Gradient estimates for degenerate quasi-linear parabolic equations

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Abstract

For a general class of divergence type quasi-linear degenerate parabolic equations with differentiable structure and lower order coefficients infinitesimally form bounded with respect to the Laplacian we obtain $L^q$-estimates for the gradients of solutions, and for the lower order coefficients from a Kato-type class we show that the solutions are Lipschitz continuous with respect to the space variable.

1 Introduction and main results

In this paper we study regularity of local weak solutions to general divergence type quasi-linear degenerate parabolic equations with measurable coefficients and lower order terms. This class of equations has numerous applications and has been attracting attention for several decades (see, e.g. the monographs \[5, 14, 25\], survey \[6\] and references therein).

Let $\Omega$ be a domain in $\mathbb{R}^N$, $T > 0$. Set $\Omega_T = \Omega \times (0, T)$. We study solutions to the equation

$$u_t - \text{div} \ A(x, t, u, \nabla u) = b(x, t, u, \nabla u), \quad (x, t) \in \Omega_T. \tag{1.1}$$

Throughout the paper we suppose that the functions $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are such that $A(\cdot, \cdot, u, z)$, $b(\cdot, \cdot, u, z)$ are Lebesgue measurable for all $u \in \mathbb{R}, z \in \mathbb{R}^N$, and $A(x, t, \cdot, \cdot)$, $b(x, t, \cdot, \cdot)$ are continuous for almost all $(x, t) \in \Omega_T$.

We also assume that the following structure conditions are satisfied:

$$A(x, t, u, z)z \geq c_1|z|^p, \quad z \in \mathbb{R}^n,$$

$$|A(x, t, u, z)| \leq c_2|z|^{p-1},$$

$$|b(x, t, u, z)| \leq g(x)|z|^{p-1} + f(x)(|u|^{p-1} + 1), \tag{1.2}$$

where $p \geq 2$, $c_1$ and $c_2$ are positive constants and $f$ and $g$ are nonnegative functions.

Let us remind the reader of the notion of a weak solution to equation (1.1). We say that $u$ is a weak solution to (1.1) if $u \in V(\Omega_T) := W^{1,p}_{\text{loc}}(\Omega_T) \cap C([0, T], L^2_{\text{loc}}(\Omega))$ and for any interval $[t_1, t_2] \subset (0, T)$ the integral identity

$$\int_{\Omega} u \psi dx|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \{ -u \partial_t \psi + A(x, t, u, \nabla u) \nabla \psi \} dx \ d\tau = \int_{t_1}^{t_2} \int_{\Omega} b(x, t, u, \nabla u) \psi dx \ d\tau \tag{1.3}$$

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for any \( \psi \in W^{1,p}_c(\Omega_T) \).

In [17] local boundedness of weak solutions to (1.1) was obtained under optimal conditions on \( f \) and \( g \) in terms of membership to the nonlinear Kato classes, which are defined below. The main thrust of the result in [17] is the presence of singular lower order coefficients in the structure conditions with optimal assumptions while not assuming anything in addition on the diffusion part.

In what follows we use the notion of the Wolff potential of a function \( f \), which is defined by

\[
W^f(x,R) := \frac{1}{R^{N-p}} \left( \frac{1}{r^{N-p}} \int_{B_r(x)} |f(y)| \, dy \right)^{\frac{1}{p-1}},
\]

where here and below \( B_r(x) = \{ z \in \Omega : |z - x| < r \} \). The non-linear (local) Kato \( K_p \) class, introduced in [1], is defined by

\[
(1.4) \quad K_p := \left\{ f \in L^1_{loc}(\Omega) : \lim_{R \to 0} \sup_{x \in \Omega^\prime} W^f(x,R) = 0 \text{ for all } \Omega^\prime \subset \Omega \right\}.
\]

As one can easily see, for \( p = 2 \), the class \( K_p \) reduces to the standard definition of the Kato class with respect to the Laplacian [21], which is extensively used in the qualitative linear theory of elliptic and parabolic second order PDEs. The nonlinear Kato class \( K_p \) turns out to be the optimal condition on the lower order coefficients also in case of nonlinear \( p \)-Laplacian type elliptic and parabolic PDEs (see [17] and the references therein). A typical example of a singular function in \( K_p \) is

\[
\beta > p - 1, \quad \text{where here and further on } 1_S \text{ stands for the characteristic function of the set } S.
\]

We further assume that

\[
(1.5) \quad f, g^\beta \in K_p,
\]

which guarantees that \( u \in L^{\infty}_{loc} \) by [17].

We are interested in the estimates of the gradients of solutions to (1.1) with differentiable structure in the diffusion part. The problem of higher regularity of solutions of quasi-linear equations (and systems) has a long history, which started from \( C^{1,\alpha}_{loc} \) results for homogeneous elliptic equations (we refer the reader to the well known monographs [11] [13] [14] [19] for the basic results, historical surveys and references). For a general structure divergence type quasi-linear elliptic equations, the Hölder continuity of the gradients of solutions were obtained by DiBenedetto [3] and Tolksdorf [24]. For the case of quasi-linear parabolic equations gradient estimates under different conditions were studied in [4] [15] [16], see also monographs [5] [25] for basic results and some historic comments. Very recently several interesting results on estimates of the gradients of solutions to quasi-linear elliptic and parabolic equations via nonlinear potentials were obtained in [7] [8] [9]. Most of the results in [7] [8] concern the elliptic equations of \( p \)-Laplacian type

\[
- \text{div } A(x,\nabla u) = \nu \quad \text{with a measure in the right hand side.}
\]

The authors give pointwise estimates of the gradients of solutions via a nonlinear Wolff potential of the measure \( \nu \), and as a consequence obtain a sufficient condition for the boundedness of the gradient. In [7] also parabolic equations were studied, and pointwise estimates of solutions and gradients were obtained, but only for the case \( p = 2 \). While the results in [7] [8] nicely cover the case of general measures on the right hand side, the situation becomes different when the measure \( \nu \) is absolutely continuous with respect to the Lebesgue measure with locally square integrable density, i.e. \( \nu = f dx \) with \( f \in L^2_{loc} \), and the condition on \( f \) in [7] turns out to be not optimal, which can be seen on explicit examples. We remark that while this paper was already in preparation, the authors were informed about the new preprint [10], where this situation was studied for the elliptic equations and systems, and with the vector field \( A \) depending on \( \nabla u \) only. The estimates obtained there are expressed in terms of a new potential which is closely related to the class \( \tilde{K}_2 \) introduced in [1] [10]. Below we make a further comparison of our results with [7] [10].

In this paper we study a general situation for equation (1.1), that is we allow for the vector field \( A \) in the diffusion part as well as for the right hand side \( b \) to depend on all the arguments. To study higher differentiability it is standard to assume that \( A \) is differentiable in \( x, u \) and \( z \) and that the following
ellipticity and growth conditions hold:

\begin{align}
(1.6) \quad \langle (\partial_z A) \mu, \mu \rangle & \geq c_0 |z|^{p-2} |\mu|^2, \quad \forall \mu, z \in \mathbb{R}^N, \\
(1.7) \quad |\partial_z A| & \leq c_1 (|z|^{p-2} + 1), \\
(1.8) \quad |\partial_x A| & \leq g_1(x) |z|^{p-2} + f_1(x), \\
(1.9) \quad |\partial_x A| & \leq g_2(x) |z|^{p-1} + f_2(x),
\end{align}

where \( f, f_1, f_2, g, g_1, g_2 \) are nonnegative functions. Without loss of generality, we do not assume dependence of \( u \) in the right hand side of (1.6) - (1.9) since \( u \) is bounded due to (1.3). In the sequel we refer to \( f, f_1, f_2, g, g_1, g_2 \) as to structure coefficients (cf., e.g. [24, Chap. VIII], see also Remark 1.6 below).

Our aim here is to reveal most general conditions on the structure coefficients guaranteeing higher integrability and boundedness of the gradients of solutions. To formulate our results, we need to introduce two classes (in fact, linear spaces).

The class \( K_2 \) of functions \( g \in L^1_{loc}(\Omega) \) is defined by the condition

\begin{equation}
\lim_{R \to 0} \sup_{r \in \Omega'} \int_0^R \left\{ \frac{1}{r^{N-2}} \int_{B_r(z)} |g(z)|dz \right\}^{\frac{1}{\alpha}} dr = 0, \quad \Omega' \subseteq \Omega.
\end{equation}

A typical example of a singular function in \( K_2 \) is

\[ \frac{1_{B_1(z)}(0)}{|x|^2 \left( \log \frac{1}{|x|} \right)^\alpha} \quad \text{with} \quad \alpha > 2. \]

We also need to introduce a class of infinitesimally form bounded with respect to the Laplacian, which we further denote by \( PK_0 \).

We say that \( F \) is infinitesimally form bounded with respect to the Laplacian and write \( F \in PK_0 \) if \( F \in L^1_{loc}(\Omega) \) and for any \( \beta > 0 \) there exists \( C(\beta) \geq 0 \) such that for all \( \theta \in C_0^\infty(\Omega) \)

\[ \left| \int_\Omega F \theta^2 dx \right| \leq \beta \int_\Omega |\nabla \theta|^2 dx + C(\beta) \int_\Omega \theta^2 dx. \]

This class became indispensable in many problems in PDE theory. Its complete characterization can be found in [20]. For comparison with the above classes, an example of a singular function in \( PK_0 \) is

\[ \frac{1_{B_1(z)}(0)}{|x|^2 \left( \log \frac{1}{|x|} \right)^\alpha} \quad \text{with} \quad \alpha > 0. \]

While our main object in this paper is the general equation, it seems worth giving an example of a simpler equation which would illustrate the results below, and which seems to be of independent interest. Let us consider the nonhomogeneous evolution \( p \)-Laplace equation \( u_t - \Delta_p u = f \). It follows from our results below that if \( f^2 \in PK_0 \) then the gradient of any weak solution \( u \) is in \( L^q_{loc} \) for any \( q < \infty \), while if \( f^2 \in K_2 \) then \( \nabla u \in L^q_{loc} \). So, for \( f(x) = \frac{1}{|x|^2 (\log \frac{1}{|x|})^\alpha} 1_{B_1(z)}(0) \) with \( \alpha > 0 \) we have that \( \nabla u \in L^q_{loc} \) for any \( q < \infty \), and with \( \alpha > 1 \), the conclusion is that \( \nabla u \in L^\infty_{loc} \), and hence every solution is locally Lipschitz continuous with respect to the spatial variables.

Our strategy is the following. We first show that under some general assumptions on the structure coefficients there exists a local weak solution to (1.1) whose space gradients are in \( L^q_{loc} \) for an arbitrary large \( q \). This constitutes an existence result. The required a-priori estimates are obtained by a finite number of iterations of Moser type. The main assumption here is that all squares of structure coefficients are infinitesimally form bounded with respect to the Laplacian. Next, under some mild additional assumption on \( f_1 \) and \( g_1 \), we prove that every weak solution to (1.1) has the same smoothness. In the proof of this result we follow the idea of Tolksdorf [24], comparing the solution \( u \) to (1.1) on a small cylinder, with a smooth solution to an auxiliary initial boundary value problem in \( Q \) with \( u \) as initial boundary value data and the equation satisfying the same structure condition as (1.1). A significant difference between our situation and that in [24] is that we do not rely on apriori H"older continuity (or even continuity) of the weak solution to (1.1) but rather on the property of \( K_p \) class (see Lemma 1.9). The next step is to obtain the supremum estimates of the gradient. This requires stronger assumptions
on the structure coefficients. The technique we use to achieve the result is a parabolic version of the Kilpeläinen–Malý technique [12], [19] (see [17, 23]).

Our first result concerns the existence of weak solutions to (1.1) with integrable powers of the gradient.

**Theorem 1.1.** Let \( Q \) denote a cylinder \( B \times (t_1, t_2) \) such that \( Q \subset \Omega_T \). Let \( v \in V(\Omega_T) \), and let \( A \) and \( b \) satisfy the structure conditions (1.2)–(1.9) with the functions \( f^2, f^1_1, f^2_1, g^2_2 \in PK_0 \) and \( f, g^2 \in KP_1 \). Then there exists a solution \( u \) to (1.1) in \( Q \) satisfying \( u = v \) on the parabolic boundary \( \partial Q \) of \( Q \), such that, for every \( l > 0 \) and \( q \geq p \) and every cylinder \( Q' = B' \times (t_1', t_2') \subset Q \) there exists a constant \( c_{lq} = c_{lq}(Q') \) such that

\[
\text{ess sup}_{t \in (t_1', t_2')} \int_{B'} |\nabla u|^{q - p + 2} \, dx + \int_Q \left| D\left(\nabla u - l\right)^{\frac{q}{2} - 1}\right|^2 \, dx \, dt \leq c_{lq}.
\]

In particular,

\[
\nabla u \in L^q_{\text{loc}}(Q) \quad \text{for any } q \in [p, \infty).
\]

The next theorem establishes the same smoothness as above, for all solutions to (1.1).

**Theorem 1.2.** Let \( A \) and \( b \) satisfy the structure conditions (1.2)–(1.9) with \( f^2, f^1_1, f^2_2, g^2_2 \in PK_0 \) and \( f, g^2 \in KP_1 \). Let \( u \) be a weak solution to (1.1) in \( \Omega_T \). Then

\[
\nabla u \in L^\infty_{\text{loc}}(0, T); L^q_{\text{loc}}(\Omega) \quad \text{and} \quad \nabla u(\nabla u - l)^{\frac{q}{2} - 1} \in L^2_{\text{loc}}(0, T); W^{1,2}_{\text{loc}}(\Omega)
\]

for all \( q \geq p \) and \( l > 0 \).

Finally, we give sufficient conditions for the boundedness of the gradient of solutions.

**Theorem 1.3.** Let \( A \) and \( b \) satisfy the structure conditions (1.2)–(1.9). Assume that \( g^p + g^q \in K_p \) and that \( f^2 + g^2 + f^1_1 + g^1_2 + f^2_2 + g^2_2 \in \tilde{K}_2 \), that is, for any \( \Omega' \subset \Omega \),

\[
\lim_{R \to \infty} \sup_{x \in \Omega'} \int_0^R \frac{dr}{r} \left( \int_{B_r(x)} \left( f(y)^2 + g(y)^2 + f_1(y)^2 + g_1(y)^2 + f_2(y)^2 + g_2(y)^2 \right) dy \right)^{\frac{1}{2}} = 0.
\]

Let \( u \) be a weak solution to (1.1). Then

\[
\nabla u \in L^\infty_{\text{loc}}(\Omega_T),
\]

i.e. all solutions to (1.1) are Lipschitz continuous with respect to the spatial variables.

**Remark 1.4.** We note that the results in [10] concern the case of elliptic equations (and systems) with restriction \( f_1 = g_1 = f_2 = g_2 = 0 \), and for the general right hand side \( b(x, u, \nabla u) \) only the existence of Lipschitz solutions is proved, while the assertion that all the solutions are Lipschitz is proved only for the particular case \( b(x, u, \nabla u) = b(x) \). On the other hand, in this case the result in [10] is stronger than our Theorem 1.3 as the authors obtain supremum estimates of the gradient of the solution via the supremum of the potential \( P^f(x, R) \). Let \( u \in L^{N,1}(\Omega_T) \) such that \( f \in L^{N,1} \Rightarrow f^2 \in \tilde{K}_2 \). We do not dwell upon this further, and refer the reader to [10] for an extensive discussion of this point.

**Remark 1.5.** As a consequence of Theorem 1.3 one can give sufficient conditions of the local boundedness of the gradient of solutions to (1.1) in terms of the structure coefficients belonging to the Lorentz spaces. This is based on an easily verifiable fact that \( f \in L^{N,1} \Rightarrow f^2 \in \tilde{K}_2 \). We do not dwell upon this further, and refer the reader to [10] for an extensive discussion of this point.

**Remark 1.6.** In all the above results structure condition (1.7) can be replaced by a more general one \( |\partial_x A| \leq c_1|x|^{p-2} + h(x) \) with the requirement \( h^2 \in PK_0 \) for Theorems 1.1, 1.2, and \( h^2 \in \tilde{K}_2 \) for Theorem 1.3 (compare this with (S_2) in [5, Chap. VIII]). We did not elaborate this further.
we have
\[ x \]
By [12, Theorem 4.8] (see also [19, Theorem 2.125]), for
\[ (1.15) \]
Testing (1.14) by \( \theta \)
\begin{align*}
\text{Lemma 1.8.} \\
\text{The following lemma provides an inequality of Hardy-type which is useful in the sequel.}
\end{align*}

1.1 Auxiliary facts

The following lemma provides an inequality of Hardy-type which is useful in the sequel.

**Lemma 1.9.** Let \( h \in W^{1,p}_0(\Omega), h > 0 \). Suppose that \( \Delta_p h \in L^1_{\text{loc}}(\Omega) \) and \( -\Delta_p h > 0 \). Then for any \( \theta \in W^{1,p}(\Omega) \)
\[ \int_\Omega (\frac{-\Delta_p h}{h^{p-1}})|\theta|^p dx \leq \int_\Omega |\nabla \theta|^p dx. \]
If in addition \( h \in L^\infty(\Omega) \) then
\[ (1.13) \]
\[ \int_\Omega (\frac{-\Delta_p h}{h^{p-1}})|\theta|^p dx \leq \|h\|_{\infty}^{p-1} \int_\Omega |\nabla \theta|^p dx. \]

**Proof.** First, by the Young inequality note that \( pa^{p-1}b - (p-1)a^p \leq b^p \) for any \( a, b > 0 \) and \( p > 1 \). Let \( \varepsilon > 0 \) and \( 0 < \theta \in C^\infty_c(\Omega) \). Then it follows that
\[ \nabla \left( \frac{\theta^p}{(h+\varepsilon)^{p-1}} \right) |\nabla h|^{p-2} \nabla h \leq p^{p-1} |\nabla h|^{p-1} |\nabla \theta| - (p-1) \left( \frac{\theta^p}{(h+\varepsilon)^{p-1}} \right)^{p-1} \leq |\nabla \theta|^p. \]

Integrating the above we obtain the required inequality for \( \theta \in C^\infty_c(\Omega) \). The assertion follows by approximation. \( \square \)

**Lemma 1.9.** Let \( f \geq 0, f \in K_p \) and \( u \) be the weak solution to
\[ (1.14) \]
\[ -\Delta_p u = f \quad \text{in } B_R, \quad u|_{\partial B_R} = 0. \]
Then there exists \( c > 0 \) such that
\[ \sup_{B_R} u(x) \leq c \sup_{B_R} W_p^f(x, 2R). \]

**Proof.** Testing (1.14) by \( u \) we obtain
\[ (1.15) \]
\[ \int_{B_R} |\nabla u|^p dy \leq \sup_{B_R} u(x) \int_{B_R} f(y) dy. \]
By [12, Theorem 4.8] (see also [19, Theorem 2.125]), for \( x_0 \in B_R \),
\[ (1.16) \]
\[ u(x_0) \leq \epsilon \left( \frac{1}{R^p} \int_{B_R(x_0) \cap B_R} u(y)^p dy \right)^{\frac{1}{p}} + c W_p^f(x_0, 2R). \]

Using the Poincaré inequality, (1.15), the Young inequality and the definition of the Wolff potential we have
\[ \left( \frac{1}{R^p} \int_{B_R} u(y)^p dy \right)^{\frac{1}{p}} \leq \left( \frac{1}{R^{n-p}} \int_{B_R} |\nabla u(y)|^p dy \right)^{\frac{1}{p}} \]
\[ \leq \left( \sup_{B_R} u(x) \right)^{\frac{1}{p}} \left( \frac{1}{R^{n-p}} \int_{B_R} f(y) dy \right)^{\frac{1}{p}} \leq \frac{1}{2} \sup_{B_R} u(x) + 4 \left( \frac{1}{R^{n-p}} \int_{B_R(x_0)} f(y) dy \right)^{\frac{1}{p-1}} \]
\[ (1.17) \]
\[ \leq \frac{1}{2} \sup_{B_R} u(x) + W_p^f(x_0, 2R), \quad x_0 \in B_R. \]
Combining (1.16) and (1.17) and taking supremum over \( B_R \) we prove the assertion. \( \square \)
As a consequence of Lemma \ref{lem:1.8} and Lemma \ref{lem:1.9} we obtain

**Corollary 1.10.** Let $\theta \in \dot{W}^{1,p}(B_R)$, $0 \leq f \in K_p$. Then there exists $\gamma > 0$ such that

\begin{equation}
\int_{B_R} f \theta^p \, dx \leq \gamma \sup_{B_{2R}} W_{\beta}^f (x, 2R)^{p-1} \int_{B_R} |\nabla \theta|^p \, dx.
\end{equation}

The following proposition shows some useful relations between the classes involved. Let us consider the following two parameter scale of Wolff potentials and corresponding Kato classes.

$$W_{\beta,p}^f(x,R) := \int_0^R \frac{dr}{r^{\beta}} \left( \frac{1}{r^{N-\beta p}} \int_{B_r(x)} |f(y)| \, dy \right)^{\frac{1}{p-1}}.$$ 

The correspondent (local) Kato class $K_{\beta,p}$ is defined by

\begin{equation}
K_{\beta,p} := \left\{ f \in L^1_{loc}(\Omega) : \lim_{R \to 0} \sup_{x \in \Omega} W_{\beta,p}^f (x,R) = 0 \text{ for all } \Omega' \subseteq \Omega \right\}.
\end{equation}

Note that $K_p = K_{1,p}$ and $\bar{K}_2 = K_{2,3}$ in this notation.

**Proposition 1.11.** The following assertions hold.

(i) Let $p, q > 1$, $\alpha, \beta > 0$ and $\varkappa > 1$. Suppose that $\varkappa > \frac{\beta p}{\alpha q}$ or $\varkappa = \frac{\beta p}{\alpha q} < \frac{p-1}{q-1}$.

If $f \in K_{\beta,p}$ then $|f|^{\frac{1}{\varkappa}} \in K_{\alpha,q}$.

(ii) $\bar{K}_2 \subset K_2 \subset PK_0$.

(iii) For $p \geq 2$, if $f^2 \in \bar{K}_2$ then $|f|^q \in K_p$ for $q \in [1,2)$.

(iv) For $1 < q < p$, $f \geq 0$, if $f^p \in K_p$ then $f^q \in K_q$.

In particular, if $p \geq 2$ and $f^p \in K_p$ then $f^2 \in PK_0$.

**Proof.** It suffices to prove (i) for $f \geq 0$. First observe that there are constants $C \geq c > 0$ dependent on $\beta, p$ and $N$ only such that, with $r_k = 2^{-k} R$, $k = 0, 1, 2, \ldots$,

$$c \sum_{k=1}^{\infty} \left( \frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) \, dy \right)^{\frac{1}{p-1}} \leq W_{\beta,p}^f (x,R) \leq C \sum_{k=0}^{\infty} \left( \frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) \, dy \right)^{\frac{1}{p-1}}.$$

Next, by the Hölder inequality,

$$\left( \frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) \, dy \right)^{\frac{1}{p-1}} \leq c \left( \frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) \, dy \right)^{\frac{1}{p-1}} \frac{r_k^{\frac{\beta p}{\alpha q}}} {r_k^{\frac{\beta p}{\alpha q} + \frac{\beta p}{\alpha q} - \frac{\beta p}{\alpha q}}}.$$

If $\varkappa(q - 1) > p - 1$ and $\varkappa > \frac{\beta p}{\alpha q}$ then, by the Hölder inequality,

$$W_{\beta,p}^f (x,R) \leq c R^{\frac{\alpha q - \beta p}{\alpha q - p}} \left( W_{\beta,p}^f \right)^{\frac{p-1}{p-1}} (x,R).$$

If $\varkappa(q - 1) < p - 1$ and $\varkappa > \frac{\beta p}{\alpha q}$ then

$$W_{\beta,p}^f (x,R) \leq c R^{\frac{\alpha q - \beta p}{\alpha q - p}} \sum_{k=0}^{\infty} \left( \frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) \, dy \right)^{\frac{1}{p-1}} \sup_k \left( \frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) \, dy \right)^{\frac{p-1}{p-1}} \left( W_{\beta,p}^f \right)^{\frac{p-1}{p-1}} (x, 2R).$$

$\square$
1.2 Approximation and a-priori estimates

In this subsection we construct an appropriate local approximation of equation (1.1) and obtain a-priori estimates.

**Approximation.** For \( \varepsilon > 0 \) let \( j_\varepsilon \) be standard mollifier in \( \mathbb{R}^N \). Denote \( A_\varepsilon = A * j_\varepsilon + \varepsilon I \), smoothing with respect to \( x \) variable only. For \( b \) we introduce \( b_\varepsilon = b \wedge \frac{1}{\varepsilon} \vee (-\frac{1}{\varepsilon}) \). Also set \( f_\varepsilon = (f * j_\varepsilon) \vee f \), \( f_{1,\varepsilon} = (f_1 * j_\varepsilon) \vee f \). Observing that

\[
\begin{align*}
\text{Lemma 1.13.} & \quad \text{Let } (1.20) \quad u_\varepsilon = \left( \begin{array}{c}
\varepsilon^p \\
\varepsilon^q
\end{array} \right) \\
& \quad \text{subject to } u_\varepsilon |_{\partial P} = v, \quad \text{where } \partial P \text{ is the parabolic boundary of } P. \quad \text{For any } \varepsilon > 0 \text{ and any sub-cylinder } \Omega = (t_1', t_2') \times B_{r'}(y) \subset \mathbb{R}^N \text{ there exists } C_\Omega(\varepsilon) > 0 \text{ such that}
\end{align*}
\]

\[
\begin{align*}
\text{Proof.} \quad \text{The existence part follows from } [18]. \quad \text{The estimate is obtained by means of the direct approach via finite differences (see, e.g. [5] Section VIII.3 and [13] Section IV.5}).
\end{align*}
\]

In the rest of this subsection we study solutions to (1.20) obtained in Proposition 1.12. Our task in the sequel is to obtain estimates which are uniform in \( \varepsilon \) and which will allow us to pass to the limit \( \varepsilon \to 0 \).

In order to simplify the notation we drop subindex \( \varepsilon \) in the rest of this subsection.

**Lemma 1.13.** Let \( u \) be a weak solution to (1.20) satisfying the assumptions of Proposition 1.12. Then for every \( \zeta \in H^1(\Omega) \) one has

\[
\begin{align*}
& -\int_Q \langle \nabla u, \partial_\zeta \rangle dxdt + \int_Q \text{tr}\{(D\zeta)(\partial_\zeta A)D^2u\}dxdt \\
& = -\int_Q \left[ \text{tr}\{(D\zeta)(\partial_\zeta A)\} + \langle (D\zeta)(\partial_\zeta A), \nabla u \rangle + b \text{div}\zeta \right]
\end{align*}
\]

(1.21)

\[
\begin{align*}
\text{Proof.} \quad \text{First, let } \zeta \in C^2(\Omega) \text{. Test equation (1.1) by } -\zeta. \quad \text{Integrating by parts we obtain}
\end{align*}
\]

\[
\begin{align*}
-\int_Q \langle \nabla u, \partial_\zeta \rangle dxdt + \int_Q \text{tr}\{(D\zeta)(\partial_\zeta A)\} = -\int_Q b \text{ div } \zeta dxdt.
\end{align*}
\]

Observing that

\[
DA = (\partial_\zeta A)D^2u + \partial_\zeta A \otimes \nabla u + \partial_\zeta A
\]

we arrive at (1.21). The general case follows by approximation.

**Remark 1.14.** Note that (1.6) implies that, for \( M \in \mathbb{R}^{N \times N} \), one has

\[
\text{tr}\{M^T(\partial_\zeta A)M\} \geq c_0|\zeta|^{p-2}|M|^2_{HS}.
\]

Indeed, let \( M = \{m_{kl}\} \). Then

\[
\text{tr}\{M^T(\partial_\zeta A)M\} = \sum_{jkl} m_{kj}(\partial_\zeta A_k)m_{lj} = \sum_j \sum_{kl} (\partial_\zeta A_k)m_{lj}m_{kj} \geq \sum_j c_0|\zeta|^{p-2} \sum_k m_{kj}^2 = c_0|\zeta|^{p-2}|M|^2_{HS}.
\]
Lemma 1.15. Let $u$, and $Q$ be as in Lemma 1.14 $Q' \subseteq Q$. Let $\xi \in C^\infty(Q')$ vanishing on its parabolic boundary $\partial Q'$. Let $\Phi \in C^0_b(\mathbb{R})$, $\Phi(0) = 0$ and $G(s) := \int_0^s \tau \Phi'(\tau) d\tau$. Let $\zeta := \nabla u \Phi(|\nabla u|) \xi$

Then, for almost all (a.a.) $t \in (t_1, t_2)$ one has

\[
\frac{1}{2} \int_{B'} G(|\nabla u|) \xi^2(t) dx + \int_{t_1}^t \int_{B'} \text{tr} \{(D\zeta)(\partial_u A)D^2u\} dx \, d\tau \leq \frac{1}{2} \int_{t_1}^t \int_{B'} G(|\nabla u|) \xi^{-1} \partial_t \xi dx \, d\tau
\]

\[
- \int_Q \left[ \text{tr} \{(D\zeta)(\partial_u A)\} + \langle (D\zeta)(\partial_u A), \nabla u \rangle + b \text{div}\xi |dx \, d\tau \right]
\]

Proof. For $h > 0$ let $T_h$ be the Steklov averaging defined by

\[
T_h g(x,t) = \frac{1}{2h} \int_{-h}^h g(x,t + \tau) d\tau.
\]

We write $u_h = T_h u$. Let $\eta \in C^\infty(\mathbb{R})$, $1_{[\tau - 1]} < \eta < 1_{[\tau - 0]}$. For $t \in (0,t)$ and $\varepsilon > 0$, let $\eta_{\varepsilon,t}(s) := \eta(t + \varepsilon - s)$ so that $1_{[\tau - 1]} < \eta_{\varepsilon,t} < 1_{[\tau - 0]}$. With notation above set $\zeta_{\varepsilon,t} = T_h \{ \nabla u_h \Phi(|\nabla u_h|) \xi \eta_{\varepsilon,t} \}$. We apply $\zeta_{\varepsilon,t}$ as the test vector function in (1.21) and then pass to limit first as $\varepsilon \to 0$ and then as $h \to 0$. The only non-trivial term is the one containing $\partial_t \zeta_{\varepsilon,t}$.

\[
- \int \langle \nabla u, \partial_t \zeta_{\varepsilon,t} \rangle = \int \langle \partial_t \nabla u_h, \nabla u_h \rangle \Phi(|\nabla u_h|) \xi \eta_{\varepsilon,t} = \frac{1}{2} \int \langle \partial_t |\nabla u_h|^2 \rangle \Phi(|\nabla u_h|) \xi \eta_{\varepsilon,t}
\]

\[
= \frac{1}{2} \int \langle \partial_t |\nabla u_h|^2 \rangle \xi \eta_{\varepsilon,t} = - \frac{1}{2} \int G(|\nabla u_h|) \xi^{-1} \partial_t \eta_{\varepsilon,t} + \frac{1}{2} \int G(|\nabla u_h|) \xi^{\varepsilon-1} \partial_t \eta_{\varepsilon,t}.
\]

Note that $\eta_{\varepsilon,t} \to 1_{[\tau - 1]}$ as $\varepsilon \to 0$ and that

\[
(\eta_{\varepsilon,t})_{t} d\tau \to -\delta(t) ds \quad \text{as} \quad \varepsilon \to 0
\]

in the sense of weak convergence of measures. Since $\sigma_\varepsilon$ is continuous w.r.t. $t$ we conclude that

\[
\lim_{\varepsilon \to 0} \int \langle \nabla u, \partial_t \zeta_{\varepsilon,t} \rangle = - \frac{1}{2} \int G(|\nabla u_h|) \xi^{-1} \partial_t \xi + \frac{1}{2} \int G(|\nabla u_h|) \xi(t).
\]

Since $\nabla u_h \to \nabla u$ a.e. as $h \to 0$, we obtain

\[
\lim_{h \to 0} \int G(|\nabla u_h|) \xi^{-1} \partial_t \xi \to \int G(|\nabla u|) \xi^{-1} \partial_t \xi, \quad \liminf_{h \to 0} \int G(|\nabla u_h|) \xi(t) \geq \int G(|\nabla u|) \xi(t),
\]

and the assertion follows.

\[
\Box
\]

Integrability of the gradient. Now we pass to a-priori estimates for solutions to (1.21). We need the following lemma.

Lemma 1.16. Let $Q' \subseteq Q'' \subseteq Q$ and let $\xi$ the standard cut-off function vanishing on the parabolic boundary of $Q''$, which is equal to 1 on $Q'$. For $l > 0$ and $\alpha \geq 0$, let

\[
\Phi_\alpha(s) = (s-l)^{2+2\alpha} s^{2}, \quad \alpha \geq 0, \quad G_\alpha(s) = \int_0^s \tau \Phi_\alpha(\tau) d\tau.
\]

Let $f, g \in K_p$. Assume that $f, f_1 \in L^2_{\text{loc}}$, $g_1^2, f_2^2 \in PK_0$. Then for $m \geq 2$, $\alpha = 0$,

\[
\text{ess sup} \int_{B'} G_\alpha(|\nabla u|) \xi^m dx + \int_{Q''} |\nabla u|^p - 2 |D^2 u|_{HS} \Phi_\alpha(|\nabla u|) \xi^m dx + \int_{Q''} |\nabla u|^{p-2} |\nabla u|^2 \Phi_\alpha(|\nabla u|) \xi^m dx + \int_{Q''} |\nabla u|^{p-2} |\nabla u|^2 \Phi_\alpha(|\nabla u|) \xi^m dx
\]

\[
\leq \gamma \int_{Q''} (f^2 + g^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2 + 1) dx d\tau + \gamma \int_{Q''} |\nabla u|^{p} \Phi_\alpha(|\nabla u|) \xi^{m-2} dx d\tau.
\]

Moreover, if $f_2, f_2^2 \in PK_0$ then (1.22) holds for all $\alpha > 0$ provided the second integral in the right hand side is finite. (Here $|D^2 u|_{HS}$ stands for the Hilbert-Schmidt norm of the Hessian.)
Proof. We give a sketch proof for the case \( l = 1 \). Note that \( g^2 \in PK_0 \) by Proposition 1.11. Also observe that \( \text{sup } W_p^f(R) \leq \text{sup } W_p^f(R) \) and \( \text{sup } W_p^{2g}(R) \leq \text{sup } W_p^{2g}(R) \). Hence, by \[17\] Theorem 1.1, the assumption \( f, g^p \in K_p \) implies that \( u \) is bounded on \( Q'' \) uniformly in \( \varepsilon \). So on \( Q'' \) one has \( |b| \leq g_0 |\nabla u|^{p-1} + \gamma f_\varepsilon \). By Proposition 1.11 we have that \( g^2 \in PK_0 \).

Note that \( \Phi(s) \leq s^{2\alpha} \) and \( \mathcal{G}(s) \leq s^{2+2\alpha} \) for large \( s \). By Lemma 1.13 and Lemma 1.15 with \( \zeta = \nabla u \Phi(|\nabla u|) \) as a test function we have
\[
\sup_t \int_{B''} \mathcal{G}(|\nabla u|)\xi^3 + \int_{Q''} \text{tr} \{(D\zeta)(\partial_2 A)D^2 u\} \leq \lambda \int_{Q''} \mathcal{G}(|\nabla u|)\xi^{q-1}\partial_t \xi + \int_{Q''} (|\partial_2 A| + |\partial_u A| |\nabla u| + |b| )|D\zeta|.
\]

The left hand side is estimated by \[1.9\] using Remark 1.14. The right hand side is estimated by the Schwartz inequality using the \( PK_0 \) condition. We omit the details.

The next proposition is an a-priori estimate for the approximate solutions (solutions to \[1.20\]) which are independent of \( \varepsilon \).

Proposition 1.17. Let \( Q, Q', v \) and \( u_{x_\varepsilon} \) be as in Proposition 1.13. Let the structure conditions \[1.2 - 1.9\] hold with \( f, f_1 \in L^1_{\text{loc}}, f \in K_p, f_2^0, g_1^0, g_2^0 \in PK_0 \) and \( g^p \in K_p \). Then, for every \( l > 0 \) there exists \( c_1 \) depending on \( Q', Q, l \) and independent of \( \varepsilon \), such that
\[
\text{ess sup}_{t \in [0, l]} \int_{B''} |\nabla u_{x_\varepsilon}|^2 \, dx + \int_{Q'} \left| D \left( \nabla \left( |\nabla u_{x_\varepsilon}| - b \right)^{\frac{q}{q-1}} \right) \right|^2 \, dx \leq c_1.
\]

If in addition \( f^2, f_1^2 \in PK_0 \), then, for every \( l > 0 \) and \( q > p \), there exists a constant \( c_0 \) depending on \( Q', Q, l \) and \( q \) such that
\[
\text{ess sup}_{t \in [0, l]} \int_{B''} |\nabla u_{x_\varepsilon}|^{q-p+2} \, dx + \int_{Q'} \left| D \left( \nabla \left( |\nabla u_{x_\varepsilon}| - b \right)^{\frac{q}{q-1}} \right) \right|^2 \, dx \leq c_0.
\]

Proof. The first assertion is an immediate corollary of Lemma 1.16 with \( \alpha = 0 \). The proof of the second assertion follows the line of the argument from \[5\] Lemma 4.1.

We will iterate with respect to \( \alpha \) as is done in \[5\] p.232–233 (with \( \beta \) in place of our \( 2\alpha \)). But first proceed formally. Let \( \Psi(s) = \int_s^1 r^{p/2-1} \sqrt{\Phi(r)} \, dr \). Note that \( \Psi(s) \leq s^{p/2+\alpha} \) for \( s \gg 1 \).
\[
\int_{Q'} |\nabla u_{x_\varepsilon}|^{p+\frac{\alpha}{1+\alpha}} \, dx \leq (2)^{p+\frac{\alpha}{1+\alpha}} |Q'| + \gamma \int_{Q'' \cap \{|\nabla u_{x_\varepsilon}| \leq 1\}} \Psi^2 G^{2/N} \, dx \, dx \leq \gamma |Q'| + \gamma \int_{Q'' \cap \{|\nabla u_{x_\varepsilon}| > 1\}} \Psi^2 G^{2/N} \, dx \, dx.
\]

By Lemma 1.16 we estimate the right hand side of the above inequality, which gives, with some \( Q'' \) such that \( Q' \subset Q'' \subset Q \),
\[
\int_{Q'} |\nabla u_{x_\varepsilon}|^{p+\frac{\alpha}{1+\alpha}} \, dx \leq \gamma \left( \int_{Q''} \left( 1 + f^2 + f_1^2 + g^2 + g_1^2 + f_2^2 + g_2^2 + |\nabla u_{x_\varepsilon}|^{p+2\alpha} \right) dx \right)^{2/N}.
\]

Now starting from \( \alpha = 0 \) and proceeding to \( \alpha = 2/N \) etc. we complete the proof. Of course, in the above argument one has to properly insert the power of the cut-off function, but we leave out these simple details.

\[\Box\]

2 Proof of Theorem 1.1

In this section we prove the existence theorem. The following lemma serves to assert the pointwise convergence of the gradient.
Lemma 2.1. Let \( \xi_n \) be a sequence of a.e. finite vector fields such that there exists \( q > 0 \) such that \( \xi_n(\frac{1}{m} - \frac{1}{m^q}) \) converges a.e. as \( n \to \infty \) for all \( m \in \mathbb{N} \). Then \( \xi_n \) converges a.e. as \( n \to \infty \).

Proof. Denote \( \eta_n := \xi_n(\frac{1}{m} - \frac{1}{m^q}) \) and \( E_{nm} := \{ x : |\xi_n| \geq \frac{1}{m} \} \).

Note that the function \( \phi_m(s) := s(s - \frac{1}{m})^q \) is a homeomorphism \( [\frac{1}{m}, \infty) \to [0, \infty) \). Let \( \psi_m \) denote the inverse map. Then one has

\[
|\xi_n\chi_{E_{nm}}| = \frac{\psi_m(|\eta_n|)}{|\eta_n|}.
\]

So there are vector fields \( \zeta_m, m \in \mathbb{N} \) such that \( \xi_n \chi_{E_{nm}} \to \zeta_m \) a.e. as \( n \to \infty \).

Let

\[
E_m = \liminf_{n \to \infty} E_{nm} = \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} E_{nm} = \{ x : \liminf_{n \to \infty} |\xi_n|(x) \geq \frac{1}{m} \},
\]

\[
E := \bigcup_{m \in \mathbb{N}} E_m = \{ x : \liminf_{n \to \infty} |\xi_n|(x) > 0 \}.
\]

Then, for every \( x \in E_m \) there exists \( N \in \mathbb{N} \) such that \( x \in E_{nm} \) for all \( n \geq N \). Hence

\[
\lim_{n \to \infty} \xi_n(x) = \lim_{n \to \infty} \xi_n(x)\chi_{E_{nm}} = \zeta_m(x) \text{ for all } x \in E_m.
\]

Thus \( \xi_n(x) \to \chi(x) \) as \( n \to \infty \) for all \( x \in E \). Note that \( |\xi|(x) \geq \frac{1}{m} \) for all \( x \in E_m \). So \( \zeta_m(x) = \xi \chi_{\{ | \xi| \geq \frac{1}{m} \}}(x) \) for all \( m \in \mathbb{N} \) and \( x \in E \).

Further,

\[
E^c := \{ x : \forall m, N \in \mathbb{N} \exists n \geq N \text{ such that } |\xi_n|(x) < \frac{1}{m} \} = \{ x : \liminf_{n \to \infty} |\xi_n|(x) = 0 \}.
\]

Therefore, for all \( x \in E^c \) and \( m \in \mathbb{N} \),

\[
|\zeta_m|(x) = \liminf_{n \to \infty} |\xi_n\chi_{E_{mn}}(x)| = 0.
\]

Now we define \( \xi(x) = 0 \) for \( x \in E^c \) so that \( \zeta_m(x) = \chi_{\{ \xi \geq \frac{1}{m} \}}(x) \) a.e. Finally, we have

\[
\limsup_{n \to \infty} |\xi_n - \xi| \leq \limsup_{n \to \infty} |\xi_n\chi_{E_{mn}} - \zeta_m| + \limsup_{n \to \infty} |\xi_n\chi_{\{ \xi_n < \frac{1}{m} \}} + |\xi\chi_{\{ \xi < \frac{1}{m} \}} < \frac{1}{m}| \to 0 \text{ as } m \to \infty.
\]

Proof of Theorem [17]. For \( \varepsilon > 0 \), let \( A, \xi, \) and \( u \) be as in Proposition [1,12]. It follows from Proposition [22] and Proposition [17] that \( \nabla u \) is uniformly bounded in \( W = L^\infty (t_1, t_2); L^p(B_R) \cap L^{p+\frac{m}{2}}(Q) \). The main result of \( [22] \) and Proposition [17] imply that there exists a subsequence \( \varepsilon_n \downarrow 0 \) such that, \( u_n = u_{\varepsilon_n} \to u \) as \( n \to \infty \) a.e. on \( Q \). Moreover, \( \nabla u_n(\frac{1}{m} - \frac{1}{m})^{p-1} \) converges as \( n \to \infty \) a.e. on \( Q \) for all \( m \in \mathbb{N} \). Then by Lemma [21] it follows that \( \nabla u_n \) converges as \( n \to \infty \) a.e. on \( Q \). Since \( \nabla u_n \) is uniformly bounded in \( W \), we conclude that \( \nabla u_n \) is \( w^* \)-convergent. Hence \( \nabla u \in W \) and \( \nabla u_n \to \nabla u \) as \( n \to \infty \) a.e. on \( Q \).

Now observe that \( \text{A}_{\varepsilon_n}(u_n, \nabla u_n) \to \text{A}(u, \nabla u) \) as \( n \to \infty \) a.e. on \( Q \) and that, due to [1,2],

\[
\sup_{\varepsilon} \| \text{A}_\varepsilon(u_n, \nabla u_n) \|^p \leq c_1 \sup_{\varepsilon} \| \nabla u_n \|^p < \infty.
\]

Hence \( \text{A}_{\varepsilon_n}(u_n, \nabla u_n) \to \text{A}(u, \nabla u) \) as \( n \to \infty \) weakly in \( L^p(Q) \).

Similarly, \( b_{\varepsilon_n}(u_n, \nabla u_n) \to b(u, \nabla u) \) a.e. on \( Q \). Then, for every \( Q'_\varepsilon \subseteq Q \), by [1,2],

\[
\sup_{n} \left( \int_{Q'} |b_{\varepsilon_n}(u_n, \nabla u_n)|^{p'} dx dt + \sup_{n} \int_{Q'} g^{p'} |\nabla u_n|^p \right) \leq c_p \int_{Q'} f^{p'}(x) dx dt + c_p \int_{Q'} g^{p'} |\nabla u_n|^p dx dt \leq c_p \int_{Q'} f^{p'}(x) dx dt + c_p |g| \int_{Q'} |\nabla u_n|^p dx dt
\]

with \( q = \frac{2(p+1)}{p-2} \). So \( b_{\varepsilon_n}(u_n, \nabla u_n) \) is weakly compact in \( L^{p'}(Q') \) and hence \( b(u, \nabla u) \in L^{p'}(Q') \) and \( b_{\varepsilon_n}(u_n, \nabla u_n) \to b(u, \nabla u) \) as \( n \to \infty \) weakly in \( L^{p'}(Q') \). Hence, for every \( \theta \in W^{1,p}(B_R) \),

\[
\int_{Q'} b_{\varepsilon_n}(u_n, \nabla u_n) \theta dx dt \to \int_{Q} b(u, \nabla u) \theta dx dt \text{ as } n \to \infty.
\]
3 Proof of Theorem 1.2

In the proof we follow the idea from [24], with required modifications. We start with the following technical lemma.

Lemma 3.1. There exist $c_p, \Gamma_p > 0$ such that, for all $(x, t) \in \Omega_T$, $\mu, \bar{\mu} \in \mathbb{R}$, $\eta, \tilde{\eta} \in \mathbb{R}^N$, one has

\[
\begin{aligned}
\langle A(x, t, \mu, \eta) - A(x, t, \bar{\mu}, \eta) - \langle 0 \rangle, \eta - \tilde{\eta} \rangle & \geq c_p (|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2 \\
- \Gamma_p \left( f_1^{\prime}(x)|\mu - \bar{\mu}|^p + g_1^{\prime}(x)|\tilde{\eta}|^{p-2}|\mu - \bar{\mu}|^2 + g_1^{\prime}(x)|\eta - \tilde{\eta}|^{p-2}|\mu - \bar{\mu}|^2 \right).
\end{aligned}
\]

(3.1)

Proof. Set $\omega_s := (x, t, s\mu + (1 - s)\bar{\mu}, sn + (1 - s)\tilde{\eta}), s \in [0, 1]$. Then

\[
\begin{aligned}
A(x, t, \mu, \eta) - A(x, t, \bar{\mu}, \eta) = \int_0^1 \partial_\omega A(\omega_s)(\eta - \tilde{\eta})ds + \frac{1}{\partial_\omega A(\omega_0)(\mu - \bar{\mu})ds}.
\end{aligned}
\]

Then, by (31), there exist $c_{0, p} > 0$ such that

\[
\int_0^1 \langle \partial_\omega A(\omega_s)(\eta - \tilde{\eta}), \eta - \tilde{\eta} \rangle ds \geq c_{0, p} (|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2.
\]

Further, by (15), there exists $C_p$ such that

\[
\begin{aligned}
\left| \int_0^1 \langle \partial_\omega A(\omega_s)(\mu - \bar{\mu}), \eta - \tilde{\eta} \rangle ds \right| & \leq f_1(x)|\mu - \bar{\mu}| |\eta - \tilde{\eta}| + C_p g_1(x)(|\eta| + |\tilde{\eta}|)^{p-2}|\mu - \bar{\mu}| |\eta - \tilde{\eta}| \\
& \leq \frac{c_{0, p}}{4} |\eta - \tilde{\eta}|^p + \frac{1}{c_{0, p}} f_1^{\prime}(x)|\mu - \bar{\mu}|^p + \frac{c_{0, p}}{4} (|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2 \\
& \quad + \frac{2^{p-2}C_p^2}{c_{0, p}} g_1^{\prime}(x)|\eta - \tilde{\eta}|^{p-2} |\mu - \bar{\mu}|^2 + \frac{4^{p-2}C_p^2}{c_{0, p}} g_1^{\prime}(x)|\eta|^{p-2} |\mu - \bar{\mu}|^2.
\end{aligned}
\]

Similar to what was done in [24] we introduce the following functions:

\[
\begin{aligned}
\hat{b}(x, t, \bar{\mu}, \tilde{\eta}) & := \Gamma_p \left( f_1^{\prime}(x)|u(x, t) - \bar{\mu}|^{p-2} + g_1^{\prime}(x)|\tilde{\eta}|^{p-2} \right)(u(x, t) - \bar{\mu}), \\
\tilde{b}(x, t, \bar{\mu}, \tilde{\eta}) & := -f(x) - g(x)|2\eta|^{p-1} \vee b(x, t, u(x, t), \nabla u(x, t)) \wedge (f(x) + g(x)|2\eta|^{p-1}).
\end{aligned}
\]

Set

\[
\begin{aligned}
\tilde{b}(x, t, \mu, \eta) = \hat{b}(x, \bar{\mu}, \tilde{\eta}) + \tilde{b}(x, \bar{\mu}, \tilde{\eta}).
\end{aligned}
\]

Consider the auxiliary the equation

\[
\partial_t \tilde{u} - \text{div} A(\tilde{u}, \nabla \tilde{u}) = \tilde{b}(x, \mu, \eta).
\]

(3.2)

Proposition 3.2. Let $Q = B_R \times (t_1, t_2) \subseteq \Omega_T$. Let $\tilde{u}$ be a weak solution to (3.2) in $Q$ such that

\[
u|_{PQ} = \tilde{u}|_{PQ},
\]

where $PQ$ is the parabolic boundary of $Q$. Then $\tilde{u} = u$ in $Q$ if $\sup_{B_R} W^{s, r}_p (\text{diam} B)$ is small enough.
Proof. Subtract (3.2) out of (1.1) and multiply the difference by \( u - \tilde{u} \). Note that the latter belongs to \( L^p((t_1, t_2) \to W^{0, p}_0(B)) \cap C_0((t_1, t_2) \to L^2(B)) \). We obtain that

\[
\frac{1}{2} \int_B |u - \tilde{u}|^2(t_2)dx + \int_Q \langle A(u, \nabla u) - A(\tilde{u}, \nabla \tilde{u}), \nabla u - \nabla \tilde{u} \rangle dxdt = \int_Q (b(u, \nabla u) - \tilde{b}(\tilde{u}, \nabla \tilde{u}))(u - \tilde{u})dxdt.
\]

By Lemma 3.1 we have

\[
\int_Q \langle A(u, \nabla u) - A(\tilde{u}, \nabla \tilde{u}), \nabla u - \nabla \tilde{u} \rangle dxdt + \int_Q \tilde{b}(\tilde{u}, \nabla \tilde{u})(u - \tilde{u})dxdt \\
\geq c_p \int_Q (|\nabla u| + |\nabla \tilde{u}|)^{p-2}|\nabla u - \nabla \tilde{u}|^2 dxdt - \Gamma_p ||\nabla u - \nabla \tilde{u}||_p^{p-2} ||g(u - \tilde{u})||_p^2.
\]

Further, note that \( b(u, \nabla u) \) is of the same sign that \( \tilde{b}(\tilde{u}, \nabla \tilde{u}) \). Also observe that \( b(u, \nabla u) \neq \tilde{b}(\tilde{u}, \nabla \tilde{u}) \) only under the condition \( |b(u, \nabla u)| > f + g|2\nabla \tilde{u}|^{p-1} \), which implies that \( |\nabla u| \geq 2|\nabla \tilde{u}| \). Hence

\[
|b(u, \nabla u) - \tilde{b}(\tilde{u}, \nabla \tilde{u})| \leq g|\nabla u|^{p-1}1_{(|\nabla u| \geq 2|\nabla \tilde{u}|)} \leq 2^{p-1}g|\nabla u - \nabla \tilde{u}|^{p-1}.
\]

Therefore

\[
\int_Q |b(u, \nabla u) - \tilde{b}(\tilde{u}, \nabla \tilde{u})||u - \tilde{u}|dxdt \leq 2^{p-1}||\nabla u - \nabla \tilde{u}||_p^{p-1} ||g(u - \tilde{u})||_p.
\]

Thus we obtain that

\[
c_p \int_Q (|\nabla u| + |\nabla \tilde{u}|)^{p-2}|\nabla u - \nabla \tilde{u}|^2 dxdt \leq \Gamma_p ||\nabla u - \nabla \tilde{u}||_p^{p-2} ||g(u - \tilde{u})||_p^{2p} + 2^{p-1} ||\nabla u - \nabla \tilde{u}||_p^{p-1} ||g(u - \tilde{u})||_p.
\]

By (1.13) this implies that

\[
c_p ||\nabla u - \nabla \tilde{u}||_p^p \leq \left\{ \Gamma_p \sup_{B_R} \left(W^{\phi^p}_p(x, 2R) \right)^{\frac{p}{m}} + 2^{p-1} \sup_{B_R} \left(W^{\phi^p}_p(x, 2R) \right)^{\frac{p}{m}} \right\} ||\nabla u - \nabla \tilde{u}||_p^p.
\]

So if \( \sup_{B_R} W^{\phi^p}_p(x, 2R) \) is small enough then \( ||\nabla u - \nabla \tilde{u}||_p^p \leq 0 \). \( \square \)

Proof of Theorem 1.3. Note that \( |\tilde{b}(x, t, \tilde{u}, \tilde{\eta})| \leq f(x) + g(x)|2\tilde{\eta}|^{p-1} \) and

\[
\tilde{b}(x, t, \tilde{u}, \tilde{\eta}) \leq \Gamma_p (g_1(x)|\tilde{\eta}|^{p-1} + (f_1(x))^{\frac{p}{m}} + g_1(x)|u(x, t) - \tilde{u}|^{p-1} + f_1(x))^{\frac{m}{p}}.
\]

Hence equation (5.2) satisfies the structural conditions (1.2)–(1.9) with \( 2^{p-1} \Gamma_p (f_1^{\frac{p}{m}} + g_1^{\frac{p}{m}} + \sup |u|)|f + g + \Gamma_p g_1| \) replacing \( f \) and \( g \), respectively. Note that by Proposition 1.11 \( g_1^p \in K_p \) implies that \( g_1^2 \in PK_0 \). Hence, by Theorem 1.1 there exists a solution \( \tilde{u} \) coinciding with \( u \) on the parabolic boundary of \( Q \), which enjoys the estimate (1.11). Since \( g^p \in K_p \), we can choose \( R \) so small that \( u = \tilde{u} \) on \( Q \), which completes the proof. \( \square \)

4 Boundedness of the gradient. Proof of Theorem 1.3.

By Proposition 1.11 \( f^2 \in \tilde{K}_2 \) implies that \( f \in K_p \) and \( f^2 \in \tilde{K}_2 \) implies that \( f^{\frac{p}{m}} \in K_p \). Hence under the conditions of Theorem 1.3 the assertions of Theorem 1.1 and Theorem 1.2 hold.

Now under the conditions \( f^2, g^2, f_1^2, g_1^2, f_2^2, g_2^2 \in \tilde{K}_2 \) it remains to obtain uniform estimates of the gradients on the sets where \( |\nabla u| > l \) for some positive \( l \). This restriction allows us to simplify the structure conditions putting \( F = f + g + f_1 + g_1 + f_2 + g_2 \) and requiring

\[
|\partial_u A| + |\partial_u A||z| + |b| \leq F(x)|z|^{p-1}
\]
To estimate the right hand side of (4.4) we note that

\[
\varphi(\sigma) := \int_{0}^{\infty} (1 + s)^{-1-\lambda} ds, \quad G(\sigma) = \int_{0}^{\infty} s \varphi(s) ds.
\]

Before formulating the next lemma let us note that \( G(\sigma) \approx \min\{\sigma^2, \sigma^3\}, \sigma \geq 0 \) and

\[
(4.2) \quad \varphi(\sigma) \geq \frac{\sigma}{(1 + \sigma)^{1+\lambda}} \quad \text{so that} \quad \partial_\sigma(\sigma \varphi(\sigma)) = \varphi(\sigma).
\]

**Lemma 4.1.** Let \( \delta > 0 \). With notation \( w = |\nabla u|^2, \sigma = \left(\frac{w}{\delta}\right)_+ \) the following inequality holds

\[
\text{ess sup}_t \int G(\sigma(t))\xi^q(t) + \int \int \text{tr}\{\text{tr}\{D\zeta(\partial_2 A)D^2 u\} \leq \frac{c_0}{2} \int \int G(\sigma(\xi)^{q-1} \partial_\sigma \xi + \int \int (|\partial_2 A| + |\partial_3 A||\nabla u| + |b|) D\zeta|\}
\]

Note that \( D\zeta = D^2 u \sigma \varphi(\sigma)\xi^q + (\nabla u \nabla)\varphi(\sigma)(\xi^{q-1} + q(\nabla u \nabla)\varphi(\sigma)\xi^{q-1} + D^2 u \nabla u = \frac{1}{2}\nabla w = \delta/2\nabla \sigma. \) By structure conditions (1.6) and (1.7), Remark 1.14 and (4.2) it follows that

\[
\text{tr}\{D\zeta(\partial_2 A)D^2 u\} \geq c_0 w^{1-\xi} \sigma^2 \varphi(\sigma)\xi^q \geq \frac{c_0}{2} w^{1-\xi}\sigma^2 \varphi(\sigma)\xi^q - \frac{c_0}{2} w^{1-\xi}\sigma^2 \varphi(\sigma)\xi^q.
\]

By the Schwartz inequality

\[
\text{tr}\{D\zeta(\partial_2 A)D^2 u\} \geq c_0 w^{1-\xi} \sigma^2 \varphi(\sigma)\xi^q \geq \frac{c_0}{4} w^{1-\xi}\sigma^2 \varphi(\sigma)\xi^q - \frac{c_0^2}{4c_0} w^{1-\xi}\sigma^2 \varphi(\sigma)\xi^q.
\]

To estimate the right hand side of (4.4) we note that

\[
(\partial_3 A)^2 \sigma^2 \varphi(\sigma)\xi^q
\]

where we used the following obvious inequality \( \frac{1}{w^{1-\xi}} \sigma \geq \sigma^2. \) Thus we have from (4.4)

\[
\delta \int G(\sigma(t))\xi^q(t) + \delta \int \int w^{1-\xi}\sigma^2 \varphi(\sigma)\xi^q
\]

To complete the proof note that \( \varphi(\sigma) \leq \frac{1}{\delta}, G(\sigma) \leq \gamma^2 \) and \( w^{1-\xi} \leq \gamma(\gamma/2+1 + \delta^{\gamma/2+1} + \sigma/2+1). \)
Lemma 4.2. Let \( h \in H_0^1(B) \cap L^\infty(B) \) be such that \(-\Delta h = F^2\). Then

\[
\delta^\frac{\sigma}{\sigma+1} \iint F^2 \sigma^\frac{\sigma}{\sigma+1} \varphi(\sigma) \xi^q \leq \gamma ||h||_\infty \iint |\nabla \varphi|^2 w^\delta \varphi(\sigma) \xi^q + \gamma ||h||_\infty \iint |\nabla \xi|^2 \sigma^\frac{\sigma}{\sigma+1} \varphi(\sigma) \xi^{q-1}.
\]

Proof. Recall that \( \varphi(\sigma) \geq \frac{\sigma}{\sigma+1} \) and apply (1.13). Then

\[
\iint F^2 \sigma^\frac{\sigma}{\sigma+1} \varphi(\sigma) \xi^q \leq \gamma \iint F^2 \sigma^\frac{\sigma}{\sigma+1} \xi^q \leq \gamma ||h||_\infty \iint |\nabla \xi|^2 \sigma^\frac{\sigma}{\sigma+1} \xi^{q-2}.
\]

Finally, note that \( \delta^\frac{\sigma}{\sigma+1} \sigma^\frac{\sigma}{\sigma+1} \leq w^\delta \). □

Now let \((x_0, t_0) \in \Omega_T\). Given \( r, \delta, l > 0 \) we denote \( \Delta := \max\{\delta, l\} \) and

\[
Q = Q_{r, \Delta}^{x_0, t_0} = B_r(x_0) \times I_\Delta \equiv B_r(x_0) \times (t_0 - \frac{r^2}{\Delta^2 - 1}, t_0 + \frac{r^2}{\Delta^2 - 1}).
\]

\( \xi \in C^1_0(B_1(0) \times (-1, 1)) \), \( 0 \leq l \leq 1 \), \( l_0 = 1 \), \( I_\Delta = \max\{l_0, \delta\} \), \( I_j = I_\Delta \), \( \xi_j = \xi_{r_j, \Delta_j} \). With this notation the next lemma is easy to check.

Lemma 4.4. If \( \delta_j > \left( \frac{1}{2} \right)^{\frac{1}{l-1}} \delta_j-1 \) then \( I_j \subset \frac{1}{2} I_{j-1} \).

Let \( I_j = \{ (x, t) \in \Omega_T : w(x, t) > l_j \} \), \( I_j = \{ x \in \Omega : w(x, t) > l_j \} \). Fix \( \varepsilon > 0 \) a small number which will be chosen later depending on the known data.

Define

\[
A_j(l) = \sup_{t \in I_j} \frac{1}{r_j} \int L_j(t) \frac{G(\frac{w-l_j}{l-l_j})}{r_j^{N+2}} \xi_j^q dx + \frac{(l-l_j)^\frac{1}{N+2}}{r_j^{N+2}} \int L_j \frac{w-l_j}{l-l_j} \xi_j^{q+1} dx dt + \frac{\Delta_j(l)}{r_j^{N+2}} \int L_j \frac{w-l_j}{l-l_j} \xi_j^{q-2} dx dt
\]

(4.7)

where \( \Delta_j(l) = \max\{l_j, l-l_j\} \).

Set

\[
\varepsilon_j = \left( \frac{1}{r_j^{N+2}} \int B_j F^2(x) dx \right)^{\frac{1}{2}}, \quad j = 1, 2, \ldots
\]

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The sequence \((l_j)_{j \in \mathbb{N}}\) is defined inductively. We set as above \(l_0 = 1\). Suppose \(l_1, \ldots, l_j\) have been defined. We show how to define \(l_{j+1}\).

First, note that \(A_j(l)\) is continuous and \(A_j(l) \to 0\) as \(l \to \infty\). If \(A(l_j + F_j) \leq \varkappa\) then we set \(l_{j+1} = l_j + F_j\). If on the other hand \(A(l_j + F_j) > \varkappa\) then there exists \(\tilde{l} > l_j + F_j\) such that \(A_j(\tilde{l}) = \varkappa\), and we set \(l_{j+1} = \tilde{l}\). In both cases

\[
A_j(l_{j+1}) \leq \varkappa.
\]

**Lemma 4.5.**

\[
\delta_j \leq \left( \frac{1}{2} \right)^{j} \frac{\varkappa}{\varkappa-\varepsilon} \delta_{j-1} + \gamma l_j F_j.
\]

**Proof.** Fix \(j \geq 1\) and suppose that \(\delta_j > \left( \frac{1}{2} \right)^{j} \frac{\varkappa}{\varkappa-\varepsilon} \delta_{j-1}\) and \(\delta_j > F_j\) since otherwise there is nothing to prove. This implies that \(A_j(l_{j+1}) = \varkappa\).

We denote \(\sigma_j := \frac{w(l_j)}{\varkappa}, \Phi_j := \Phi(\sigma_j), \Psi_j := \Psi(\sigma_j)\).

**Claim.** \(\sup_{t \in [l_j, l_{j+1}]} |L_j(t)| \leq \gamma \varkappa\). Indeed, for \((x, t) \in L_j\) one has

\[
\frac{w(x, t) - l_{j-1}}{\delta_{j-1}} = 1 + \frac{w(x, t) - l_j}{\delta_{j-1}} \geq 1.
\]

Note that yields \(\xi_{j-1} = 1\) on \(Q_j\). Hence

\[
\begin{align*}
 r_j^{-N} \sup_{t \in [l_j, l_{j+1}]} |L_j(t)| &\leq r_j^{-N} \sup_{t \in [l_j, l_{j+1}]} \int_{L_j(t)} G \left( \frac{w - l_{j-1}}{\delta_{j-1}} \right) \xi_{j-1}^q dx \\
 &\leq 2r_j^{-N} \sup_{t \in [l_j, l_{j+1}]} \int_{L_j(t)} G \left( \frac{w - l_{j-1}}{\delta_{j-1}} \right) \xi_{j-1}^q dx \leq 2N \varkappa.
\end{align*}
\]

which proves the claim.

Now decompose \(L_j\) as \(L_j = L'_j \cup L''_j\),

\[
L'_j = \left\{ (x, t) \in L_j : \frac{w(x, t) - l_j}{\delta_j} < \varepsilon \right\}, \quad L''_j = L_j \setminus L'_j,
\]

where \(\varepsilon\) depending on the data is small enough to be determined later. Then

\[
\delta_j^{\frac{q-1}{2}} \int_{L'_j} \left( \int_{L_j} \Phi_j^{\frac{q}{2}} \xi_j^{q-2} dx \right) dt + \delta_j^{\frac{q-1}{2}} \int_{L''_j} \left( \int_{L_j} \left( \Phi_j^{\frac{q}{2}} \right)^{2} dx \right) dt \leq \gamma \varepsilon^{2} (1 + \varepsilon^{p/2-1}) \sup_{t \in [l_j, l_{j+1}]} \frac{1}{|L_j(t)|} \leq \gamma \varepsilon^{2} \varkappa.
\]

Now recall that \(\Phi(\sigma) = \min\{\sigma^{\frac{q}{2}} + 1, \frac{\mu}{\nu} \}\). So \(\sigma_j^{rac{q}{2} - 1} \leq \gamma (\varepsilon) \Phi_j^2\) on \(L''_j\). So we have

\[
\delta_j^{\frac{q-1}{2}} \int_{L''_j} \left( \int_{L_j} \left( \frac{1}{\Phi_j^{\frac{q}{2}}} \right)^{2} dx \right) dt \leq \gamma (\varepsilon) \delta_j^{\frac{q-1}{2}} \int_{L''_j} \left( \int_{L_j} \left( \Phi_j^{\frac{q}{2}} \xi_j^{q-2} dx \right) dt \leq \gamma (\varepsilon) \frac{\delta_j^{\frac{q-1}{2}}}{r_j^{N+2}} \int_{L_j} \left( \int_{L_j(t)} \left( \Phi_j^{\frac{q}{2}} \right)^{2} dx \right) dt
\]

\[
\leq \gamma (\varepsilon) \frac{\delta_j^{\frac{q-1}{2}}}{r_j^{N+2}} \left( \sup_{t \in [l_j, l_{j+1}]} |L_j(t)| \right)^{\frac{1}{2}} \int_{L_j} \left| \nabla \left( \Phi_j^{\frac{q}{2}} \xi_j^{q-2} dx \right) \right| dt
\]

\[
\leq \gamma (\varepsilon) \delta_j^{\frac{q-1}{2}} \int_{L_j} \left( \sup_{t \in [l_j, l_{j+1}]} |L_j(t)| \right)^{\frac{1}{2}} \delta_j^{\frac{q-1}{2}} \int_{L_j} \left| \nabla \left( \Phi_j^{\frac{q}{2}} \xi_j^{q-2} dx \right) \right| dt
\]

\[
\leq \gamma (\varepsilon) \varkappa^{\frac{q}{2}} \delta_j^{\frac{q-1}{2}} \int_{L_j} \left| \nabla \left( \Phi_j^{\frac{q}{2}} \xi_j^{q-2} dx \right) \right| dt.
\]
Similarly, if \( l_j \geq \delta_j \),

\[
\frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int_{L_j}^{N+2} \sigma_j^2 \xi_j^{l_j-2} dx dt \leq \gamma(\varepsilon) \left( \sup_{t \in L_j} \frac{1}{r_j} |L_j(t)| \right) \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int \left| \nabla (\Psi_j \xi_j^{\frac{1}{2}}) \right|^2 dx dt \\
\leq \gamma(\varepsilon) \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int \left| \nabla (\Psi_j \xi_j^{\frac{1}{2}}) \right|^2 dx dt.
\]

(4.14)

Using Corollary 4.3 we have

\[
\frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int_{L_j}^{N+2} \sigma_j^2 \xi_j^{l_j-2} dx dt + \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int_{L_j}^{N+2} \sigma_j^2 \xi_j^{l_j-2} dx dt \\
\leq \gamma(\varepsilon) \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int \left| \nabla (\Psi_j \xi_j^{\frac{1}{2}}) \right|^2 dx dt.
\]

(4.15)

Now we estimate the first term in the right hand side of (4.7) using Corollary (4.3) and the Claim.

\[
\sup_{t \in L_j} \int G \left( \frac{w - l_j}{1 - l_j} \right) \xi_j^3 dx \\
\leq \gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) \kappa + \gamma(\varepsilon) \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int_{B_j} F^2 dx.
\]

(4.16)

Collecting (4.12) – (4.16) we obtain

\[
\kappa \leq \gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) \kappa + \gamma(\varepsilon) \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int_{B_j} F^2 dx.
\]

Now first choosing \( \varepsilon \) by the condition

\[
\gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) = \frac{1}{4},
\]

and then \( \kappa \) such that

\[
\gamma(\varepsilon) \frac{\Delta \xi_j^{l_j-1}}{r_j^{N+2}} \int_{B_j} F^2 dx = \frac{1}{4},
\]

we arrive at (4.9).

Summing up the inequalities (4.9) with respect to \( j \) from 1 to \( J - 1 \) we obtain

\[
l_J \leq \gamma \delta_0 + \gamma l_J \sum_{j=1}^{J-1} F_j.
\]

Choosing \( r_0 \) small enough so that \( \int_{B_{r_0}} \frac{dx}{r_0} \left( \frac{1}{r_0} \int_{B_{r_0}} F^2(y) dy \right)^{\frac{1}{2}} < \frac{\delta}{2} \), we arrive at

\[
l_J \leq \gamma \delta_0.
\]

It remains to estimate \( \delta_0 \). From (4.7) we have

\[
\delta_0 \leq \left( \frac{1}{r_0^{N+1}} \sup_{t} \int_{B_0} |\nabla u|^2 \xi_0 dx \right)^{\frac{1}{2}} + \left( \frac{1}{r_0^{N+p}} \int_{Q_0} |\nabla u|^{p+2} \xi_0^{q-2} dx d\tau \right)^{\frac{1}{2}}.
\]

The finiteness of the right hand side follows now from Theorem 1.2.
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References


