<tr>
<td>$G_6$</td>
<td>[0.93,0.96,1.00]</td>
<td>[0.72,0.76,0.83]</td>
<td>[0.83,0.87,0.94]</td>
<td>[0.86,0.93,0.97]</td>
<td>[0.84,0.93,0.96]</td>
</tr>
<tr>
<td>$G_7$</td>
<td>[0.31,0.44,0.62]</td>
<td>[0.69,0.82,0.93]</td>
<td>[0.69,0.82,1.00]</td>
<td>[0.59,0.72,0.86]</td>
<td>[0.34,0.48,0.62]</td>
</tr>
<tr>
<td>$G_8$</td>
<td>[0.62,0.75,0.86]</td>
<td>[0.48,0.59,0.72]</td>
<td>[0.52,0.65,0.79]</td>
<td>[0.65,0.79,0.90]</td>
<td>[0.75,0.90,1.00]</td>
</tr>
</tbody>
</table>
Step 6. Rank all the alternatives \( x_j (j = 1, 2, 3, 4, 5) \) according to the closeness coefficients \( cc_j (j = 1, 2, 3, 4, 5) \):

\[
x_4 > x_1 > x_5 > x_3 > x_2
\]

Thus, the most preferred alternative is \( x_4 \).

However, the decision makers sometimes have different weights, and they may provide the information about attribute weights with value ranges or order relations as described in Section 2. Moreover, in the process of decision making, a decision maker often needs to interact with group members (or analysts) by providing and modifying his/her incomplete preference information gradually. Clearly, the extended TOPSIS is unsuitable for dealing with these situations. For example, we suppose that \( \lambda = (0.4, 0.3, 0.3)^T \) is the weight vector of decision makers, and the information about attribute weights, given by the decision makers, are as follows, respectively:

\[
d_1: w_1 \leq 0.1, \ 0.2 \leq w_3 \leq 0.5, \ w_5 \leq 0.3;
\]
\[
d_2: 0.1 \leq w_2 \leq 0.2, \ w_4 - w_6 \geq 0.1;
\]
\[
d_3: w_3 - w_6 \geq w_6 - w_7, \ w_7 \leq w_5, \ 0.1 \leq w_6 \leq 0.4.
\]

Then

\[
H = \left\{ w_1 \leq 0.1, \ 0.2 \leq w_2 \leq 0.5, \ 0.2 \leq w_3 \leq 0.5, \ w_5 \leq 0.3, \ w_4 - w_6 \geq 0.1, \ 0.1 \leq w_6 \leq 0.4, \ w_7 \leq w_5, \ w_3 - w_8 \geq w_6 - w_7, \ w_i \geq 0, \ i = 1, 2, \ldots, 8, \sum_{i=1}^{8} w_i = 1 \right\}
\]

In this case, we can utilize Procedure 1 to select the most preferred alternative, which is shown as follows:

Step 1. Suppose that the decision makers \( d_k (k = 1, 2, 3) \) give the indices of rating attitude \( \eta_1 = 0.770, \ \eta_2 = 0.805, \ \eta_3 = 0.752 \), respectively. Then we utilize the decision matrices \( \hat{A}^{(k)} (k = 1, 2, 3) \) and (1) to get the expected decision matrices \( \hat{A}^{(k)} \) (see Tables 8–10).

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>4.3350</td>
<td>2.2350</td>
<td>4.2700</td>
<td>4.2350</td>
<td>3.2350</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>6.4850</td>
<td>5.2850</td>
<td>4.7350</td>
<td>5.1850</td>
<td>6.2700</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>0.9270</td>
<td>0.5270</td>
<td>0.5655</td>
<td>0.8270</td>
<td>0.7270</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>36.350</td>
<td>71.350</td>
<td>66.350</td>
<td>41.350</td>
<td>56.350</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>0.4270</td>
<td>0.4155</td>
<td>0.8270</td>
<td>0.9270</td>
<td>0.6270</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>96.350</td>
<td>76.350</td>
<td>86.350</td>
<td>91.350</td>
<td>89.965</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0.4270</td>
<td>0.8270</td>
<td>0.8540</td>
<td>0.7270</td>
<td>0.5270</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>0.7270</td>
<td>0.5270</td>
<td>0.6270</td>
<td>0.8270</td>
<td>0.9270</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>4.6330</td>
<td>2.1122</td>
<td>5.3050</td>
<td>4.4233</td>
<td>4.2893</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>6.7622</td>
<td>3.2305</td>
<td>4.7818</td>
<td>5.1635</td>
<td>7.3050</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>0.8305</td>
<td>0.4708</td>
<td>0.5610</td>
<td>0.8708</td>
<td>0.6305</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>37.318</td>
<td>71.330</td>
<td>68.525</td>
<td>42.318</td>
<td>57.025</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>0.4208</td>
<td>0.5305</td>
<td>0.8305</td>
<td>0.8708</td>
<td>0.6305</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>95.915</td>
<td>76.330</td>
<td>86.428</td>
<td>91.830</td>
<td>91.928</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0.5208</td>
<td>0.8208</td>
<td>0.7708</td>
<td>0.8305</td>
<td>0.4305</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>0.8305</td>
<td>0.6305</td>
<td>0.7305</td>
<td>0.7305</td>
<td>0.8305</td>
</tr>
</tbody>
</table>
Step 2. Normalize the expected decision matrices \( A^k \) \((k = 1, 2, 3)\), and then get the normalized matrices \( R^k \) \((k = 1, 2, 3)\) by using (2) and (3) (see Tables 11–13).

Step 3. Utilize the HWA operator (let \( v = (0.2429, 0.5142, 0.2429) \) be its associated vector, please see [36] for more details) to aggregate the normalized expected decision matrices \( R^k \) \((k = 1, 2, 3)\) into a complex decision matrix \( R \) (see Table 14).

### Table 10
Expected decision matrix \( A^{(3)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>4.4636</td>
<td>2.3760</td>
<td>5.5040</td>
<td>4.6388</td>
<td>3.2264</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>6.4012</td>
<td>5.2764</td>
<td>4.9760</td>
<td>5.4024</td>
<td>6.5040</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>0.9252</td>
<td>0.5880</td>
<td>0.6628</td>
<td>0.8504</td>
<td>0.7504</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>37.384</td>
<td>73.260</td>
<td>69.388</td>
<td>44.636</td>
<td>58.636</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>0.4628</td>
<td>0.4252</td>
<td>0.8252</td>
<td>0.9128</td>
<td>0.6504</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>94.756</td>
<td>79.636</td>
<td>90.760</td>
<td>93.760</td>
<td>93.388</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0.5004</td>
<td>0.8252</td>
<td>0.9252</td>
<td>0.6504</td>
<td>0.4752</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>0.7128</td>
<td>0.6628</td>
<td>0.6504</td>
<td>0.8128</td>
<td>0.9128</td>
</tr>
</tbody>
</table>

### Table 11
Normalized expected decision matrix \( R^{(1)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>0.1583</td>
<td>0.3070</td>
<td>0.1607</td>
<td>0.1620</td>
<td>0.2121</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>0.1700</td>
<td>0.2086</td>
<td>0.2329</td>
<td>0.2126</td>
<td>0.1759</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>0.2594</td>
<td>0.1475</td>
<td>0.1582</td>
<td>0.2314</td>
<td>0.2034</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>0.2792</td>
<td>0.1422</td>
<td>0.1530</td>
<td>0.2455</td>
<td>0.1801</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>0.1325</td>
<td>0.1289</td>
<td>0.2566</td>
<td>0.2876</td>
<td>0.1945</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>0.2188</td>
<td>0.1734</td>
<td>0.1961</td>
<td>0.2074</td>
<td>0.2043</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0.1270</td>
<td>0.2460</td>
<td>0.2540</td>
<td>0.2162</td>
<td>0.1568</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>0.2000</td>
<td>0.1450</td>
<td>0.1725</td>
<td>0.2275</td>
<td>0.2550</td>
</tr>
</tbody>
</table>

### Table 12
Normalized expected decision matrix \( R^{(2)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>0.1614</td>
<td>0.3541</td>
<td>0.1410</td>
<td>0.1691</td>
<td>0.1744</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>0.1483</td>
<td>0.3104</td>
<td>0.2097</td>
<td>0.1942</td>
<td>0.1373</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>0.2499</td>
<td>0.1417</td>
<td>0.1688</td>
<td>0.2198</td>
<td>0.2198</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>0.2775</td>
<td>0.1452</td>
<td>0.1511</td>
<td>0.2447</td>
<td>0.1816</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>0.1282</td>
<td>0.1616</td>
<td>0.2530</td>
<td>0.2651</td>
<td>0.1920</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>0.2168</td>
<td>0.1725</td>
<td>0.1953</td>
<td>0.2076</td>
<td>0.2078</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0.1544</td>
<td>0.2433</td>
<td>0.2285</td>
<td>0.2462</td>
<td>0.1276</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>0.2213</td>
<td>0.1680</td>
<td>0.1947</td>
<td>0.1947</td>
<td>0.2213</td>
</tr>
</tbody>
</table>

### Table 13
Normalized expected decision matrix \( R^{(3)} \)

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>0.1657</td>
<td>0.3113</td>
<td>0.1344</td>
<td>0.1594</td>
<td>0.2292</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>0.1764</td>
<td>0.2140</td>
<td>0.2269</td>
<td>0.2090</td>
<td>0.1736</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>0.2450</td>
<td>0.1557</td>
<td>0.1755</td>
<td>0.2252</td>
<td>0.1987</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>0.2838</td>
<td>0.1448</td>
<td>0.1529</td>
<td>0.2377</td>
<td>0.1809</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>0.1413</td>
<td>0.1298</td>
<td>0.2519</td>
<td>0.2786</td>
<td>0.1985</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>0.2095</td>
<td>0.1761</td>
<td>0.2007</td>
<td>0.2073</td>
<td>0.2065</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0.1482</td>
<td>0.2444</td>
<td>0.2740</td>
<td>0.1926</td>
<td>0.1407</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>0.1900</td>
<td>0.1767</td>
<td>0.1734</td>
<td>0.2167</td>
<td>0.2433</td>
</tr>
</tbody>
</table>
Table 14
Complex decision matrix $\bar{R}$

<table>
<thead>
<tr>
<th>$G_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>0.1581</td>
<td>0.3214</td>
<td>0.1415</td>
<td>0.1603</td>
<td>0.2060</td>
</tr>
<tr>
<td>$G_2$</td>
<td>0.1636</td>
<td>0.2434</td>
<td>0.2187</td>
<td>0.2011</td>
<td>0.1616</td>
</tr>
<tr>
<td>$G_3$</td>
<td>0.2448</td>
<td>0.1460</td>
<td>0.1642</td>
<td>0.2197</td>
<td>0.2044</td>
</tr>
<tr>
<td>$G_4$</td>
<td>0.2734</td>
<td>0.1402</td>
<td>0.1484</td>
<td>0.2368</td>
<td>0.1761</td>
</tr>
<tr>
<td>$G_5$</td>
<td>0.1320</td>
<td>0.1407</td>
<td>0.2469</td>
<td>0.2707</td>
<td>0.1905</td>
</tr>
<tr>
<td>$G_6$</td>
<td>0.2099</td>
<td>0.1697</td>
<td>0.1727</td>
<td>0.2018</td>
<td>0.2009</td>
</tr>
<tr>
<td>$G_7$</td>
<td>0.1409</td>
<td>0.2380</td>
<td>0.2508</td>
<td>0.2191</td>
<td>0.1387</td>
</tr>
<tr>
<td>$G_8$</td>
<td>0.2022</td>
<td>0.1608</td>
<td>0.1783</td>
<td>0.2092</td>
<td>0.2353</td>
</tr>
</tbody>
</table>

Step 4. By Theorem 1, we identify whether the alternative $x_1$ is a dominated alternative or not. To do so, we first establish the following linear programming model:

$$F_1 = \max_{w, \theta}(0.1581w_1 + 0.1636w_2 + 0.2448w_3 + 0.2734w_4 + 0.1320w_5 + 0.2099w_6 + 0.1409w_7 + 0.2022w_8 + \theta_1 - \theta_2)$$

s.t.  
- $0.3214w_1 + 0.2434w_2 + 0.1460w_3 + 0.1402w_4 + 0.1407w_5 + 0.1697w_6 + 0.2380w_7 + 0.1608w_8 + \theta_1 - \theta_2 \leq 0$
- $0.1415w_1 + 0.2187w_2 + 0.1642w_3 + 0.1484w_4 + 0.2469w_5 + 0.1727w_6 + 0.2508w_7 + 0.1783w_8 + \theta_1 - \theta_2 \leq 0$
- $0.1603w_1 + 0.2011w_2 + 0.2197w_3 + 0.2368w_4 + 0.2707w_5 + 0.2018w_6 + 0.2191w_7 + 0.2092w_8 + \theta_1 - \theta_2 \leq 0$
- $0.2060w_1 + 0.1616w_2 + 0.2044w_3 + 0.1761w_4 + 0.1905w_5 + 0.2009w_6 + 0.1387w_7 + 0.2353w_8 + \theta_1 - \theta_2 \leq 0$
- $w_1 \leq 0.1, \ 0.2 \leq w_3 \leq 0.5, \ w_5 \leq 0.3, \ 0.1 \leq w_6 \leq 0.2, \ w_4 - w_6 \geq 0.1$
- $w_3 - w_8 \geq w_6 - w_7, \ w_7 \leq w_5, \ 0.1 \leq w_6 \leq 0.4$
- $w_i \geq 0 \ (i = 1, 2, \ldots, 8), \ \sum_{i=1}^{8} w_i = 1, \ \theta_1 \geq 0, \ \theta_2 \geq 0$

where $\theta_1$ and $\theta_2$ are two unconstrained auxiliary variables, which have no actual meaning. Solving this model, we have

$\theta_1 = 0, \ \theta_2 = 0.1396, \ w_1 = 0, \ w_2 = 0.1, \ w_3 = 0.2667, \ w_4 = 0.3667, \ w_5 = 0, \ w_6 = 0.2667, \ w_7 = 0, \ w_8 = 0, \ F_1 = 0.1437 > 0$

and thus, $x_1$ is a non-dominated alternative. Similarly, we can get that $x_4$ is a non-dominated alternative, $x_2, x_3$ and $x_5$ are three dominated alternatives, and thus $\mathcal{X} = \{x_1, x_4\}$.

Step 5. Interact with the decision makers. Suppose that the decision makers add the weight information $w_5 \geq 0.4$ to the set $H$, then by Theorem 1, it follows that

(1) For the alternative $x_1$, we have

$\theta_1 = 0, \ \theta_2 = 0.2434, \ w_1 = 0, \ w_2 = 0.1, \ w_3 = 0.2, \ w_4 = 0.3, \ w_5 = 0, \ w_6 = 0, \ w_7 = 0, \ w_8 = 0, \ F_4 = -0.0432 < 0$

(2) For the alternative $x_4$, we have

$\theta_1 = 0, \ \theta_2 = 0.2004, \ w_1 = 0, \ w_2 = 0.1, \ w_3 = 0.2, \ w_4 = 0.2, \ w_5 = 0.4, \ w_6 = 0.1, \ w_7 = 0, \ w_8 = 0, \ F_1 = 0.0395 > 0$

From (1) and (2), we know that $x_1$ is a dominated alternative, and $x_4$ is a non-dominated alternative, that is, there is only the alternative $x_4$ left in the set $\mathcal{X}$. Therefore, the most preferred alternative is $x_4$. 
5. Concluding remarks

In this paper, we have developed an interactive method to solve fuzzy multiple group decision making problems. The method transforms the fuzzy decision matrices into their expected decision matrices and constructs the normalized expected decision matrices. The normalized expected decision matrices can preserve all the information that the original fuzzy decision matrices contain. Moreover, the transformation is straightforward, and can be performed on computer easily. Although the WAA and OWA operators are two of the most common operators for aggregating information, both the operators have their disadvantages, i.e., the WAA operator only weights the argument itself, and the OWA operator only weights the ordered position of the argument. Therefore, in this paper we have adopted a practical operator called the HWA operator to aggregate the decision information given by each decision maker, which can weight both the given argument and the ordered position of the argument. In the process of interactions, the decision makers provide and modify their preference information such that the dominated alternatives can be diminished gradually until the most preferred alternative is obtained. Theoretical analysis and the numerical results have shown that the number of linear programming models to be solved is no more than \( n \) at one interaction, and the number of linear programming models to be solved decreases as the interactive time steadily increases. The interactive method is somewhat computationally complex, but it has some desirable advantages over the extended TOPSIS, including: (1) the interactive method can be applied more widely since the information about attribute weights can be constructed in various forms; (2) the interactive method can not only reflect the importance of the given arguments and the ordered positions of the arguments, but also relieve the influence of unfair arguments on the decision result; (3) by using the interactive method, the decision makers can provide and modify their preference information gradually in the course of decision making, and thus make the decision result more reasonable.

The developed interactive method is illustrated using an example of determining what kind of air-conditioning systems should be installed in a library. It can also be applicable to group decision making problems in many other fields, such as negotiation processes, the high technology project investment of venture capital firms, supply chain management, etc. Furthermore, in this paper, we only consider the situations where the weights of decision makers are expressed in exact numerical values or triangular fuzzy numbers. Under some conditions, however, the information about the weights of decision makers might also be linear-inequality-typed information, which is a difficult but promising research problem that needs to be answered in the future.

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References


