Decision Support

Two consensus models based on the minimum cost and maximum return regarding either all individuals or one individual

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\textbf{abstract}

In some important group decision making, a moderator representing the collective interest, who has pre-determined, and possesses an effective leadership and strong interpersonal communication and negotiation skills, is crucial to the consensus reaching. In the process of consensus reaching, the moderator needs to persuade each individual to change his/her opinion towards a consensus opinion by paying a minimum cost, while the individuals have to modify and to gradually approach this consensus opinion by expecting to obtain a maximum compensation. This paper, which proposes two kinds of minimum cost models with regard to all the individuals and one particular individual respectively, shows the economic significance of these two models by exploring their dual models grounded in the primal–dual linear programming theory, and builds the conditions under which these two models have the same optimal consensus opinion. The validity of the theoretical analysis is confirmed by numerical examples.

1. Introduction

Group decision making (GDM) (Arrow, 1963; Palomares, Liu, Xu, & Martínez, 2012) requires the subjective judgment of a number of decision makers (DMs) to solve complex and unstructured problems, such as negotiations and conflict resolutions. In the process of GDM, different DMs may represent different interest groups, and may have different values or preferences even they have the same interest. In a GDM, most DMs may eventually arrive at a certain degree of consensus associated with the most relevant alternatives after thought-provocative discussions and many round of negotiations. The consensus decision making (Eklund, Rusinowska, & De Swart, 2007, 2008; French, 1981; Lehrer & Wagner, 1981; Liu & Zhang, 2013; Palomares, Martínez, & Herrera, 2014) is the base of making group choices. In recent years, abundant achievements have been made in the fields of consensus measure and consensus modeling.

1.1. Consensus measure

Consensus measure is mainly about the similarity or dissimilarity among DMs’ opinions (preferences, interests). The early literatures suggest a “hard” approach (Bezdek, Spillman, & Spillman, 1978; Spillman, Bezdek, & Spillman, 1979) to measure the consensus level of a group, where the value of consensus level is between 0 and 1. The closer to 1 the index is, the higher consensus level is achieved; and conversely, the closer to 0 the index is, the lower consensus level is. The “hard” approach to consensus modeling is based on the premise that a full agreement within the group has been arrived at, which is also called a Utopian consensus by Tapia García, Del Moral, Martínez, and Herrera-Viedma (2012). It is difficult to achieve such a complete consensus (Cabrerizo, Moreno, Pérez, & Herrera-Viedma, 2010). Kacprzyk and Fedrizzi (1986, 1988, 1989), Kacprzyk and Fedrizzi (1989, 1992, 1997) and Fedrizzi, Kacprzyk, and Numi (1993) propose a “soft” method instead of the “hard” approach to measure the consensus level which is also referred as “soft” consensus degree level (Chiclana, Tapia García, del Moral, & Herrera-Viedma, 2013). Since the key elements in GDM are based on human thinking and subjective judgment, most experts only expect to reach a fuzzy-majority-sense consensus at the best. The development of soft decision making theories such as fuzzy decision making theory and linguistic decision making theory provides a rich tool to the “soft” approach-oriented research over recent years (Ben-Arieh & Chen, 2006; Bezdek et al., 1978; Cabrerizo, Alonso, Pérez, & Herrera-Viedma, 2008; Carlsson et al., 1992; Dong, Xu, & Li, 2008; Fedrizzi, Kacprzyk, & Zadrozny, 1988, 1999, 2007; Tapia García et al., 2012; Herrera-Viedma, Martínez, Mata, & Chiclana, 2005; Kacprzyk & Fedrizzi, 1989; Kacprzyk et al., 1997; Parreiras, Ekel, & Morais, 2012; Xu, Wu, & Zhang, 2014; Xu, Li, & Wang, 2014).
1.2. Consensus modeling

The optimization consensus modeling is based on the assumption that there exists an optimum consensus opinion such that deviations between this opinion and individual DMs’ opinions should be as small as possible. The aggregation model supposes that there exists a suitable aggregation operator that would be able to aggregate all the individual DMs’ opinions to the consensus opinion (Ben-Arie and Easton, 2007; Ben-Arie, Easton, Evans, 2009; Dong, Xu, Li, & Feng, 2010; Dong, Li, Xu, & Gu, 2014; Fu & Yang, 2010, 2011, 2012; Xu et al., 2013; Xu, 2009, 2012; Xu & Cai, 2011; Zhang, Dong, Xu, & Li, 2011; Zhang, Dong, & Xu, in press). Technically, consensus models are mostly constructed by using methods of optimization, and belong to the “hard” approach. However, each optimization model is constructed on the assumption that individual DMs’ opinions do not exceed a tolerated error of consensus opinion after many times of dynamically revisions and modifications (Bryson, 1996, 1997; Bryson & Joseph, 1999; Dong et al., 2014; Zhang et al., in press). It means that consensus modeling is actually a combination of “soft” and “hard” approaches.

In the last few years, the rapid development of web technologies provides much more convenient platforms for larger number of users from all over the world to freely communicate, share and exchange ideas. Therefore, consensus modeling also needs to incorporate the feedback mechanism during consensus decision making: Alonso, Pérez, Cabrerizo, and Herrera-Viedma (2013) explore a novel linguistic consensus model for Web 2.0 communities, which increases the speed of consensus convergence; Pérez, Cabrerizo, Alonso, and Herrera-Viedma (2014) build up a new consensus model, which specially considers the heterogeneity of DMs; and Pérez, Wikström, Mezei, Carlsson, and Herrera-Viedma (2013) develop a consensus model by using the power of a fuzzy ontology, which deals with the psychology of negotiation. In many consensus decision making, it takes time, requires efforts, and then needs to pay cost to convince DMs to shift their opinions during the feedback process. To model this kind of consensus decision making, Ben-Arie and Easton (2007) develop a minimum cost consensus model to obtain the optimal convergence point of all DMs: A moderator who represents the collective interest to help reach the consensus is introduced during consensus process, where he/she has been predetermined and possesses an effective leadership and strong interpersonal communication and negotiation skills (Bryson, 1996; Cabrerizo et al., 2008, 2010; Herrera, Herrera-Viedma, & Verdegay, 1996; Herrera-Viedma et al., 2005, Herrera-Viedma, Alonso, Chiclana, & Herrera, 2007, 2014; Mata, Martinez, & Herrera-Viedma, 2009; Palomares et al., 2012, 2014; Pérez et al., 2013; Tapia García et al., 2012). On one hand, the moderator tries his/her best to convince most of the individuals to conform to the collective interest or value by spending all possible forms of resources, such as material, financial, human, and information. He/She always wishes that the amount of resources he/she spends is as small as possible (Ben-Arie & Easton, 2007; Ben-Arie et al., 2009; Zhang et al., 2011). On the other hand, every individual DM has an eye on his/her own benefit. Each individual DM hopes that his/her opinion deserves to be particularly considered, or he/she should show the significance and value of himself/herself by playing an important role in the consensus decision making. When they have to change their opinions or they offer more useful opinions, they deserve to be compensated or to be rewarded. Each individual DM always hopes that his/her return is as big as possible. The minimum cost and the maximum return are respectively, the moderator’s optimum objective and the individual DMs’ optimum objective, and they are dual to each other mathematically, making it helpful to further explore the consensus reaching problem by considering both minimum cost and maximum return.

Considering the moderator’s interest, Ben-Arie and Easton (2007) and Ben-Arie et al. (2009) suggest a consensus model with linear minimum cost and a consensus models with quadratic cost respectively, to obtain the optimum consensus opinion. Recently, Zhang et al. (2011) and Zhang, Dong, and Xu (2013) generalized Ben-Arie and Easton’s work by proposing a novel consensus model with aggregation operators to obtain the maximum consensus degree under the given cost budget. However, there is few research on consensus model considering the individuals’ interests. Actually, the process of consensus reaching needs balancing both the moderator’s and the individuals’ interests. The theories of primal–dual optimal programming will help to discuss how to obtain an optimum consensus opinion by preserving the benefits of both sides.

This paper discusses two kinds of consensus decision making problems by constructing primal–dual linear programming models. The first is that when all individuals are taken into account as a whole, a primal problem of minimum cost and its dual problem of maximum return for reaching the greatest consensus regarding all the individuals are developed. Secondly, when most individuals’ opinions do not exceed the tolerated error (or mathematically, in the neighborhood) of consensus opinion as suggested by the moderator, the individual DMs accept the consensus opinion but expect nothing about the return, while only a few DMs insist on their opinions unless the moderator pays more to them, this means that they accept the consensus opinion conditionally. For convenience, we suppose that there is only one individual who needs to be paid. Hence, a primal problem of minimum cost and its dual problem of maximum return for reaching the greatest consensus regarding one individual are also investigated.

This paper is structured as follows. Section 2 discusses the description of our problem. Section 3 constructs the primal–dual models based on the minimum cost consensus problem and the maximum return regarding all individuals. Section 4 discusses the economic significance of the primal–dual models by introducing its dual properties and exploring their relationship. Similarly, Section 5 establishes the primal–dual models based on the minimum cost and the maximum return regarding only one individual, and investigates the economic significance of these models. Section 6 builds the conditions under which these two kinds of primal–dual models have the same optimal consensus opinion. Lastly, conclusion and problems for the future research are provided in Section 7.

2. Problem description

Suppose that there are $m$ decision makers (DMs) $D = \{d_1, \ldots, d_m\}$, that take part in GDM. Let $o_i \in R$ represent the opinion of DM $d_i$ ($i \in M = \{1, 2, \ldots, m\}$) in GDM. Without loss of generality, we always suppose that $o_1 \leq o_2 \leq \ldots \leq o_m = 0$. According to the American Heritage Dictionary, consensus is defined as “an opinion or position reached by a group as a whole”. This means in group decision making, the ideal state is that there exists an ideal opinion $\bar{o}$ such that $o_1 = o_2 = \ldots = o_m = \bar{o}$. When such an ideal opinion is derived, we get a full and unanimous agreement or a Utopian consensus. However, according to Ness and Hoffman (1998), consensus represents “a decision that has been reached when most members of the team agree on a clear option and the few who oppose it think they have had a reasonable opportunity to influence that choice; all team members agree to support
the decision”. This means that such an ideal opinion is absolute, and it is difficult to obtain, then the moderator has to suggest a relatively satisfactory opinion $o'$ to meet the most individuals’ preferences. We call such an $o'$ as an acceptable consensus opinion (or, simply, a consensus opinion). Firstly, we suppose that $o'$ exists. In fact, it can be solved by a programming model constructed later: Let $f_i(o) = |o - o_i|$ be the deviation between the opinion $o_i$ of individual $i$ ($i \in M$) and the consensus opinion $o'$. Obviously, the smaller $f_i(o')$ is, the closer the individual’s opinion is to the consensus opinion. Let $w_i$ denote a unit cost that paid by the moderator to persuade individual $i$ ($i \in M$) to change his/her opinion. Then $w_if_i(o')$ denotes the cost that paid by the moderator to persuade individual $i$ ($i \in M$). The smaller this value is, the closer the distance between the individual’s opinion and the consensus opinion, and the lower the cost to individual $i$ ($i \in M$) is.

For all individuals, as they are required to present valuable opinions, they also have to dynamically adjust opinions to conform to the consensus opinion $o'$, and thus they deserve to gain some return according to the change $o_i - o'$. Let $y_i$ be the unit return expected by the individual $i$, then $y_i(o_i - o')$ denotes the total return of $i$ for changing his/her opinion. For each individual, the greater the value $y_i(o_i - o')$ is, the higher the total return is expected by the individual. We suppose that all individuals are rational, and that they only need an appropriate value of unit return $y_i$ that will also contribute to reaching consensus. We also prove that this unit return $y_i$ is actually a shadow price (Jensen & Bard, 2003) in an economic sense.

From the viewpoint of the moderator, he/she hopes to achieve the greatest consensus while paying the minimum cost to all the individuals. And from the viewpoint of each individual, he/she expects to gain the maximum compensation for his/her changing opinions. Mathematically, these two goals are dual to each other.

Next, we construct two consensus models of mathematical programming, and explore the relationship between these models.

### 3. Primal problem of minimum cost and its dual of maximum return for reaching the greatest consensus

In Section 2, if we add all the costs $w_i f_i(o')$ paid by the moderator to persuade the individuals $i$ ($i \in M$), then we get a weighted arithmetic mean value $\sum_{i=1}^{m} w_i f_i(o')$. It denotes the total cost paid by the moderator to persuade all the individual DMs for arriving at the consensus. For the moderator, the smaller this value is, the closer the distance between individuals’ opinion and the consensus opinion, and the lower total cost to all individuals.

If we add all the returns $y_i(o_i - o')$ expected by $i$, for changing his/her opinion, then $\sum_{i=1}^{m} y_i(o_i - o')$ denotes the total return that all individuals expect for changing their opinions. For all these individuals, the greater the value $\sum_{i=1}^{m} y_i(o_i - o')$ is, the higher the total return expected by all individuals.

Now $\sum_{i=1}^{m} w_i f_i(o')$ can be regarded as the total cost (resource) paid by the moderator to obtain a consensus. The smaller the value $\sum_{i=1}^{m} w_i f_i(o')$ is, the greater degree of consensus will be. Thus, we construct a nonlinear optimization model $P(w)$ under the premise that there is a consensus opinion so that the total cost to obtain the consensus is the minimum:

$$
P(w) : \min \phi = \sum_{i=1}^{m} w_i f_i(o') \quad (1)
$$

s.t. $\{ o' \in O \quad (1 - 1) \}$

In the $P(w)$ model, if $f_i(o') = |o' - o_i|$, then there must exist $u_i \geq 0$, $v_i \geq 0$, and $u_i + v_i = 0$, such that $|o' - o_i| = u_i + v_i$, $o' - o_i = u_i - v_i$. In fact, if we let $u_i = |o' - o_i| + (|o' - o_i|)/2$, $v_i = |o' - o_i| - (|o' - o_i|)/2$, then the above conditions hold. The linear programming format $LP(w)$ of the nonlinear model $P(w)$ is as follows:

$$
LP(w) : \min \phi = \sum_{i=1}^{m} (w_i u_i + w_i v_i) \quad (2)
$$

s.t. $\{ o' - u_i + v_i = o_i, \quad i \in M \quad (2 - 1) \}$

$$
\{ o' \geq 0, \quad u_i \geq 0, \quad v_i \geq 0, \quad i \in M \quad (2 - 2) \}
$$

which is called the weighted linear consensus problem.

It is easy to prove that the set of feasible solutions $X = \{ o', u_1, v_1, \ldots, u_n, v_n, v_{n0} \}$ to Model (2) is nonempty. It can also be shown that the number of basic feasible solutions of Model (2) is finite and the optimal solution to Model (2) can be solved easily (Jensen & Bard, 2003).

In Model (2), the objective function $\phi = \sum_{i=1}^{m} (w_i u_i + w_i v_i)$ can be considered to be the minimum total cost for obtaining the greatest consensus. The restriction condition (2–1) denotes the limits of the deviation between the consensus opinion and individual’s opinion. Obviously, the cost of arriving at the greatest consensus is as small as possible under the restriction conditions (2–1) and (2–2). Next, we further explore the specific meaning in economics by discussing the dual problem of Model (2). With the primal–dual theory of linear programming, the dual problem of Model (2) is as follows:

$$
DLP(w) : \max \psi = \sum_{i=1}^{m} y_i \quad (3)
$$

s.t. $\{ \sum_{i=1}^{m} y_i \leq 0 \quad (3 - 1) \}$

$$
\{ |y_i| \leq w_i, \quad i \in M \quad (3 - 2) \}
$$

Model (3) is the dual problem of Model (2). It is easy to prove that the set of feasible solutions $Y = \{ y_1, \ldots, y_n \}$ to Models (3) is nonempty. The number of basic feasible solutions is finite, and $|y_i| \leq w_i$ ($i \in M$) is bounded. We would like to further mention that if there exists an optimal solution $X' = \{ o', u'_1, v'_1, \ldots, u'_n, v'_n, v'_{n0} \}$ to Model (2), then obviously $o' > 0$. Based on the complementary slackness property (Jensen & Bard, 2003) of dual linear programming theory, the equality of restriction condition (3–1) holds, that is, the condition $\sum_{i=1}^{m} y_i o_i = 0$ holds. Thus, the objective function of Model (3) is equivalent to $\sum_{i=1}^{m} y_i o_i = 0$ and Model (3) is equivalent to

$$
DLP(w) : \max \psi = \sum_{i=1}^{m} y_i o_i - o' \quad (4)
$$

s.t. $\{ \sum_{i=1}^{m} y_i = 0 \quad (4 - 1) \}$

$$
\{ |y_i| \leq w_i, \quad i \in M \quad (4 - 2) \}
$$

We call Model (2) the primary problem ($LP(w)$), and Model (4) the dual problem ($DLP(w)$).

In Model (4), the variable $|y_i|$ indicates the unit return (profit) that individual $i$ expects to obtain for changing his/her original opinion to the consensus opinion. On the basis of dual theory, the unit return is referred as a shadow value or shadow profit. We show in more detail later that if $y_i$ is a negative value, it means that the individual’s original opinion $o_i$ is smaller than the optimal consensus opinion $o'$; if $y_i$ is a positive value, it means that the individual’s original opinion $o_i$ is larger than the consensus opinion $o'$. Hence, $y_i < 0$ denotes a negative unit return for the individual to change his/her opinion to consensus opinion when it is smaller than the optimal consensus opinion, while $y_i > 0$ denotes that the positive unit return for changing his/her opinion toward the consensus opinion when it is larger than the optimal consensus opinion. In Model (4), the restriction condition (4–1) can be understood as restricting the sum of all the unit returns (shadow profits) to be equal to 0; the restriction condition (4–2) can be understood as restricting the unit return $|y_i|$ paid to $i$ to not exceed the cost $w_i$. 

paid by the moderator, or restricting the upper limit of the unit return \( y_j \) paid to \( d_i \) to the cost \( w_i \) paid by the moderator.

The optimal objective function \( \psi = \sum_{i=1}^{m} y_i (o_i - o') \) is considered to be the total return that is expected by all individuals for changing their opinions toward the consensus. It is obvious that what all the individuals want is for the return to be as large as possible. The value \( \max \psi \) denotes the maximum return of all the individuals. Because the “return” is actually a shadow price, its real meaning is referred to as the “expected return” but not the true return.

### 4. Relation between the primal problem \( \text{LP}(w) \) and its dual \( \text{DLP}(w) \)

Given the analysis above, the following three questions arise naturally:

- (4.1a) What is the relation between the maximum return \( \max \psi \) of all the individuals and the minimum cost \( \min \phi \) of the moderator?
- (4.1b) Do the consensus opinion and the unit return (shadow profit) have practical significance (do optimal solutions of Models (2) and (4) exist)?
- (4.1c) What are the connections among the unit return \( y_j \), \( j \in M \), unit cost \( w_i \), \( i \in M \), the individual’s original opinion \( o_i \), \( i \in M \) and the optimal consensus opinion \( o' \)?

In the following, the Principles of Weak Duality and Optimality Criterion (Jensen & Bard, 2003) answer Question (4.1a); the Principles of Sufficient Optimality Criterion and Strong Duality (Jensen & Bard, 2003) answer Question (4.1b); and Theorems 1–3 answer Question (4.1c).

#### 4.1. The Principle of Weak Duality

Let \( X \) be a primal feasible solution of the primal problem \( \text{LP}(w) \), and \( \phi(X) \) the corresponding value of the primal function that is to be minimized. Let \( Y \) be a dual feasible solution of the dual problem \( \text{DLP}(w) \), and let \( \psi(Y) \) the corresponding value of the dual function that is to be maximized. Then the objective value \( \phi(X) \) for a feasible solution to the primal problem \( \text{LP}(w) \) will always be larger than or equal to the objective value \( \psi(Y) \) for a feasible solution to the dual problem \( \text{DLP}(w) \). That is, \( \phi(X) \geq \psi(Y) \).

The economic meaning implied in the Principle of Weak Duality is that the total return \( \psi \) expected by all the individual DMs for changing their opinions is less than or equal to the cost \( \phi \) paid by the moderator for reaching a consensus. That is, the cost the moderator incurs can sufficiently satisfy the total demand all the individuals collectively claim, where we suppose that all individual DMs are relatively rational and do not claim more than they expect. Clearly, this conclusion holds in many real-life situations.

#### 4.2. The Principle of Optimality Criterion

If both the primal problem \( \text{LP}(w) \) and the dual problem \( \text{DLP}(w) \) have optimal feasible solutions, then the two optimal objective values are equal. That is, \( \max \psi = \min \phi \).

The economic meaning implied in the Principle of Optimality Criterion is that the total return \( \max \psi \) expected by all the individual DMs for changing their opinions equals the cost \( \min \phi \) paid by the moderator for reaching a consensus. In other words, the optimal result of reaching consensus is that all the individuals obtain the maximum return that they expect, while the moderator pays the minimum resources that he/she is prepared to pay.

#### 4.3. The Principle of Sufficient Optimality Criterion

If \( \mathbf{X}^* \) and \( \mathbf{Y}^* \) are feasible solutions to the primal problem \( \text{LP}(w) \) and its dual problem \( \text{DLP}(w) \), respectively, and if the primal objective function \( \sum_{i=1}^{m} w_i (u_i + v_i) \) and the dual objective function \( \sum_{i=1}^{m} (o_i - o') y_i \) satisfy \( \sum_{i=1}^{m} w_i (u'_i + v'_i) = \sum_{i=1}^{m} (o_i - o') y_i \), then \( \mathbf{X}^* \) and \( \mathbf{Y}^* \) are the optimal solutions to \( \text{LP}(w) \) and \( \text{DLP}(w) \), respectively.

The economic meaning implied in the Principle of Sufficient Optimality Criterion is that if the total return \( \max \psi \) expected by all individual DMs for changing their opinions is equal to the cost \( \min \phi \) paid by the moderator for reaching a consensus, then the consensus opinion and the unit return (shadow profit) have practical economic meaning (optimal solutions exist), or the economic meaning can be interpreted.

#### 4.4. The Principle of Strong Duality

If either the primal problem \( \text{LP}(w) \) or the dual problem \( \text{DLP}(w) \) has an optimal feasible solution, then so does the other problem and the two optimal objective values are equal. That is, \( \max \psi = \min \phi \).

The economic meaning implied in the Principle of Strong Duality is that:

- once a consensus is reached (i.e., the optimal solution of the primal problem exists and hence the consensus opinion exists), the unit returns expected by all the individual DMs for changing their opinions have economic meaning (the shadow profit exists);
- if the unit returns expected by all individual DMs for changing their opinions are solved (i.e., the optimal solution to the dual problem exists and hence the shadow profit exists), then the optimal solution to the primal problem exists and hence consensus has been reached and vice versa;
- as long as either the unit return or the consensus opinion has been solved, the total return expected by all the individual DMs for changing their opinions is equal to the cost paid by the moderator for reaching the consensus.

Let us now look at an example that further interprets the relationship between the primal problem Model (2) and its dual problem Model (4). The data are originated from the reference by Ben-Arieh et al. (2009). Example 1. Suppose there are four DMs \( d_1, d_2, d_3 \) and \( d_4 \) in a GDM and their corresponding opinions are \( o_1 = 0, o_2 = 3, o_3 = 6 \) and \( o_4 = 10 \), respectively. We also suppose that the unit cost paid to the four DMs by the moderator are \( w_1 = 1, w_2 = 2, w_3 = 3 \) and \( w_4 = 1 \), respectively, and the consensus opinion is supposed to be \( o' \). The optimal consensus decision making model based on minimum cost is constructed as follows:

\[
P(w_1) : \quad \begin{align*}
\min \quad & \phi = 1 \cdot |o' - 0| + 2 \cdot |o' - 3| + 3 \cdot |o' - 6| \\
& + 1 \cdot |o' - 10| \\
\text{s.t.} \quad & o' \in O \quad (5-1)
\end{align*}
\]

The unique solution \( o' = 6 \) to Model (5) can be solved easily. By letting \( o' - o_i = u_i + v_i, o' - o_i = u_i - v_i, i = 1, 2, 3, 4 \), Model (5) is equivalent to the following linear programming model:

\[
\text{LP}(w_1) : \quad \begin{align*}
\min \quad & \phi = u_1 + v_1 + 2u_2 + 2v_2 + 3u_3 + 3v_3 + u_4 + v_4 \\
\text{s.t.} \quad & \begin{cases}
0' - u_1 + v_1 = 0 \\
0' - u_2 + v_2 = 3 \\
0' - u_3 + v_3 = 6 \\
0' - u_4 + v_4 = 10
\end{cases} \\
& o' \in O, u_i, v_i \geq 0, \quad i = 1, 2, 3, 4 \quad (6-2)
\end{align*}
\]
The unique solution to Model (6) is $\mathbf{X}^* = (6 6 0 3 0 0 0 0 4)^T$, and the optimal value of the objective function of Model (6) is $\min \phi = 16$, which means that the total cost paid by the moderator is 16. Obviously, the optimal consensus opinion $o^*$ to Model (6) is $o^* = 6$. Then the deviation between 4 individuals’ opinions and the consensus opinion are $o_1 - o^* = -6$, $o_2 - o^* = -3$, $o_3 - o^* = 0$ and $o_4 - o^* = 4$, respectively.

The dual problem of Model (6) is

$$DLP(w_1): \max \quad \psi = 3y_1 + 6y_2 + 10y_4 - 6y_1 + y_2 + y_3 + y_4$$

s.t.

$$\begin{cases}
  y_1 < 1 \\
  y_2 < 2 \\
  y_3 < 3 \\
  y_4 < 1
\end{cases}$$

(7)

Model (7) also has a unique solution $\mathbf{Y}^* = (-1 -2 2 1)^T$, and its optimal objective is $\max \phi = 16$. This means that the unit return by 4 individual DMs are respectively $y_1 = -1$, $y_2 = -2$, $y_3 = 2$ and $y_4 = 1$, and the total return expected by all these individuals are 16. The total maximum return obtained by Model (7) is equal to the total minimum compensation obtained from Model (6). In model (6), the opinions of the first and second DMs are both smaller than the consensus opinion 6. So, the corresponding unit returns or shadow price ($y_1 = -1$ and $y_2 = -2$) of the individual $d_1$ and $d_2$ are both negative; while the opinions of the third and fourth DMs are equal or greater than the consensus opinion 6, respectively. So, the corresponding unit returns or shadow prices ($y_3 = 2$ and $y_4 = 1$) of the individual $d_3$ and $d_4$ are both positive.

Next, we will construct a sufficient and necessary condition under which the unit return or shadow price $|y_j|$ expected by individual $d_j$ is equal to the unit cost or compensation $w_j$ paid by moderator to the individual $d_j$. That is, if $o^*$ is the optimal solution to $LP(w)$, and if there exist a $t_0 \in M$ such that $o_{t_0} < o^* < o_{t_0+1}$, and

$$\sum_{j=t_0}^{t_0+1} w_j = \sum_{j=t_0+1}^{m} w_j$$

(8)

if and only if $DLP(w)$ have optimal solutions, and one of the optimal solutions is $(-w_1, \ldots, -w_{t_0}, w_{t_0+1}, \ldots, w_m)^T$.

Proof. For the reason that $o^*$ is the optimal solution to primal problem $LP(w)$, there must exist a $t_0 \in M$ such that $o_{t_0} < o^* < o_{t_0+1}$. Without loss of generality, we suppose that $o_1 \leq o_2 \leq \cdots \leq o_m$. Obviously, there must exist a $t \in M$ such that $o_t < o^* < o_{t+1}$.

- If $0 < o^* < o_t$ holds true for all $o_t \in M$, then there must exist an $o^*$ such that $0 < o^* < o_t < o_t$. So, we have $|o_t - o^*| < |o_t - o_t|$, and $\sum_{j=t_0}^{t_0+1} w_j |o_t - o^*| < \sum_{j=t_0+1}^{m} w_j |o_t - o^*|$, which contradicts the hypothesis that $o^*$ is the optimal solution to $LP(w)$ problem.
- If $o^* > o_t$ holds true for all $o_t \in M$, then there must exist an $o^*$, such that $o^* > o_t > o_t$. Obviously, we have $|o_t - o^*| < |o_t - o_t|$, and $\sum_{j=t_0+1}^{m} w_j |o_t - o^*| < \sum_{j=t_0}^{t_0} w_j |o_t - o^*|$, which also contradicts the hypothesis that $o^*$ is the optimal solution to $LP(w)$ problem.

The objective function of $LP(w)$ attains its minimum on $o^*$, and its minimum value satisfies

$$\min \phi = \sum_{t=t_0}^{t_0+1} w_j (o^* - o_t) = \sum_{t=t_0}^{t_0+1} w_j (o_t - o_t) + \sum_{t=t_0+1}^{m} w_j (o_t - o^*)$$

$$= \left(\sum_{t=t_0}^{t_0+1} w_j - \sum_{j=t_0+1}^{m} w_j\right) o^* + \left(\sum_{t=t_0}^{t_0+1} w_j o_t + \sum_{j=t_0+1}^{m} w_j o_t\right)$$

When condition (8) holds true, the minimum value of $LP(w)$ is

$$\min \phi = \left(-\sum_{t=t_0}^{t_0+1} w_j o_t + \sum_{t=t_0+1}^{m} w_j o_t\right).$$

According to the Principle of Strong Duality, there must exist an optimal solution to $DLP(w)$ which can be obtained by letting

$$y_j = -w_i, \quad i = 1, 2, \ldots, t_0; \quad y_j = w_j, \quad j = t_0 + 1, \ldots, m$$

(9)

Condition (9) which satisfies the restrictions (3–1) and (3–2), means that it is a feasible solution to the dual problem $DLP(w)$. Let $y_i (i \in M)$ be an optimal solution to the dual problem $DLP(w)$, then we have $-\sum_{t=t_0}^{t_0+1} w_j o_t + \sum_{t=t_0+1}^{m} w_j o_t \leq \sum_{t=t_0+1}^{m} o_t y_t$. According to Principle of the Weak Duality Theory, we have $\sum_{t=t_0}^{t_0+1} o_t y_t \leq (\sum_{t=t_0+1}^{m} w_j o_t + \sum_{t=t_0+1}^{m} w_j o_t)$. Thus, $\sum_{t=t_0+1}^{m} w_j o_t = \sum_{t=t_0+1}^{m} o_t y_t$. That is, Eq. (9) is an optimal solution to $DLP(w)$.

It is easy to prove that the proof process above is reversible. This leads to Theorem 1.

Theorem 1. Suppose that the individual DMs’ opinions satisfy $o_1 \leq \ldots \leq o_t \leq \ldots \leq o_m$. If $o^*$ is the optimal solution to the primal problem $LP(w)$, then there must exist a $t_0 \in M$ such that $o_{t_0} < o^* < o_{t_0+1}$, and

$$\sum_{t=1}^{t_0} w_j = \sum_{j=t_0+1}^{m} w_j$$

(10)

if and only if $DLP(w)$ have optimal solutions, and one of the optimal solution is $(-w_1, \ldots, -w_{t_0}, w_{t_0+1}, \ldots, w_m)^T$.

Aiming at problem (4.1c) above, Theorem 1 actually presents a sufficient and necessary condition that an individual DM’s unit return or shadow profit is equal to the real unit compensation paid by the moderator. That is, when the sum of one part of individual DMs’ unit returns is exactly equal to the sum of the rest part of individual DMs’ unit returns, the unit return or shadow price $|y_j|$ expected by individual $d_j$ for abandoning his/her own original opinion is equal to the unit cost or compensation $w_j$ that would be paid by the moderator to individual $d$.

Theorem 2. The statement $y_j = -w_i < 0$ holds when $o^* > o_i$ holds; and $y_j = w_i > 0$ holds when $o^* < o_i$ holds. This denotes $|y_j| = w_i$ holds when $o^* \neq o_i$ holds; $-w_i \leq y_j \leq w_i$ when $o^* = o_i$ holds.

Proof. If $o^* > o_t$, then $o^* > o_t$. For the reason that $u_t \geq 0$, $n_t \geq 0$, $u_t n_t = 0$, we have $n_t \geq 0$, $u_t > 0$. According to the principle of complementary slackness, we have $y_j = -w_i$. Likewise, if $o^* < o_t$, we have $y_j = w_i > 0$; if $o^* = o_i$ holds true, we also have $-w_i \leq y_j \leq w_i$.

Theorem 2 implies that when the consensus opinion is greater than an individual DM’s opinion, then the value of the expected unit return of this individual DM is negative; and when the consensus opinion is lower than an individual DM’s opinion, the value of the expected unit return of this individual DM is positive. Under these two circumstances, the absolute value of the expected unit return of the individual DM must be equal to the real compensation paid to him/her.

Theorem 2 actually presents a condition under which a unit return expected by an individual DM is equal to a unit compensation paid by the moderator. Thus we have the following corollary.

Corollary 1. If $o^* \notin \{o_1, \ldots, o_m\}$, then $|y_j| = w_i$ hold for all $i \in M$.

This result means that if the optimization solution to $LP(w)$ is not equal to any of the individual DMs’ opinions, then the moderator has to pay more effort and cost to persuade the individuals to change their opinions. And only when the individual DMs’ expected unit returns $|y_j|$ attains the upper limit value of the shadow profit $w_i (|y_j| = w_i, i \in M)$, does the optimal consensus opinion solve.
Theorem 3. The statement \( o^* \leq o_i \) holds when \( y_i = w_i; \) \( o^* \geq o_i \) holds when \( y_i = -w_i; \) and \( o^* = o_i \) holds when \( -w_i < y_i < w_i. \)

Proof. If \( y_i = w_i, \) then there must be \( -y_i < w_i. \) According to the principle of complementary slackness, we have \( u_i = 0, \) and thus \( o^* = o_i + v_i \leq o_i; \) If \( y_i = -w_i, \) then there must be \( y_i < w_i. \) According to the principle of complementary slackness, we have \( v_i = 0, \) and thus \( o^* = o_i + v_i \geq o_i; \) If \( -w_i < y_i < w_i, \) we have \( u_i - v_i = 0 \) according to the principle of complementary slackness, and thus we have \( o^* = o_i. \)

Theorem 3 indicates that if an individual DM's expected unit return is lower than the real unit compensation paid by the moderator, the individual DM's opinion must be equal to the consensus opinion. This result actually provides a condition under which an individual DM's opinion is exactly equal to the consensus opinion.

5. A consensus model based on the minimum cost on the k-th DM

In some group decision making, the moderator proposes a relatively ideal opinion after many rounds of negotiation and communication. But he/she has to consider the following two cases:

(i) Most individual DMs feel that the moderator's opinion is within the deviation limits of their own opinions. So they are happy to accept this opinion. And the moderator does not need to pay out any compensation. Mathematically, there exists an allowed deviation \( \varepsilon_j \) such that the deviation \( f_j(o^*) \) between the opinion of the moderator and the opinion of the j-th individual DM satisfies \( f_j(o^*) \leq \varepsilon_j. \) In this case, we also say that opinion \( o_j \) is acceptable.

(ii) Only a few individual DMs insist on their opinions because they perceive that there exist divergences between their opinions and the moderator's opinion. So the moderator has to make additional effort to persuade these individuals to accept his/her opinion. Mathematically, the moderator pays individual DM \( d_k \) an amount \( r_kf_k(o^*) \) of compensation according to the deviation \( f_k(o^*) = |o_k - o^*| \) between the moderator's opinion and the k-th individual DM's opinion, where \( r_k \) is the unit compensation paid to individual \( d_k \) by the moderator.

In this section, we will discuss a consensus model based on the minimum cost of the k-th individual DM.

5.1. The \( e \) restriction problem \( P_s(e) \) based on the minimum cost of the k-th individual DM and its dual problem \( DP_s(e) \)

For the sake of convenience of communication, all the individual DMs' opinions are acceptable except for the individual DM \( d_k. \) From the viewpoint of the moderator, on the one hand, he/she hopes that the smaller the value of \( f_k(o^*) = |o_k - o^*| \) is, the closer the deviation between \( o^* \) and \( o_k \), and the lower the total compensation \( r_kf_k(o^*) \) that needs to be paid according to the value of the unit cost \( r_k \) and the value of deviation \( f_k(o^*). \) On the other hand, the moderator does not need to pay any compensations if the moderator's opinion \( o^* \) is allowed in the deviation limits of the most individual DMs' opinions. I.e., the restriction conditions \( f_j(o^*) \leq \varepsilon_j \) hold for all \( j \in M, j \neq k. \)

Thus, a nonlinear programming model \( P_s(e) \) is constructed as follows, where its objective function satisfies that the total compensation \( r_kf_k(o^*) \) is as small as possible, and the restriction condition \( f_j(o^*) \) is limited within the allowed deviation \( \varepsilon_j, j \in M, i \neq k. \)

\[
\begin{align*}
\text{min} \quad & Z = r_kf_k(o^*) \\
\text{s.t.} \quad & f_j(o^*) \leq \varepsilon_j, \quad j \in M, \quad j \neq k \quad (11-1)
\end{align*}
\]

In \( P_s(e) \) Model, for \( f_k(o^*) = |o_k - o^*|, k = 1, 2, \ldots, m, \) there exist \( u_k, v_k \geq 0 \) such that \( u_k + v_k = 0, \) and \( |o_k - o^*| = u_k + v_k. \) In fact, by letting \( u_k = |o_k - o^*| + (o_k - o^*)/2, \) \( v_k = |o_k - o^*| - (o_k - o^*)/2, \) the above conditions hold true.

\( P_s(e) \) is equivalent to the following linear programming model \( LP_s(e): \)

\[
\begin{align*}
\text{min} \quad & Z = r_ku_k + r_kv_k \\
\text{s.t.} \quad & o^* = 0 \quad (12-1) \\
& o^* \leq o_j + \varepsilon_j, \quad j \in M, \quad j \neq k \quad (12-2) \\
& o^* \geq o_j - \varepsilon_j, \quad j \in M, \quad j \neq k \quad (12-3) \\
& -u_k + v_k = o_k \quad (12-4) \\
& o^* \geq 0, \quad u_k \geq 0, \quad v_k \geq 0 \quad (12-5)
\end{align*}
\]

Let's reconsider the feasible field of model (12). For all \( j \in M, j \neq k, \) there must exist \( s, t \in M, \) \( s \neq k, \) such that the set \( \{o^* \leq o_j + \varepsilon_j, o^* \geq o_j - \varepsilon_j, j \in M, j \neq k\} \)

is equal to the set \( \{o^* \leq o_j + \varepsilon_j, o^* \geq o_j - \varepsilon_j, j \in M, j \neq k\} \)

where \( o^* + \varepsilon_j = \min(o_j + \varepsilon_j, j \in M, j \neq k), o^* - \varepsilon_j = \max(o_j - \varepsilon_j, j \in M, j \neq k). \)

Therefore, model (12) can be further simplified into

\[
\begin{align*}
\text{min} \quad & Z = r_ku_k + r_kv_k \\
\text{s.t.} \quad & o^* = 0 \quad (13-1) \\
& o^* \leq o_j + \varepsilon_j, \quad t \in M, \quad t \neq k \quad (13-2) \\
& o^* \geq o_j - \varepsilon_j, \quad s \in M, \quad s \neq k \quad (13-3) \\
& -u_k + v_k = o_k \quad (13-4) \\
& o^* \geq 0, \quad u_k \geq 0, \quad v_k \geq 0 \quad (13-5)
\end{align*}
\]

According to the theory of linear programming, the dual problem of Model (13) is given as follows:

\[
\begin{align*}
\text{max} \quad & S = (o_j - o_k)x_j - (o_j + \varepsilon_j)x_j + o_kx_k \\
\text{s.t.} \quad & x_k - x_j + x_k \leq 0 \quad (14-1) \\
& -x_k \leq r_k \quad (14-2) \\
& x_j \leq r_k \quad (14-3) \\
& x_k, x_j, x_k \geq 0 \quad (14-4)
\end{align*}
\]

Suppose that the optimal solution \( X^* = (o^*_j, \ v_j, \ v_j)^* \) to the primal problem (13) exists. Then there must be \( o^* > 0. \) According to the principle of complementary slackness of the primal–dual linear programming, Model (14) must have its optimal solution, and the equality of its restriction condition \( x_k - x_j + x_k \leq 0 \) holds true, that is,

\[
x_k - x_j + x_k = 0 \quad (15)
\]

We next discuss the economic meaning of Model (14) by simplifying its objective function. According to the principle of complementary slackness, the restriction conditions (13-2) and (13-3) are helpful to simplify the dual Model (14). Suppose that the optimal solution \( X^* = (o^*_j, \ v_j, \ v_j)^* \) to the primal problem (13) exists, then the optimal solution to the dual problem (14) also exists according to the Sufficient Optimality Criterion of the primal–dual linear programming problem (As proved in next section).

(i) If the restriction conditions satisfy \( o^* < o_j + \varepsilon_j, \) \( o^* > o_j - \varepsilon_j, \) then its dual variables satisfy \( x_k = x_j = 0. \) From Eq. (15), it follows that \( x_k = 0. \) Thus, the optimal objective function of the dual problem (14) is equivalent to \( S = 0. \)
(ii) If the restriction conditions satisfy \( o' = o_{ij} \), \( o' > o_{ij} \), then its dual variables satisfy \( x_k = 0 \). And so, Eq. (15) is equivalent to \(-x_k + x_0 = 0\). Thus, the optimal objective function of the dual problem (14) is equivalent to \( S = (o_k - o_{ij})x_k = (o_k - o')x_k \).

(iii) If the restriction conditions satisfy \( o' < o_{ij} \), \( o' = o_{ij} \), then its dual variables satisfy \( x_k = 0 \). Then Eq. (15) is equivalent to \( x_k = x_0 = 0 \). Thus, the optimal objective function of dual problem (14) is equivalent to \( S = (o_k - o_{ij})x_k = (o_k - o')x_k \).

(iv) If the restriction conditions satisfy \( o' = o_{ij} \), \( o' = o_{ij} \), then we have \( o_{ij} + e_i = o_{ij} - e_i \). Thus, the optimal objective function of the dual problem (14) is equivalent to \( S = o'x_k - o'x_k + x_0o_k = (o_k - o')x_k \).

In case (i), for \( x_k = 0 \), we have that the inequalities of restriction conditions (14–2) and (14–3) in the dual problem (14) hold true. According to the principle of complementary slackness, the restriction conditions (13–4) in primal problem (13) is equivalent to \( o' = o_k \). Therefore, the optimal objective function in dual problem is equivalent to \( S = (o_k - o')x_k \).

This leads to the following result.

**Theorem 4.** The equation \( (o_k - e_i)x_k - (o_k + e_i)x_k + o_kx_k = (o_k - o')x_k \) always holds true under the condition that the objective function in Model (14) attains its maximum value.

Given all that, Model (14) is equivalent to

\[
\text{DLP}_e(x): \quad \text{max} \quad S = (o_k - o')x_k \\
\text{s.t.} \quad \begin{cases} x_k - x_0 + x_0 = 0 & (16-1) \\ -x_k \leq r_k & (16-2) \\ x_k \leq r_k & (16-3) \\ x_k = x_0 \geq 0 & (16-4) \end{cases}
\]

If the primal Model (13) has its optimization solution, then its dual Models (14) and (16) are equivalent.

The economic meaning of Model (16) is given as follows: In GDM, the individual DM \( d_k \) may insist on his/her opinion, or may care about him/her being recognized as serious. However, he/she has to sacrifice his/her interests to get closer to the moderator's opinion. So the moderator needs to compensate for his/her loss. From the viewpoint of the individual DM \( d_k \), he/she expects the maximum compensation from the moderator. In model (16), the meaning of the variable \( x_k \) can be understood as a unit return (profit) expected by the individual DM \( d_k \) for changing his/her original opinion. According to the primal–dual linear programming theory, \( x_k \) is actually a shadow price or profit.

In Model (16), the restriction condition (16–1) can be understood as restricting the sum of all the unit returns (shadow profits) to be equal to 0; the restriction conditions (16–2) and (16–3) can be understood as restricting the unit return \( x_k \) paid to \( d_k \) to not exceed the cost \( r_k \) paid by the moderator, or restricting the upper limit of the unit return \( x_k \) paid to \( d_k \) to the cost \( r_k \) from the moderator. The optimal objective function \( \max S = (o_k - o')x_k \) can be considered as the maximum compensation expected by the individual DM \( d_k \) according to the product of the shadow profit and the deviation between \( o' \) and \( o_k \).

### 5.2. The relation between the primal problem \( \text{LP}_e(x) \) and its dual problem \( \text{DLP}_e(x) \)

Given the analysis of Section 5.1, the following three questions arise naturally:

1. **(5.2a)** What is the relation between the maximum return \( \max S \) of the individual DM \( d_k \) and the minimum cost \( \min Z \) paid to/him by the moderator?

2. **(5.2b)** Do the consensus opinion and the unit return (shadow profit) have practical significance (do optimal solutions of Models (13) and (14) exist)?

3. **(5.2c)** What are the connections among the unit return \( x_k \), unit cost \( r_k \), the \( k \)-th individual DM’s opinion \( o_k \), and the optimal consensus opinion \( o' \)?

In the following, the principles of Weak Duality and Optimality Criterion (Jensen & Bard, 2003) answer Question (5.2a); the Principles of Sufficient Optimality Criterion and Strong Duality (Jensen & Bard, 2003) address Question (5.2b) (where we omit the explanation of economic meanings of theses principles); and Theorems 5 considers Question (5.2c).

#### 5.2.1. The Principle of Weak Duality

Let \( X \) be a primal feasible solution of the primal problem \( \text{LP}_e(x) \), and \( Z(X) \) the corresponding value of the primal function that is to be minimized. Let \( Y \) be a dual feasible solution of the dual problem \( \text{DLP}_e(x) \), and \( S(Y) \) the corresponding value of the dual function that is to be maximized. Then the objective value \( Z(X) \) for a feasible solution to the primal problem \( \text{LP}_e(x) \) will always be larger than or equal to the objective value \( S(Y) \) for a feasible solution to the dual problem \( \text{DLP}_e(x) \). That is, \( S(Y) \leq Z(X) \).

#### 5.2.2. The Principle of Optimality Criterion

If both the primal problem \( \text{LP}_e(x) \) and the dual problem \( \text{DLP}_e(x) \) have optimal feasible solutions, then the two optimal objective values are equal. That is, \( \max S = \min Z \).

#### 5.2.3. The Principle of Sufficient Optimality Criterion

If \( \bar{X} = (o', \bar{u}, \bar{v}_0)^T \) and \( \bar{Y} = (x', \bar{X}, \bar{X})^T \) are feasible solutions to the primal problem \( \text{LP}_e(x) \) and its dual problem \( \text{DLP}_e(x) \), respectively, and if the primal objective function \( r_ku_k + r_vv_k \) and the dual objective function \( (o_k - o')r_k \) satisfy \( r_ku_k + r_vv_k = (o_k - o')r_k \), then \( \bar{X} \) and \( \bar{Y} \) are the optimal solutions to \( \text{LP}_e(x) \) and \( \text{DLP}_e(x) \), respectively.

#### 5.2.4. The Principles of Strong Duality

If either the primal problem \( \text{LP}_e(x) \) or the dual problem \( \text{DLP}_e(x) \) has an optimal feasible solution, then so does the other problem and the two optimal objective values are equal. That is, \( \max S = \min Z \).

According to the principle of complementary slackness of the primal–dual problem, the following Theorem 5 can be easily established.

**Theorem 5.** If both \( \text{LP}_e(x) \) and \( \text{DLP}_e(x) \) have their optimization solutions, then the relation between the unit return or the shadow profit \( x_k \) expected by \( d_k \) and the compensation \( r_k \) paid by the moderator is given as follows:

1. **(i)** \( x_k - x_0 + x_0 = 0 \) holds true.
2. **(ii)** \( x_k + r_k < 0 \) holds true when \( o' > o_k \).
3. **(iii)** \( x_k = x_0 \) holds true when \( o' = o_k \).

### Proof.
The details are similar to those given in the proof of Theorem 2, and are omitted here. □
the value of unit return or shadow profit expected by $d_i$ is positive. In either of these cases, the absolute value of unit return or shadow profit expected by $d_i$ must be equal to the real unit cost or compensation paid by the moderator. This can also be understood as the reason why $d_i$ has to modify his/her opinion $o_i$ into the consensus opinion $o^*$, so he/she expects the greatest unit return or shadow profit $|x_i| = r_i$.

Theorem 5 (iii) indicates that, when the unit return or shadow profit expected by $d_i$ is lower than the real unit cost or compensation paid by the moderator, the $k$-th individual DM’s opinion must be equal to the consensus opinion. This can also be understood as the reason why $d_i$ is willing to accept his/her unit return or shadow profit lower than the unit cost or compensation $r_i$, because his/her opinion is identical to that of the moderator’s.

5.3. Numerical examples

This section presents a numerical example to show the relationship between the primal model (12) and its dual model (16) and the validity of the theoretical analysis of Theorem 5.

Example 2. Suppose that there are four DMs $d_1$, $d_2$, $d_3$, $d_4$ in a GDM and their corresponding opinions are $o_1 = 0$, $o_2 = 3$, $o_3 = 6$ and $o_4 = 10$, respectively. Except for the third individual, the allowed deviations of the opinions of other three DMs are $e_1 = 5$, $e_2 = 4$, $e_4 = 6$, respectively. We also suppose that the unit cost paid to the third individual by the moderator is $w_3 = 3$ and the consensus opinion is supposed to be $o^*$. The optimal consensus decision making model based on minimum cost is constructed as follows:

\[
P(w_2): \quad \min Z = 3 + |o' - 6| \tag{17-1}
\]

\[
\text{s.t.} \quad \begin{cases} |o'| \leq 5 \\ |o' - 3| \leq 4 \\ |o' - 10| \leq 6 \\ o' \geq 0 \\
\end{cases} \tag{17-2}
\]

By letting $|o' - 6| = u_3 + v_1$ and $o' - 6 = u_3 - v_1$, Model (17) can be transformed into the following linear programming model:

\[
LP_2(w_2): \quad \min Z = 3u_3 + 3v_1 \tag{18-1}
\]

\[
\text{s.t.} \quad \begin{cases} 4 \leq u_3 \leq 5 \\ o' - u_3 + v_3 = 6 \\ o' \geq 0, \quad u_3 \geq 0, \quad v_3 \geq 0 \quad (18-3)
\end{cases}
\]

Model (18) has the unique solution $X^* = (5 0 1)^T$, and its minimum objective function is $\min Z = 3$. So, the consensus opinion solved from Model (18) is $o^* = 5$, and the deviations between the consensus opinion and the four individuals’ opinions are respectively $o' - o_1 = 5$, $o' - o_2 = 2$, $o_3 - o^* = 1$ and $o_4 - o^* = 5$. The total cost paid by the moderator for arriving at the consensus is 3.

The dual model of the primal Model (18) is

\[
DLP_2(w_2): \quad \max S = 4x_4 - 5x_1 + 6x_3 \tag{19-1}
\]

\[
\text{s.t.} \quad \begin{cases} x_1 - x_1 + x_3 = 0 \\ -x_3 \leq 3 \\ x_1 \leq 3 \\ x_1, \quad x_4 \geq 0 \quad (19-3)
\end{cases}
\]

Model (19) also has the unique solution $X^* = (3 3 0 0)^T$, and its maximum objective function is $\max S = 3$. The third individual’s unit return and total compensation is $x_3 = 3$ and $S = 3$. The total return expected by the third individual is equal to the total compensation paid by the moderator. The opinion $o_3 = 6$ of the third individual is greater than the consensus opinion $o^* = 5$. The unit return or the shadow profit expected by the third individual is equal to the compensation paid by the moderator ($x_3 = w_3 = 3$).

6. The link between $P(w)$ and $P_k(a)$

Both $P(w)$ model and $P_k(a)$ model may be solved for different optimal consensus opinions. Next, let us explore the conditions under which the optimal consensus opinions solved out of these two models are the same.

In $P_k(a)$ Model (11), for a given point $o^*$, we use the symbol $P_k(o^*)$ to represent the problem $P_k(a)$, where $v_i = v_j = f_j(o^*)$, $j \in M$, $j \neq k$.

Theorem 6 ([Vira & Haimes, 1983]). For any given $k$, let $o^*$ be the optimal consensus opinion solved out of $P_k(o^*)$. Then there exists $w^* \in W$, $w^* \geq 0$, such that $o^*$ is also the optimal consensus opinion solved out of $P(w^*)$.

Theorem 7. If there exists $w \in W$ such that $o^*$ is the optimization solution to $P(w)$, then either (1) or (2) of the following holds true:

(1) if $w_k > 0$, then $o^*$ is also the optimization solution to $P_k(o^*)$;
(2) if $o^*$ is the unique optimization solution to $P(w)$, then $o^*$ is also the optimization solution to $P_k(o^*)$, for all $k$, $k \in M$.

Proof. Let $o^*$ be the optimization solution to $P(w)$ for some $w \in W$. Then, for all $o' \in O$, we have

\[
\sum_{j=1}^{m} w_j [f_j(o^*) - f_j(o')] \geq 0 \tag{20}
\]

(1) Suppose that $o^*$ is not the optimization solution to $P_k(o^*)$, then there exists an $o \in O$ such that $f_k(o) < f_k(o^*)$ and $f_j(o) \leq f_j(o^*)$, $j \neq k$. Since $w_k > 0$, by the hypothesis and $w_j \geq 0$, $j \in M$, $j \neq k$, we have

\[
w_k [f_k(o) - f_k(o^*)] + \sum_{j=k}^{m} w_j [f_j(o) - f_j(o^*)] < 0
\]

which contradicts inequality (20). Therefore, $o^*$ is also the optimization solution to $P_k(o^*)$.

(2) Suppose that $o^*$ is the unique optimization solution to $P(w)$. Then for all $o' \in O$, $o^* \neq o'$, we have

\[
\sum_{j=1}^{m} w_j [f_j(o^*) - f_j(o')] > 0 \tag{21}
\]

Assume that there exists a $k$ such that $o^*$ is not the optimization solution to $P_k(o^*)$. Then there must exist $o'$, satisfying $o' \neq o^*$, such that $f_k(o') < f_k(o^*)$ and $f_j(o') \leq f_j(o^*)$, $j \in M$, $j \neq k$. Thus we have

\[
\sum_{j=1}^{m} w_j [f_j(o') - f_j(o^*)] = \sum_{j=1}^{m} w_j [f_j(o') - f_j(o^*)] < 0
\]

which contradicts inequality (21). Therefore, for all $k$, $k \in M$, $o^*$ is the optimization solution to $P_k(o^*)$.

Theorem 8. Assume that there exists an optimal solution $o^*$ to $LP(w)$, and the restriction conditions of $LP_k(a)$ attain the upper limitations $e_j = |o^* - o_j|$, $j \in M$, $j \neq k$. Then for all $j \in M$, $o^*$ must be the unique optimal solution to $LP_k(a)$.

Proof. Assume that $o^*$ is an optimal solution to $LP(w)$, and the restriction conditions of $LP_k(a)$ attain the upper limitations $e_j = |o^* - o_j|$, $j \in M$, $j \neq k$. Without loss of generality, we assume $o_1 \leq o_2 \leq \cdots \leq o_m$. Obviously, there must exist $t \in M$, such that $o_t \leq o^* \leq o_{t+1}$. Otherwise,
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7. Conclusions and future research

A kind of consensus model regarding all individual DMs and a kind of consensus model regarding only one individual DM have been investigated in this paper: A minimum cost primal model and its dual model – a maximum return model for reaching greatest consensus – have been developed from the standpoint of all the individual DMs. Our results show that once a consensus is arrived at, the maximum return expected by all the individual DMs for changing their original opinions and the minimum cost paid by the moderator for persuading all individual DMs to accept the consensus opinion are equal; the existence of optimization solutions to the primal–dual models with all individual DMs implicates that both the consensus opinion and the shadow profits have economic significance. The optimality criterion and the principle of complementarity slackness of the primal–dual models clarify the interrelation between the unit return or shadow profit expected by each individual DM and the unit cost or compensation paid by the moderator.

An $\varepsilon$ restriction problem $P_\varepsilon(x)$ based on the minimum cost with a particular individual DM and its dual problem have also been explored from the standpoint of the particular individual DM. It is proven that once a consensus is arrived at, the maximum return expected by the particular individual DM for changing his/her original opinion and the minimum cost paid by the moderator for persuading this DM to accept the consensus opinion are equal; the existence of optimization solutions to the primal–dual models with respect to the particular individual DM explains that both the consensus opinion and the shadow profits have economic significance. The optimality criterion and the principle of complementarity slackness of the primal–dual models explain the interrelation between the unit return or shadow profit expected by the particular individual DM and the unit cost or compensation paid by the moderator. The close interrelation between these two kinds of consensus models is also investigated, showing that when certain conditions met, the optimal consensus opinions derived by these two models are identical.

This paper focuses on the cost model of consensus reaching. In our consensus models, distance measure can more intuitively reflect the deviation between the individual DMs' opinions and the consensus opinion. The equivalent linear programming model transformed by nonlinear programming model $P(w)$ (Ben-Arieh et al., 2009) has two merits: It is easier to get the optimal solution of a linear programming than that of a nonlinear programming; It is more convenient to analyze the economic significance of the primal–dual linear programming models. That is, we can not only build up relation between the minimum cost paid by the moderator and the maximum return expected by the individuals, but also explore the relation between the consensus opinion, the shadow profit expected by the individual, and the unit cost paid by the moderator. In a word, the economic analysis of our primal–dual linear programming model considers both sides of the interests: the moderator's cost and the individuals' return, so it conforms to the process of practical consensus reaching.

However, consensus reaching is so complex in many GDM that a simple model can not fully simulate the whole process of consensus. Consensus reaching consists of different stages, and the individual DMs' opinions may change gradually during each stage. Obtaining a satisfied consensus needs to consider different stages and different conditions of consensus decision making, the teams of Herrera-Viedma et al. (Alonso et al., 2013; Chiclana et al., 2013; Pérez et al., 2013, 2014) have done a series work on consensus in GDM, consensus measures, consensus and fuzzy ontologies, consensus software tools, and etc. In the future, we will construct a dynamic cost model to simulate the process of consensus reaching.

Acknowledgements

The authors are grateful to the editors and the three anonymous reviewers for their insightful comments and suggestions. In addition, this research was partly supported by the National Natural Science Foundation of China (71171115, 71173116, 70901043), the reform Foundation of Postgraduate Education and Teaching in Jiangsu Province (JGKTK0034), Qing Lan Project, Natural Science Foundation of Higher Education of Jiangsu Province of China under Grant (08KJD630002), the Project of Philosophy and Social Science Research in Colleges and Universities in Jiangsu (2012SJD630037), and the Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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