An Overview of Operators for Aggregating Information

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In this work, we first make a survey of the existing main aggregation operators and then propose some new aggregation operators such as the induced ordered weighted geometric averaging (IOWGA) operator, generalized induced ordered weighted averaging (GIOWA) operator, hybrid weighted averaging (HWA) operator, etc., and study their desirable properties. Finally, we briefly classify all of these aggregation operators. © 2003 Wiley Periodicals, Inc.

1. INTRODUCTION

At present, many aggregation operators have been developed to aggregate information such as the max and min operators,1–24 arithmetic averaging (AA) operator,16–21 weighted AA (WAA) operator,20–34 fuzzy weighted averaging (FWA) operator,35–39 geometric averaging (GA) operator,40–44 weighted geometric averaging (WGA) operator,45–49 ordered weighted averaging (OWA) operator,16–18,50–60 induced ordered weighted averaging (IOWA) operator,21,61 generalized ordered weighted averaging (GOWA) operator,62 weighted ordered averaging (WOWA) operator,20 weighted order statistic averaging (WOSA) operator,63 etc. These operators have been used in a wide range of applications such as engineering,30 neural networks,57,64 database systems,56,66 fuzzy logic controllers,19,67 decision making,16,27,29,31,68–70 expert systems,71 market research,72 linguistic quantified propositions,55,73 mathematical programming,56 lossless image compression,58 etc. In this article, first, we reviewed the existing main aggregation operators and then proposed some new aggregation operators and studied their desirable properties. Finally, we briefly classified all these aggregation operators.
2. THE EXISTING MAIN AGGREGATION OPERATORS

In the following, we review the existing main aggregation operators. For simplicity, we let \( M = \{1, 2, \ldots, m\} \) and \( N = \{1, 2, \ldots, n\} \).

2.1. Max Operator\(^{1–24}\)

Let \( \{a_1, a_2, \ldots, a_n\} \) be a collection of arguments, if

\[
 f(a_1, a_2, \ldots, a_n) = \max_i \{a_i\} \quad (1)
\]

then \( f \) is called the max operator.

Especially, if \( a_i \in S \), and \( f \) is called the linguistic max operator, where \( S \) is a limited set of linguistic labels. For example, \( S \) can be defined so its elements are distributed uniformly on a scale on which a total order is defined as\(^{74}\)

\[
 S = \{s_1 = \text{none}, s_2 = \text{very low}, s_3 = \text{low}, s_4 = \text{medium}, s_5 = \text{high}, s_6 = \text{very high}, s_7 = \text{perfect}\}
\]

where \( s_i < s_j \) iff \( i < j \).

2.2. Min Operator\(^{1–22}\)

Let \( \{a_1, a_2, \ldots, a_n\} \) be a collection of arguments, if

\[
 f(a_1, a_2, \ldots, a_n) = \min_i \{a_i\} \quad (2)
\]

then \( f \) is called the min operator.

Especially, if \( a_i \in S \), then \( f \) is called the linguistic min operator.

2.3. AA Operator\(^{16–21}\)

Let \( \{a_1, a_2, \ldots, a_n\} \) be a collection of arguments, and let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), if

\[
 f(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i \quad (3)
\]

then \( f \) is called the AA operator.

2.4. WAA Operator\(^{20–34}\)

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), if

\[
 f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \omega_i a_i \quad (4)
\]
where $\mathbf{w} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i (i \in N)$, and $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$, then $f$ is called the WAA operator.

Especially, if $\mathbf{w} = (1/n, 1/n, \ldots, 1/n)^T$, then $f$ is reduced to the AA operator.

2.5. FWA Operator\textsuperscript{35–39}

Let $f : \Theta^n \rightarrow \Theta$, where $\Theta$ is the set of fuzzy numbers, if

$$f(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \sum_{i=1}^n \hat{\omega}_i \hat{a}_i / \sum_{i=1}^n \hat{\omega}_i$$

(5)

where $\hat{\mathbf{w}} = (\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_n)^T$ is the weighting vector of the $\hat{a}_i (i \in N)$, and $\hat{\omega}_j$, $\hat{a}_j \in \Theta$, then $f$ is called the FWA operator.

2.6. GA Operator\textsuperscript{40–44}

Let $f : R^+_n \rightarrow R^+$, if

$$f(a_1, a_2, \ldots, a_n) = \left( \prod_{i=1}^n a_i \right)^{1/n}$$

(6)

then $f$ is called the GA operator.

2.7. WGA Operator\textsuperscript{45–49}

Let $f : R^+_n \rightarrow R^+$, if

$$f(a_1, a_2, \ldots, a_n) = \prod_{i=1}^n a_i^{\omega_i}$$

(7)

where $\mathbf{w} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the exponential weighting vector of the $a_i (i \in N)$, and $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$, then $f$ is called the WGA operator.

Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then $f$ is reduced to the GA operator.

2.8. Max–Min Operator\textsuperscript{9,19,72,75–83}

Let $\{ a_1, a_2, \ldots, a_n \}$ be a collection of arguments, if

$$f(a_1, a_2, \ldots, a_n) = \max_{i} \min \{ \omega_i, a_i \}$$

(8)

where $\mathbf{w} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i (i \in N)$, and $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$, then $f$ is called the max–min operator.

Especially, if $\omega_i, a_i \in S$, then we call $f$ the linguistic max–min operator.
2.9. Min-Max Operator\textsuperscript{79–82,84,85}

If we replace Equation 8 with
\[ f(a_1, a_2, \ldots, a_n) = \min_{i} \max \{ \omega_i, a_i \} \]  \hspace{1cm} (9)
then we get a min–max operator.

2.10. OWA Operator\textsuperscript{16–18,50–60}

An OWA operator of dimension \( n \) is a mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated \( n \) vector \( \mathbf{w} = (w_1, w_2, \ldots, w_n)^T \) such that \( w_j \in [0, 1] \), \( \sum_{j=1}^{n} w_j = 1 \). Furthermore,
\[ f(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \]  \hspace{1cm} (10)
where \( b_j \) is the \( j \)th largest of the \( a_i \).

Especially, if \( \mathbf{w} = (1, 0, \ldots, 0)^T \), then \( f \) is reduced to the max operator; if \( \mathbf{w} = (0, 0, \ldots, 1)^T \), then \( f \) is reduced to the min operator; if \( \mathbf{w} = (1/n, 1/n, \ldots, 1/n)^T \), then \( f \) is reduced to the AA operator.

The fundamental aspect of the OWA operator is the reordering step, in particular, an argument \( a_i \) is not associated with a particular weight \( w_i \) but rather a weight \( w_i \) is associated with a particular ordered position \( i \) of the arguments.

2.11. S-OWA-OR Operator\textsuperscript{18,19,76}

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), if
\[ f(a_1, a_2, \ldots, a_n) = (1 - \beta) \frac{1}{n} \sum_{i=1}^{n} a_i + \beta \max_{i} \{ a_i \} \]  \hspace{1cm} (11)
where \( 0 \leq \beta \leq 1 \), then \( f \) is called the S-OWA-OR operator.

Especially, if \( \beta = 0 \), then \( f \) is reduced to the AA operator; if \( \beta = 1 \), then \( f \) is reduced to the max operator.

2.12. S-OWA-AND Operator\textsuperscript{18,19,76}

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), if
\[ f(a_1, a_2, \ldots, a_n) = (1 - \alpha) \frac{1}{n} \sum_{i=1}^{n} a_i + \alpha \min_{i} \{ a_i \} \]  \hspace{1cm} (12)
where \( 0 \leq \alpha \leq 1 \), then \( f \) is called the S-OWA-AND operator.

Especially, if \( \alpha = 0 \), then \( f \) is reduced to the AA operator; if \( \alpha = 1 \), then \( f \) is reduced to the min operator.
2.13. Uncertain OWA Operator

Let $\Omega$ be the set of interval numbers. An uncertain OWA (UOWA) operator of dimension $n$ is a mapping $f: \Omega^n \rightarrow \Omega$ that has an associated $n$ vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. Furthermore,

$$f(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \sum_{j=1}^n w_j \bar{b}_j$$

(13)

where $\bar{b}_j$ is the $j$th largest of the $\bar{a}_i$, and the $\bar{a}_i$'s ($i \in N$) are interval numbers.

2.14. Ordered WGA Operator

An ordered WGA (OWGA) operator of dimension $n$ is a mapping $f: R^+ \rightarrow R^+$ which has associated with it an exponential weighting vector $w = (w_1, w_2, \ldots, w_n)^T$, with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, such that

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \prod_{j=1}^n b_j^{w_j}$$

(14)

where $b_j$ is the $j$th largest of the $\alpha_i$.

Especially, if $w = (1, 0, \ldots, 0)^T$, then $f$ is reduced to the max operator; if $w = (0, 0, \ldots, 1)^T$, then $f$ is reduced to the min operator; if $w = (1/n, 1/n, \ldots, 1/n)^T$, then $f$ is reduced to the GA operator.

The OWGA operator has many desirable properties similar to those of the OWA operator (Ref. 89).

2.15. Linguistic OWA Operator

In Ref. 74, Bordogna, etc. gives a linguistic OWA (LOWA) operator as follows:

An LOWA operator is a mapping $f: S^n \rightarrow S$, which has associated with it a weighting vector $w = (w_1, w_2, \ldots, w_n)^T$, with $w_i \in S$, and $w_i \geq w_j$, for $i > j$, such that

$$f(a_1, a_2, \ldots, a_n) = \max_{i} \min \{ w_i, b_i \}$$

(15)

where $b_j$ is the $j$th highest label among the $\alpha_i$.

In Refs. 91–96, Herrera et al. give another LOWA operator as follows: An LOWA operator is defined as

$$f(a_1, a_2, \ldots, a_n) = w \cdot B^T = \phi^n\{w_k, b_k, k = 1, 2, \ldots, n\}$$

$$= w_1 \otimes b_1 \oplus (1 - w_1) \otimes \phi^{n-1}\{\beta_h, b_h, h = 2, 3, \ldots, n\}$$

(16)
where $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $\mathbf{w}_j \in [0, 1]$, $\sum_{i=1}^n w_j = 1$, $\beta_h = \mathbf{w}_j / \sum_{k=1}^n w_k$, $h = 2, 3, \ldots, n$, and $B = (b_1, b_2, \ldots, b_n)^T$ is the associated ordered label vector. Each element $b_j \in B$ is the $i$th largest label in the collection $a_1, a_2, \ldots, a_n$. The term $\phi^n$ is the convex combination operator of $n$ labels and if $n = 2$, then it is defined as

$$
\phi^2(\mathbf{w}_i, b_i, i = 1, 2) = w_1 \otimes s_j \oplus (1 - w_1) \otimes (1 - w_i) \otimes s_i
$$

such that $k = \min\{n, i + \text{round}(w_i(j - i))\}$, where round is the usual round operator, and $b_1 = s_j$, and $b_2 = s_i$; if $w_j = 1$ and $w_i = 0 (i \neq j)$, then the convex combination is defined as $\phi^n(\mathbf{w}_k, b_k, k = 1, 2, \ldots, n) = b_j$.

### 2.16. Induced OWA Operator

Yager and Filev\textsuperscript{21} present an induced OWA (IOWA) operator in which the ordering of the $a_i (i \in N)$ is induced by other variables $u_i (i \in N)$ called the order inducing variables, where $u_i$ and $a_i (i \in N)$ are the components of the OWA pairs $\langle u_i, a_i \rangle (i \in N)$. The IOWA operator was defined as follows:

$$
f(\langle u_1, a_1 \rangle, \ldots, \langle u_n, a_n \rangle) = \sum_{j=1}^n \mathbf{w}_j b_j
$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $\mathbf{w}_j \in [0, 1]$, $\sum_{i=1}^n w_j = 1$, $b_j$ is the $i$th value of the OWA pair having the $i$th largest $u_j (i \in N)$, and $u_i$ in $\langle u_i, a_i \rangle$ is referred to as the order inducing variable and $a_i$ as the argument variable.

Especially, if $\mathbf{w} = (1/n, 1/n, \ldots, 1/n)^T$, then $f$ is reduced to the AA operator; if $\forall i \in N$ and $u_i = a_i$, then $f$ is reduced to the OWA operator; if $\forall i \in N$, $v_i = \text{No. } i$, where No. $i$ is the ordered position of the $a_i$, then $f$ is the WAA operator.

### 2.17. GOWA Operator

A GOWA operator of Ref. 62 is defined as follows.

Consider a set of arguments $a_i (i \in N)$. Assume that each $a_i$ has a priority $b_i$. Rearrange the argument $a_i$ according to the priorities $b_i$, such that the arguments of higher priority come first. Suppose $a_{(k)}$ and $b_{(k)}$ are the rearranged arguments and priorities. Then, if $b_i$ is the $k$th largest priority, $b_{(k)} = b_i$, and then $a_{(k)} = a_i$. In this case, the GOWA is given by

$$
f(a_1, a_2, \ldots, a_n) = \mathbf{w}^T \mathbf{P} \mathbf{A}
$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector, such that $\mathbf{w}_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, $\mathbf{P}$ is the permutation matrix whose elements $\mathbf{P}_{ki}$, $i$, and $k \in N$, satisfy $\mathbf{P}_{ki} \in [0, 1]$, $\sum_{k=1}^n \mathbf{P}_{ki} = 1$, $\sum_{i=1}^n \mathbf{P}_{ki} = 1$, and $\mathbf{A} = (a_1, a_2, \ldots, a_n)^T$. 
2.18. **WOWA Operator**

Let \( p = (p_1, p_2, \ldots, p_n)^T \) and \( w = (w_1, w_2, \ldots, w_n)^T \) be weighting vectors of dimension \( n \), such that

(i) \( p_j \in [0, 1], \sum_{j=1}^{n} p_j = 1 \);
(ii) \( w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1 \).

In this case, a mapping \( f : \mathbb{R}^n \to \mathbb{R} \) is a WOWA operator of dimension \( n \) if

\[
f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}
\]

where \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) is a permutation of \( (1, 2, \ldots, n) \) such that \( a_{\sigma(i-1)} \geq a_{\sigma(i)}, \forall i \in N \), and the weight \( \omega_i \) is defined as:

\[
\omega_i = w^* \left( \frac{\sum_{j=i}^{n} p_{\sigma(j)}}{\sum_{j=1}^{i} p_{\sigma(j)}} \right) - w^* \left( \frac{\sum_{j=1}^{n} p_{\sigma(j)}}{\sum_{j=1}^{n} p_{\sigma(j)}} \right)
\]

with \( w^* \) a monotone increasing function that interpolates the points \( (i/n, \sum_{j=i}^{n} w_j) \) together with the point \((0, 0)\). The term \( w^* \) is required to be a straight line when the points can be interpolated in this way.

Especially, if \( w = (1/n, 1/n, \ldots, 1/n)^T \), then \( f \) is reduced to the WAA operator; if \( p = (1/n, 1/n, \ldots, 1/n)^T \), then \( f \) is reduced to the OWA operator.

2.19. **WOSA Operator**

A WOSA operator is a symmetric mean of the form

\[
f(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j
\]

where weights \( w_j \)'s are real numbers, the \( j \)th order statistic \( b_j \) is defined by arranging arguments \( a_1, a_2, \ldots, a_n \) in the increasing order \( b_1 \leq b_2 \leq \cdots \leq b_n \).

Ovchinnikov\(^{63}\) points out that the OWA operator is a special case of the WOSA operator.

2.20. **Fuzzy OWGA Operator**

A fuzzy OWGA (FOWGA) operator of dimension \( n \) is a mapping \( f : \Theta^n \to \Theta \) that has an associated \( n \) vector \( w = (w_1, w_2, \ldots, w_n)^T \) such that \( w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1 \). Furthermore,

\[
f(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \prod_{j=1}^{n} (\hat{b}_j)^{w_j}
\]

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where \( b_j \) is the \( j \)th largest of the \( \hat{a}_i \), and the \( \hat{a}_i \)'s \((i \in N)\) are fuzzy numbers.

3. SOME NEW AGGREGATION OPERATORS

In this section, we introduce four new aggregation operators.

**Definition 3.1.** An induced OWGA (IOWGA) operator is defined as

\[
f(\langle u_1, a_1 \rangle, \ldots, \langle u_n, a_n \rangle) = \prod_{j=1}^{n} b_j^{w_j}
\]

where \( w = (w_1, w_2, \ldots, w_n)^T \) is the associated exponential weighting vector such that \( w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1 \), and \( b_j \) is the \( a \) value of \( \langle u_i, a_i \rangle \) having the \( j \)th largest \( u \) value. The term \( u_i \) is referred as the order inducing variable and \( a_i \) is referred as the argument variable.

Especially, if \( w = (1/n, 1/n, \ldots, 1/n)^T \), then \( f \) is reduced to the GA operator; if \( \forall i \in N, u_i = a_i \), then \( f \) is reduced to the OWGA operator; if \( \forall i \in N, v_i = No. i \), where No. \( i \) is the ordered position of the \( a_i \), then \( f \) is the WGA operator.

The IOWGA operator has many desirable properties similar to those of the IOWA operator.

Both the IOWA and the IOWGA operators, which essentially aggregate objects that are pairs, provide a very general family of aggregation operators. Particularly noteworthy is their ability to provide for aggregations in environments that mix linguistic and numeric variables. However, in some situations, when we need to provide more information about the objects, i.e., each object may consist of three components, a direct locator, an indirect locator, and a prescribed value, it is unsuitable to use the IOWA and IOWGA operators as an aggregation tool. Motivated by this issue, in the following, we present two more general aggregation techniques called the generalized IOWA (GIOWA) operator and generalized IOWGA (GIOWGA) operator.

**Definition 3.2.** A GIOWA operator is given by

\[
f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j
\]

where \( w = (w_1, w_2, \ldots, w_n)^T \) is the associated weighting vector with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \); the object \( \langle v_i, u_i, a_i \rangle \) consists of three components, where the first component \( v_i \) represents the importance degree or character of the second component \( u_i \), and the second component \( u_i \) is used to induce an ordering through the first component \( v_i \) over the third components \( a_i \), which are then aggregated. Here, \( b_j \) is the \( a_i \) value of the object having the \( j \)th largest \( v_i \) \((i \in N)\). In discussing these objects \( \langle v_i, u_i, a_i \rangle \) \((i \in N)\), because of its role we shall refer to the \( v_i \) as the direct
order inducing variable, the $u_i$ as the indirect order inducing variable, and $a_i$ as the argument variable.

If we replace Equation 24 with

$$g((v_1, u_1, a_1), \ldots, (v_n, u_n, a_n)) = \prod_{j=1}^{n} b_j^{w_j}$$

then by Definition 3.2, we get a GIOWGA operator.

The following simple example illustrates the use of both the GIOWA and the GIOWGA operators.

**Example.** Consider the collection of the objects

- $\langle$No. 2, Johnson, 50$\rangle$, $\langle$No. 1, Brown, 40$\rangle$,
- $\langle$No. 4, Smith, 20$\rangle$, $\langle$No. 3, Anderson, 80$\rangle$

By the first component, we get the ordered objects

- $\langle$No. 1, Brown, 40$\rangle$, $\langle$No. 2, Johnson, 50$\rangle$,
- $\langle$No. 3, Anderson, 80$\rangle$, $\langle$No. 4, Smith, 20$\rangle$

The ordering induces the ordered arguments

$$b_1 = 40, \quad b_2 = 50, \quad b_3 = 80, \quad \text{and} \quad b_4 = 20$$

If the weighting vector for this aggregation is $w = (0.2, 0.3, 0.1, 0.4)^T$, then we get

$$f(\langle$No. 2, Johnson, 50$\rangle$, $\langle$No. 1, Brown, 40$\rangle$, $\langle$No. 4, Smith, 20$\rangle$),
- $\langle$No. 3, Anderson, 80$\rangle$) = (0.2)(40) + (0.3)(50) + (0.1)(80) + (0.4)(20) = 39

$$g(\langle$No. 2, Johnson, 50$\rangle$, $\langle$No. 1, Brown, 40$\rangle$, $\langle$No. 4, Smith, 20$\rangle$),
- $\langle$No. 3, Anderson, 80$\rangle$) = $(40^{0.2})(50^{0.3})(80^{0.1})(20^{0.4}) = 34.74$.

Especially, if there exists two objects $\langle v_i, u_i, a_i \rangle$ and $\langle v_j, u_j, a_j \rangle$ such that $v_i = v_j$, then we can follow the policy presented by Yager and Filev, 21 i.e., to replace the arguments of the tied objects by the average of the arguments of the tied objects. If $k$ items are tied, we replace these by $k$ replica’s of their average.

In the following, let us first look at some desirable properties associated with the GIOWA operator.

**Theorem 3.1 (Commutativity).** Assume $f$ is the GIOWA operator, then

$$f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = f(\langle v'_1, u'_1, a'_1 \rangle, \ldots, \langle v'_n, u'_n, a'_n \rangle)$$

where $((v'_1, u'_1, a'_1), \ldots, (v'_n, u'_n, a'_n))$ is any permutation of $((v_1, u_1, a_1), \ldots, (v_n, u_n, a_n))$. 

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Proof. Let

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j \]  

and

\[ f(\langle v'_1, u'_1, a'_1 \rangle, \ldots, \langle v'_n, u'_n, a'_n \rangle) = \sum_{j=1}^{n} w_j b'_j \]

Since \((\langle v'_1, u'_1, a'_1 \rangle, \ldots, \langle v'_n, u'_n, a'_n \rangle)\) is a permutation of \((\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle)\), we have \(b_j = b'_j \) (\(j \in \mathbb{N}\)), and then

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = f(\langle v'_1, u'_1, a'_1 \rangle, \ldots, \langle v'_n, u'_n, a'_n \rangle) \]  

\[ \blacksquare \]

Theorem 3.2 (Idempotency). Assume \(f\) is the GIOWA operator, if \(\alpha_j = \alpha \) \(\forall j \in \mathbb{N}\), then

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = a \]  

(27)

Proof. Since \(a_i = a, \forall i\), we have

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j = \sum_{j=1}^{n} w_j a = a \sum_{j=1}^{n} w_j = a \]  

\[ \blacksquare \]

Theorem 3.3 (Monotonicity). Assume \(f\) is the GIOWA operator, if \(a_i \leq \hat{a}_i, \forall i\), then

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) \leq f(\langle v_1, u_1, \hat{a}_1 \rangle, \ldots, \langle v_n, u_n, \hat{a}_n \rangle) \]  

(28)

Proof. Let

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j \]  

and

\[ f(\langle v_1, u_1, \hat{a}_1 \rangle, \ldots, \langle v_n, u_n, \hat{a}_n \rangle) = \sum_{j=1}^{n} w_j \hat{b}_j \]

Since \(a_i \leq \hat{a}_i, \forall i\), it follows that \(b_i \leq \hat{b}_i\), and then

\[ f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) \leq f(\langle v_1, u_1, \hat{a}_1 \rangle, \ldots, \langle v_n, u_n, \hat{a}_n \rangle) \]  

\[ \blacksquare \]

Theorem 3.4 (Bounded). Assume \(f\) is the GIOWA operator, then
\[
\min_i (a_i) \leq f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) \leq \max_i (a_i) \quad (29)
\]

**Proof.** Let \(\max_i (a_i) = b\) and \(\min_i (a_i) = a\), then

\[
f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j \leq \sum_{j=1}^{n} w_j b = b \sum_{j=1}^{n} w_j = b
\]

\[
f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j a_j \geq \sum_{j=1}^{n} w_j a = a \sum_{j=1}^{n} w_j = a
\]

hence,

\[
\min_i (a_i) \leq f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) \leq \max_i (a_i) \quad \blacksquare
\]

**Theorem 3.5 (AA).** If the weighting vector \(w = (1/n, 1/n, \ldots, 1/n)^T\), and then the GIOWA operator is reduced to the AA operator, i.e.,

\[
f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \frac{1}{n} \sum_{j=1}^{n} a_j \quad (30)
\]

**Proof.** Since the weighting vector \(w = (1/n, 1/n, \ldots, 1/n)^T\), it follows that

\[
f(\langle v_1, u_1, a_1 \rangle, \ldots, \langle v_n, u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j = \frac{1}{n} \sum_{j=1}^{n} b_j = \frac{1}{n} \sum_{j=1}^{n} a_j \quad \blacksquare
\]

By appropriate selection of the direct ordering variable, many different types of argument aggregation can be modeled within this framework of the GIOWA operator:

We first show that if \(v_i = a_i, \forall i \in N\), then the GIOWA operator is reduced to the OWA operator. In fact, we consider \(\langle v_i, u_i, a_i \rangle\), where \(v_i = a_i\). Under our approach \(b_j\) is the \(\alpha_i\) value corresponding to the \(j\)th largest \(v_i\) \((i \in N)\); however, since \(v_i = a_i\), then \(b_j\) is the \(\alpha_i\) value of object having the \(j\)th largest \(\alpha_i\) \((i \in N)\). Thus, the OWA operator is a special case of the GIOWA operator. Similarly, we can show that if \(v_i = u_i, \forall i \in N\), then the GIOWA operator is reduced to the IOWA operator. Therefore, the IOWA operator is a special case of the GIOWA operator.

Similarly to the proof given by Yager and Filev,\(^21\) we can easily prove that the WAA operator also is a special case of this more general technique of the GIOWA operator.
The GIOWGA operator has many desirable properties similar to those of the GIOWA operator.

In the following, we define another aggregation operator called the hybrid weighted averaging (HWA) operator, which combines the advantages of both the OWA and the WAA operators.

**Definition 3.3.** An HWA operator is mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}$, which has associated with it a weighting vector $w = (w_1, w_2, \ldots, w_n)^T$, with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, such that

$$f(a_1, a_2, \ldots, a_n) = \sum_{j=1}^n w_j b_j$$  \hspace{1cm} (31)

where $b_j$ is the $j$th largest of the weighted arguments $n\omega a_i$ ($i \in N$) and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i$ ($i \in N$), with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$, and $n$ is the balancing coefficient.

**Theorem 3.6.** The WAA operator is a special case of the HWA operator.

*Proof.* Let $w = (1/n, 1/n, \ldots, 1/n)^T$, and then

$$f(a_1, a_2, \ldots, a_n) = \sum_{j=1}^n w_j b_j = \frac{1}{n} \sum_{j=1}^n b_j = \frac{1}{n} \sum_{i=1}^n \omega_i a_i$$

which completes the proof of Theorem 3.6.

**Theorem 3.7.** The OWA operator is a special case of the HWA operator.

*Proof.* Let $\omega = (1/n, 1/n, \ldots, 1/n)^T$, and then

$$n \omega a_i = a_i, \quad i \in N$$

which completes the proof of Theorem 3.7.

From Theorems 3.6 and 3.7, we know that the HWA operator generalizes both the OWA and the WAA operators and reflects the importance degrees of both the given argument and the ordered position of the argument.

### 4. Classification of Aggregation Operators

In this section, we briefly classify all the given aggregation operators into four categories, i.e.,

(i) The operators that can only be used in situations where the arguments are exact numeric variables including the AA operator, GA operator, OWA operator, OWGA operator, S-OWA-OR operator, S-OWA-AND operator, WAA operator, WGA operator, GOWA operator, WOWA operator, and HWA operator.
(ii) The operators that can only be used in situations where the arguments are inexact numeric variables (such as interval values, fuzzy values, etc.) including the FWA operator, UOWA operator, and FOWGA operator.

(iii) The operators that can be used only in situations where the arguments are linguistic numeric variables, including the LOWA operators.

(iv) The operators that can provide for aggregations in environments, which mix linguistic and numeric variables, including the max operator, min operator, max–min operator, min–max operator, IOWA operator, IOWGA operator, GIOWA operator, and GIO-WGA operator.

5. CONCLUSIONS

In this article, we have reviewed the existing main aggregation operators and presented some new aggregation operators such as the IOWGA operator, GIOWA operator, HWA operator, etc. We have studied some desirable properties of the proposed operators and briefly classified all of the given aggregation operators. We plan to give several applications of the developed operators in the near future.

Acknowledgments
This work was supported by the National Natural Science Foundation of China (NSFC) under Project 79970093, and the Ph.D. Dissertation Foundation of Southeast University—NARI Relays Electric Co., Ltd.

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