Triple positive solutions for some second-order boundary value problem on a measure chain

Zhanbing Bai\textsuperscript{a,*}, Xiangqian Liang\textsuperscript{a}, Zengji Du\textsuperscript{b}

\textsuperscript{a} Institute of Mathematics, Shandong University of Science and Technology, Qingdao 266510, People’s Republic of China

\textsuperscript{b} Department of Mathematics, Xuzhou Normal University, Xuzhou, Jiangsu 221116, People’s Republic of China

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Abstract

In this paper, we study the existence of three positive solutions for the second-order two-point boundary value problem on a measure chain,

\[ x^{\Delta\Delta}(t) + p(t)f\left(t, x(\sigma(t)), x^{\Delta}(t)\right) = 0, \quad t \in [t_1, t_2], \]

\[ a_1 x(t_1) - a_2 x^{\Delta}(t_1) = 0, \quad a_3 x(\sigma(t_2)) + a_4 x^{\Delta}(\sigma(t_2)) = 0, \]

where \( f : [t_1, \sigma(t_2)] \times [0, \infty) \times R \rightarrow [0, \infty) \) is continuous and \( p : [t_1, \sigma(t_2)] \rightarrow [0, \infty) \) a nonnegative function that is allowed to vanish on some subintervals of \([t_1, \sigma(t_2)]\) of the measure chain. The method involves applications of a new fixed-point theorem due to Bai and Ge [Z.B. Bai, W.G. Ge, Existence of three positive solutions for some second order boundary-value problems, Comput. Math. Appl. 48 (2004) 699–707]. The emphasis is put on the nonlinear term \( f \) involved with the first order delta derivative \( x^{\Delta}(t) \).

Keywords: Boundary value problem; Positive solution; Multiplicity; Measure chains; Fixed point theorem

1. Introduction

Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have been studied extensively. The main tools used are fixed-point theorems. Fixed-point theorems and their applications to nonlinear problems have a long history, some of which is documented in Zeidler’s book [1], and the recent book by Agarwal, O’Regan and Wong [2] contains an excellent summary of the current results and applications. In addition, since Hilger’s [3,4] initial paper unifying the continuous and discrete calculus, the measure chain calculus has been developed considerably. And there has been some merging of fixed-point theorems in seeking multiple positive solutions of boundary value problems for differential equations on measure chains, see [5–12].
In [13], the author and Ge generalized the fixed-point theorem of Leggett-Williams by using theory of fixed-point index. An application of the theorem is given to prove the existence of three positive solutions to the following second-order boundary value problem

\[ x''(t) + f(t, x(t), x'(t)) = 0, \quad \text{for all } t \in (0, 1), \]

\[ x(0) = x(1) = 0, \]

where \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) is continuous and nonnegative. In [14], the authors generalized the results to four-point boundary value problems.

More recently, Ma and Luo [15] obtained the existence of one solution, not necessarily be positive, for nonlinear two-point boundary value problems on time scale

\[ x^\Delta(t) = f \left( t, x(t), x(\sigma(t)) \right), \quad t \in [0, 1], \]

\[ x(0) = 0, \quad x(\sigma(1)) = 0. \]

The main tools used are the Leray–Schauder principle and an existence of the barrier strips.

However, for boundary value problem on measure chains, we note there is little literature referred to the existence of positive solutions when the nonlinear term is involved with delta derivative explicitly. Inspired by all the above works, in this paper, we consider the existence of positive solutions for the boundary value problem on a measure chain

\[ x^{\Delta\Delta}(t) + p(t) f \left( t, x(\sigma(t)), x^\Delta(t) \right) = 0, \quad t \in [t_1, t_2] \]

\[ a_1 x(t_1) - a_2 x^\Delta(t_1) = 0, \quad a_3 x(\sigma(t_2)) + a_4 x^\Delta(\sigma(t_2)) = 0, \]

where \( f : [t_1, \sigma(t_2)] \times [0, \infty) \times \mathbb{R} \to [0, \infty) \) is continuous, \( p \) a nonnegative function that is allowed to vanish on some subintervals of \([t_1, \sigma(t_2)]\) of the measure chain \( \mathbb{T} \), \( a_1, a_2, a_3, a_4 \geq 0 \) and \( k := a_2 a_3 + a_1 a_4 + a_1 a_3 (\sigma(t_2) - t_1) \geq 0 \).

The emphasis is put on the nonlinear term \( f \) involved with the first order delta derivative \( x^\Delta(t) \).

2. Background materials and definitions

To understand (1.1) and (1.2), we need to recall some notations about measure chains, which could be found in [1, 2, 5].

A measure chain \( \mathbb{T} \) is a nonempty closed subset of the real numbers \( \mathbb{R} \).

**Definition 2.1.** Define the forward (respectively, backward) jump operator at \( t \) for \( t < \sup \mathbb{T} \) (respectively, for \( t > \inf \mathbb{T} \)) by

\[ \sigma(t) = \inf \{ s > t : s \in \mathbb{T} \} \quad \text{for all } t \in \mathbb{T} \]

(respectively, \( \rho(t) = \sup \{ s < t : s \in \mathbb{T} \} \)).

The point \( t \in \mathbb{T} \) is left-dense, left-scattered, right-dense, right-scattered respectively, if \( \rho(t) = t \), \( \rho(t) < t \), \( \sigma(t) = t \), \( \sigma(t) > t \), respectively. The set \( \mathbb{T}^r \) is defined to be \( \mathbb{T} \) if \( \mathbb{T} \) does not have a left-scattered maximum; otherwise it is \( \mathbb{T} \) without this left-scattered maximum.

**Definition 2.2.** For \( x : \mathbb{T} \to \mathbb{R} \) and \( t \in \mathbb{T} \) (if \( t = \sup \mathbb{T} \), assume \( t \) is not left-scattered), define the delta derivative of \( x \) at \( t \), denoted \( x^\Delta(t) \), is the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that

\[ |x(\sigma(t)) - x(s) - x^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in U. \]

**Definition 2.3.** A function \( f : \mathbb{T} \to \mathbb{R} \) is right-dense continuous or rd-continuous provided it is continuous at right-dense points in \( \mathbb{T} \) and its left-sided limits exist (finite) at left-dense points in \( \mathbb{T} \). If \( f \) is rd-continuous, then there is a function \( F(t) \) such that \( F^\Delta(t) = f(t) \). In this case, we define

\[ \int_a^b f(t) \Delta t = F(b) - F(a). \]
Lemma 2.1 ([15]). Suppose that \( f : [a, b] \to \mathbb{R} \) is \( \Delta \)-differentiable on \([a, b]\), then \( f \) is non-decreasing (non-increasing) on \([a, b]\) if and only if
\[
 f^\Delta(t) \geq 0 \quad (f^\Delta(t) \leq 0), \quad t \in [a, b]^k.
\]

From [8], we have that the problem \(-x^\Delta(t) = 0, t \in [t_1, t_2]\), with (1.2) has the Green’s function
\[
 G(t, s) = \begin{cases} 
 \frac{1}{k}[a_1(t - t_1) + a_2][a_3(\sigma(t_2) - \sigma(s)) + a_4], & t \leq s \leq t_2, \\
 \frac{1}{k}[a_1(\sigma(s) - t_1) + a_2][a_3(\sigma(t_2) - t) + a_4], & t_1 \leq \sigma(s) \leq t,
\end{cases}
\]
for \( t \in [t_1, \sigma(t_2)] \) and \( s \in [t_1, t_2] \), where \( a_1, a_2, a_3, a_4 \geq 0 \) and \( k = a_2a_3 + a_1a_4 + a_1a_3(\sigma(t_2) - t_1) > 0 \).

Lemma 2.2 ([8]). For all \( s \in [t_1, t_2] \) and \( t \in (t_1, \sigma(t_2)) \),
\[
g(t)G(\sigma(s), s) \leq G(t, s) \leq G(\sigma(s), s),
\]
where
\[
g(t) := \min \left\{ \frac{a_1(t - t_1) + a_2}{a_1(\sigma(t_2) - t_1) + a_2}, \frac{a_3(\sigma(t_2) - t) + a_4}{a_3(\sigma(t_2) - \sigma(t_1)) + a_4} \right\}.
\]

The following definitions come from the cone theory on ordered Banach spaces.

Definition 2.4. Let \( E \) be a real Banach space over \( R \). A nonempty convex closed set \( P \subset E \) is said to be a cone provided that

(i) \( au \in P \) for all \( u \in P \) and all \( a \geq 0 \), and

(ii) \( u, -u \in P \) implies \( u = 0 \).

Note that every cone \( P \subset E \) induces an ordering in \( E \) given by \( x \leq y \) if \( y - x \in P \).

Definition 2.5. The map \( \psi \) is said to be a nonnegative continuous concave functional on \( P \) provided that \( \psi : P \to [0, \infty) \) is continuous and
\[
\psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y)
\]
for all \( x, y \in P \) and \( 0 \leq t \leq 1 \). Similarly, we say the map \( \alpha \) is a nonnegative continuous convex functional on \( P \) provided that \( \alpha : P \to [0, \infty) \) is continuous and
\[
\alpha(tx + (1 - t)y) \leq t\alpha(x) + (1 - t)\alpha(y)
\]
for all \( x, y \in P \) and \( 0 \leq t \leq 1 \).

Definition 2.6. Let \( r > a > 0, L > 0 \) be given and \( \psi \) be a nonnegative continuous concave functional and \( \alpha, \beta \) be nonnegative continuous convex functionals on the cone \( P \). Define convex sets
\[
P(\alpha, r; \beta, L) = \{ x \in P \mid \alpha(x) < r, \beta(x) < L \},
\]
\[
\overline{P}(\alpha, r; \beta, L) = \{ x \in P \mid \alpha(x) \leq r, \beta(x) \leq L \},
\]
\[
P(\alpha, r; \beta, L; \psi, a) = \{ x \in P \mid \alpha(x) < r, \beta(x) < L, \psi(x) > a \},
\]
\[
\overline{P}(\alpha, r; \beta, L; \psi, a) = \{ x \in P \mid \alpha(x) \leq r, \beta(x) \leq L, \psi(x) \geq a \}.
\]

Suppose the nonnegative continuous convex functionals \( \alpha, \beta \) on the cone \( P \) satisfy

(B1) there exists \( M > 0 \) such that \( \|x\| \leq M \max[\alpha(x), \beta(x)] \), for \( x \in P \);

(B2) \( P(\alpha, r; \beta, L) \neq \emptyset \), for any \( r > 0, L > 0 \).

Lemma 2.3 ([13]). Let \( E \) be a Banach space, \( P \subset E \) a cone and \( r_2 \geq d > b > r_1 > 0, L_2 \geq L_1 > 0 \) constants. Assume that \( \alpha, \beta \) are nonnegative continuous convex functionals such that (B1) and (B2) are satisfied,
\( \psi \) is a nonnegative continuous concave functional on \( P \) such that \( \psi(x) \leq \alpha(x) \) for all \( x \in \overline{P}(\alpha, r_2; \beta, L_2) \) and let \( T: \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2) \) be a completely continuous operator. Suppose

(C1) \( \{ x \in \overline{P}(\alpha; d, \beta; L_2; \psi, b) \mid \psi(x) > b \} \neq \emptyset, \psi(Tx) > b \) for \( x \in \overline{P}(\alpha; d, \beta; L_2; \psi, b) \),

(C2) \( \alpha(Tx) < r_1, \beta(Tx) < L_1 \) for all \( x \in \overline{P}(\alpha, r_1; \beta, L_1) \),

(C3) \( \psi(Tx) > b \) for all \( x \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \) with \( \alpha(Tx) > d \).

Then \( T \) has at least three fixed points \( x_1, x_2 \) and \( x_3 \) in \( \overline{P}(\alpha, r_2; \beta, L_2) \). Further,

\[
\begin{align*}
x_1 & \in P(\alpha, r_1; \beta, L_1), \quad x_2 \in \{ \overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \mid \psi(x) > b \}, \\
x_3 & \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).
\end{align*}
\]

3. Existence of triple positive solutions

Let the Banach space \( E = C_{rd}[t_1, \sigma(t_2)] = \{ x : [t_1, \sigma(t_2)] \to \mathbb{R} \mid x \text{ is continuous on } [t_1, \sigma(t_2)] \text{, and } x^\Delta \text{ is rd-continuous on } [t_1, t_2] \} \) be endowed with the norm

\[
\| x \| = \max \left\{ \max_{t \in [t_1, \sigma(t_2)]} |x(t)|, \sup_{t \in [t_1, t_2]} \left| x^\Delta(t) \right| \right\}.
\]

Define the cone \( P \subset E \) by

\[
P = \left\{ x \in E : x(t) \geq g(t) \max_{t \in [t_1, \sigma(t_2)]} |x(t)|, t \in [t_1, \sigma(t_2)] \right\},
\]

where \( g \geq 0 \) is given in (2.2).

For the rest of this section we suppose thoroughly that \( p : [t_1, \sigma(t_2)] \to [0, \infty) \) is a nonnegative, right-dense continuous function satisfying

\[
0 < \int_{t_1}^{t_2} G(\sigma(s), s)p(s)\Delta s < \infty.
\]

Let \( t_1 < \xi < \eta < t_2 \) be chosen from \( \mathbb{T} \) such that

\[
\min \left\{ \int_{\xi}^{\eta} G(\xi, s)p(s)\Delta s, \int_{\eta}^{\sigma(t_2)} G(\eta, s)p(s)\Delta s \right\} > 0,
\]

as \( p \) is a nonnegative function, this allows \( p \) to vanish on some subintervals of \([t_1, \sigma(t_2)]\).

Let the nonnegative continuous convex functionals \( \alpha, \beta \) and the nonnegative continuous concave functional \( \psi \) be defined on the cone \( P \) by

\[
\begin{align*}
\alpha(x) & = \max_{t \in [t_1, \sigma(t_2)]} |x(t)|, \quad \beta(x) = \sup_{t \in [t_1, t_2]} \left| x^\Delta(t) \right|, \quad \psi(x) = \min_{t \in [\xi, \sigma(t_2)]} |x(t)|.
\end{align*}
\]

Then \( \alpha, \beta, \psi : P \to [0, \infty) \) are three continuous nonnegative functionals such that \( \|x\| = \max\{\alpha(x), \beta(x)\} \), and (B1), (B2) hold; \( \alpha, \beta \) are convex, \( \psi \) is concave and there holds \( \psi(x) \leq \alpha(x) \), for all \( x \in P \).

Denote

\[
\begin{align*}
e & = \min_{t \in [\xi, \sigma(t_2)]} g(t), \\
M & = \max_{t \in [t_1, \sigma(t_2)]} \int_{t_1}^{t_2} G(t, s)p(s)\Delta s, \\
N & = \max \left\{ \int_{t_1}^{t_2} \left| G^\Delta(t, s) \right|_{t=t_1} p(s)\Delta s, \int_{t_1}^{t_2} \left| G^\Delta(t, s) \right|_{t=t_2} p(s)\Delta s \right\}, \\
C & = \min \left\{ \int_{\xi}^{\eta} G(\xi, s)p(s)\Delta s, \int_{\eta}^{\sigma(t_2)} G(\eta, s)p(s)\Delta s \right\}.
\end{align*}
\]
Theorem 3.1. Suppose there exist constants \( r_2 \geq b/e > b > r_1 > 0, L_2 \geq L_1 > 0 \) such that \( b/C \leq \min\{r_2/M, L_2/N\} \). If the following assumptions hold

(A1) \( f(t, u, v) < \min\{\frac{1}{M}, \frac{L_1}{N}\}, \) for \( (t, u, v) \in [t_1, \sigma(t_2)] \times [0, r_1] \times [-L_1, L_1]; \)

(A2) \( f(t, u, v) > \frac{b}{e}, \) for \( (t, u, v) \in [\xi, \sigma(\eta)] \times [b, b/e] \times [-L_2, L_2]; \)

(A3) \( f(t, u, v) \leq \min\{\frac{1}{M}, \frac{L_1}{N}\}, \) for \( (t, u, v) \in [t_1, \sigma(t_2)] \times [0, r_2] \times [-L_2, L_2], \)

then Problem (1.1) and (1.2) has at least three positive solutions \( x_1, x_2, \) and \( x_3 \) with

\[
\begin{align*}
\max_{t \in [t_1, t_2]} x_1(t) &< r_1, \\
\sup_{t \in [t_1, t_2]} |x_1^\Delta(t)| &< L_1; \\
\min_{\frac{b}{e} \leq t \leq \sigma(\eta)} x_2(t) &\leq \max_{t \in [t_1, \sigma(t_2)]} x_2(t) \leq r_2, \\
\sup_{t \in [t_1, t_2]} |x_2^\Delta(t)| &\leq L_2; \\
\min_{\frac{b}{e} \leq t \leq \sigma(\eta)} x_3(t) &\leq \frac{b}{e}, \\
\max_{t \in [t_1, \sigma(t_2)]} x_3(t) &\leq \sup_{t \in [t_1, t_2]} |x_3^\Delta(t)| \leq L_2.
\end{align*}
\]

Proof. It is clear that \( x(t) \) is a solution of problem (1.1) and (1.2), if and only if

\[
x(t) = \int_{t_1}^{t_2} G(t, s)p(s)f\left(s, x(\sigma(s)), x^\Delta(s)\right) \Delta s, \quad t \in [t_1, \sigma(t_2)].
\]

Now we define the operator \( T : P \rightarrow E \) by

\[
(Tx)(t) = \int_{t_1}^{t_2} G(t, s)p(s)f\left(s, x(\sigma(s)), x^\Delta(s)\right) \Delta s.
\]

We claim that the operator \( T \) maps \( P \) into itself. For this, let \( x \in P, t \in [t_1, \sigma(t_2)], \)

\[
(Tx)(t) = \int_{t_1}^{t_2} G(t, s)p(s)f\left(s, x(\sigma(s)), x^\Delta(s)\right) \Delta s \leq \int_{t_1}^{t_2} G(\sigma(s), s)p(s)f\left(s, \sigma(s), x^\Delta(s)\right) \Delta s.
\]

Then

\[
\max_{t \in [t_1, \sigma(t_2)]} |Tx(t)| \leq \int_{t_1}^{t_2} G(\sigma(s), s)p(s)f\left(s, x(\sigma(s)), x^\Delta(s)\right) \Delta s. \tag{3.1}
\]

Again from (2.1) and (3.1), we obtain

\[
(Tx)(t) = \int_{t_1}^{t_2} G(t, s)p(s)f\left(s, x(\sigma(s)), x^\Delta(s)\right) \Delta s \geq g(t)\int_{t_1}^{t_2} G(\sigma(s), s)p(s)f\left(s, x(\sigma(s)), x^\Delta(s)\right) \Delta s \geq g(t)\max_{t \in [t_1, \sigma(t_2)]} |Tx(t)|.
\]

Thus \( T : P \rightarrow P \). Moreover, \( T \) is completely continuous by a typical application of the Arzela–Ascoli Theorem.

We now show that all the conditions of Lemma 2.3 are fulfilled.

If \( x \in P(\alpha, r_2; \beta, L_2) \), then \( \alpha(x) \leq r_2, \beta(x) \leq L_2, \) and assumption (A3) implies \( f(t, x(\sigma(t)), x^\Delta(t)) \leq \min\{r_2/M, L_2/N\} \) for \( t \in [t_1, t_2]. \) Consequently,

\[
\alpha(Tx) = \max_{t \in [t_1, \sigma(t_2)]} \left| \int_{t_1}^{t_2} G(t, s)p(s)f(s, x(\sigma(s)), x^\Delta(s)) \Delta s \right| \leq \frac{r_2}{M} \max_{t \in [t_1, \sigma(t_2)]} \left| \int_{t_1}^{t_2} G(t, s)p(s) \Delta s \right| = r_2.
\]
On the other hand, for \( x \in P \), there is \( (Tx)^{A\Delta} (t) \leq 0 \) for \( t \in [t_1, t_2] \). With the use of Lemma 2.1, one has that \((Tx)^{A\Delta}\) is non-increasing on \([t_1, t_2]\). Thus
\[
\sup_{t \in [t_1, t_2]} |(Tx)^{A\Delta}(t)| = \max_{t \in [t_1, t_2]} |(Tx)^{A\Delta}(t)| = \max \left\{ \left| (Tx)^{A\Delta}(t_1) \right|, \left| (Tx)^{A\Delta}(t_2) \right| \right\},
\]
and
\[
\beta(Tx) = \sup_{t \in [t_1, t_2]} |(Tx)^{A\Delta}(t)| = \max \left\{ \int_{t_1}^{t_2} \left| G^{A\Delta}(t, s) \right| p(s) f \left( s, x(\sigma(s)), x^{A\Delta}(s) \right) ds, \right. \\
\left. \int_{t_1}^{t_2} \left| G^{A\Delta}(t, s) \right| p(s) \Delta s \right\} \\
\leq \frac{L_2}{N} \cdot \max \left\{ \int_{t_1}^{t_2} \left| G^{A\Delta}(t, s) \right| p(s) ds, \int_{t_1}^{t_2} \left| G^{A\Delta}(t, s) \right| p(s) \Delta s \right\} \\
= \frac{L_2}{N} \cdot N = L_2.
\]
Hence, \( T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2) \). In the same way, if \( x \in \overline{P}(\alpha, r_1; \beta, L_1) \), then assumption (A1) yields \( f(t, x(\sigma(t))), x^{A\Delta}(t) < \min \{ \eta(t), \beta \}, t_1 \leq t \leq t_2 \). As in the argument above, we can obtain that \( T : \overline{P}(\alpha, r_1; \beta, L_1) \to P(\alpha, r_1; \beta, L_1) \). Therefore, condition (C2) of Lemma 2.3 is satisfied.

To check condition (C1) of Lemma 2.3, we choose \( x(t) = b/e \), \( t_1 \leq t \leq \sigma(t_2) \). It is easy to see that \( x(t) = b/e \in \overline{P}(\alpha, b/e; \beta, L_2; \psi, b) \), \( \psi(x) = \psi(b/e) > b \), and consequently \( \{ x \in \overline{P}(\alpha, b/e; \beta, L_2; \psi, b) \mid \psi(x) > b \} \neq \emptyset \). Hence, if \( x \in \overline{P}(\alpha, b/e; \beta, L_2; \psi, b) \), then \( b \leq x(t) \leq b/e \) for \( \xi \leq t \leq \eta \). From assumption (A2), we have \( f(t, x(\sigma(t))), x^{A\Delta}(t) \geq b/B \) for \( \xi \leq t \leq \eta \). In order to estimate \( \psi(Tx) \), we have to distinguish two cases, (i) \( \psi(Tx) = (Tx)(\xi) \) and (ii) \( \psi(Tx) = (Tx)(\sigma(\eta)) \).

In case (i),
\[
\psi(Tx) = (Tx)(\xi) = \int_{\xi}^{t_2} G(\xi, s) p(s) f \left( s, x(\sigma(s)), x^{A\Delta}(s) \right) ds \\
> \int_{\xi}^{\eta} G(\xi, s) p(s) f \left( s, x(\sigma(s)), x^{A\Delta}(s) \right) ds \\
\geq \frac{b}{C} \int_{\xi}^{\eta} G(\xi, s) p(s) ds \geq b.
\]
In case (ii),
\[
\psi(Tx) = (Tx)(\sigma(\eta)) = \int_{t_1}^{\sigma(\eta)} G(\sigma(\eta), s) p(s) f \left( s, x(\sigma(s)), x^{A\Delta}(s) \right) ds \\
> \int_{\xi}^{\eta} G(\sigma(\eta), s) p(s) f \left( s, x(\sigma(s)), x^{A\Delta}(s) \right) ds \\
\geq \frac{b}{C} \int_{\xi}^{\eta} G(\sigma(\eta), s) p(s) ds \geq b
\]
\( \psi(Tx) > b \), for all \( x \in \overline{P}(\alpha, b/e; \beta, L_2; \psi, b) \).

This shows that Condition (C1) of Lemma 2.3 is satisfied.

We show finally that (C3) of Lemma 2.3 holds, too. Suppose that \( x \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \) with \( \alpha(Tx) > b/e \). Then, by the definition of \( \psi \) and \( Tx \in P \), we have
\[
\psi(Tx) = \min_{\xi \leq t \leq \sigma(\eta)} \left| (Tx)(t) \right|
\]
\[ \geq e \cdot \max_{t_1 \leq t \leq \sigma(t_2)} |(T x)(t)| \]
\[ = e \cdot \alpha(T x) \]
\[ > e \cdot b/e = b. \]

So, the condition (C3) of Lemma 2.3 is also satisfied. Therefore, Lemma 2.3 yields that Problem (1.1) and (1.2) has at least three positive solutions \( x_1, x_2, x_3 \) in \( \mathcal{P}(\alpha, r_2; \beta, L_2) \) with
\[ x_1 \in \mathcal{P}(\alpha, r_1; \beta, L_1), \quad x_2 \in (\mathcal{P}(\alpha, r_2; \beta, L_2; \psi, b) \mid \psi(x) > b) \]
and
\[ x_3 \in \mathcal{P}(\alpha, r_2; \beta, L_2) \setminus \left( \mathcal{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \mathcal{P}(\alpha, r_1; \beta, L_1) \right). \]

The fact that the functionals \( \alpha \) and \( \psi \) on \( P \) satisfy an additional relation
\[ e\alpha(x) \leq \psi(x), \quad \text{for } x \in P, \]
completes our proof. \( \square \)

An Example
Finally, we present an example to check our results. Let the measure chain \( \mathbb{T} = [0, 1] \cup \{2, 3, 4\} \) and \( t_1 = 0, t_2 = 3 \). Consider the boundary value problem
\[ x^\Delta(t) + f(t, x(\sigma(t)), x^\Delta(t)) = 0, \quad t \in [t_1, t_2], \]
\[ x(t_1) = x(\sigma(t_2)) = 0, \]
where
\[ f(t, u, v) = \begin{cases} \frac{1}{3} \left( \frac{1}{3} \frac{u}{2} \right) + \left( \frac{v}{100} \right)^3, & \text{for } u \leq 2, \\ \frac{1}{3} \left( \frac{1}{3} \frac{u}{6} \right) + \left( \frac{v}{100} \right)^3, & \text{for } u \geq 2. \end{cases} \]

Set \( \xi = 1/2, \eta = 2 \), we have
\[ e = \min_{t \in [\xi, \sigma(\eta)]} g(t) = \frac{1}{8}, \]
\[ M = \max_{t \in [t_1, t_2]} \int_{t_1}^{t_2} G(t, s) \Delta s = \frac{7}{4}, \]
\[ N = \max \left\{ \int_{t_1}^{t_2} \left| G^\Delta(t, s) \right|_{t=t_1} \Delta s, \int_{t_1}^{t_2} \left| G^\Delta(t, s) \right|_{t=t_2} \Delta s \right\} = \frac{13}{8}, \]
\[ C = \min \left\{ \int_{\xi}^{\eta} G(\xi, s) \Delta s, \int_{\xi}^{\eta} G(\sigma(\eta), s) \Delta s \right\} = \frac{29}{64}. \]

Choose \( r_1 = 1, b = 2, r_2 = 16, L_1 = 2, L_2 = 10 \), then \( \min\{r_1/M, L_1/N\} = 4/7, b/C = 128/29, \min\{r_2/M, L_2/N\} = 80/13 \). Consequently, \( f(t, u, v) \) satisfy
\[ f(t, u, v) < 4/7, \quad \text{for } t_1 \leq t \leq t_2, 0 \leq u \leq 1, -2 \leq v \leq 2; \]
\[ f(t, u, v) > 128/29, \quad \text{for } \xi \leq t \leq \sigma(\eta), 2 \leq u \leq 16, -10 \leq v \leq 10; \]
\[ f(t, u, v) < 80/13, \quad \text{for } t_1 \leq t \leq \sigma(t_2), 0 \leq u \leq 16, -10 \leq v \leq 10. \]

Then all the assumptions of Theorem 3.1 hold. Thus, with Theorem 3.1, Problem (3.2) and (3.3) has at least three positive solutions \( x_1, x_2, x_3 \) such that
\[ \max_{t_1 \leq t \leq \sigma(t_2)} x_1(t) \leq 1, \quad \sup_{t_1 \leq t \leq t_2} |x_1^\Delta(t)| \leq 2; \]
\[ 2 < \min_{\xi \leq t \leq \sigma(\eta)} x_2(t) \leq \max_{t_1 \leq t \leq t_2} x_2(t) \leq 16, \quad \sup_{t_1 \leq t \leq t_2} |x_2^\Delta(t)| \leq 10; \]
\[ \max_{t \in [t_1, \sigma(t_2)]} x_3(t) \leq 16, \quad \min_{\xi \leq t \leq \sigma(\eta)} x_3(t) \leq 2, \quad \sup_{t_1 \leq t \leq t_2} |x_3^\Delta(t)| \leq 10. \]

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References