Eccentric connectivity index of graphs with subdivided edges

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Abstract
We consider four classes of graphs arising from a given graph via different types of edge subdivisions. We present explicit formulas expressing their eccentric connectivity index in terms of the eccentric connectivity index of the original graph and some auxiliary invariants.

Keywords: Eccentric connectivity index, subdivided graph.
1 Introduction

The eccentric connectivity index is a distance-related topological invariant whose potential of predicting biological activity of certain classes of chemical compounds made it very attractive for use in QSAR/QSPR studies. We refer the reader to a number of recent papers [7,11,12,13,15] that demonstrate its suitability for the task. In spite of its increasing importance, mathematical properties of this invariant have been largely left unexplored; only recently they started to attract more attention from the mathematical community [17,10,4,5,3]. An interested reader could consult a very useful summary of recent results by Ilić [9]. Most of the papers cited there were concerned with finding extremal values of the eccentric connectivity index in various classes of graphs, as well as with the study of its behavior under several graph products. The aim of this paper is to continue this line of research by studying the relationship between the eccentric connectivity index of a given graph and of four new graphs that arise via subdivision of its edges.

In the next section we define the types of edge subdivisions considered in the rest of the paper and quote some relevant results on the distances in such graphs. Section 3 presents the explicit formulas for the eccentric connectivity index of subdivided graphs in terms of the eccentric connectivity index of original graph and some auxiliary invariants. Finally, in Section 4 we discuss some possible directions for further research.

2 Definitions and preliminary results

All graphs in this paper are finite, simple and connected. For terms and concepts not defined here we refer the reader to any of several standard monographs such as, e.g., [8].

Let $G$ be a graph on $n$ vertices. We denote the vertex and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. For two vertices $u$ and $v$ of $V(G)$ their distance $d(u,v)$ is defined as the length of any shortest path connecting $u$ and $v$ in $G$. For a given vertex $u$ of $V(G)$ its eccentricity $\varepsilon(u)$ is the largest distance between $u$ and any other vertex $v$ of $G$. Hence, $\varepsilon(u) = \max_{v \in V(G)} d(u,v)$. The
eccentric connectivity index $\xi(G)$ of a graph $G$ is defined as

$$\xi(G) = \sum \delta_G(u) \varepsilon(u),$$

where $\delta_G(u)$ denotes the degree of vertex $u$ in $G$, i.e., the number of its neighbors in $G$.

It is often interesting to consider the sum of eccentricities of all vertices of a given graph $G$. We call this quantity the total eccentricity of the graph $G$ and denote it by $\zeta(G)$. Hence,

$$\zeta(G) = \sum_{u \in V(G)} \varepsilon(u).$$

For a $k$-regular graph $G$ we have $\xi(G) = k \zeta(G)$.

For a given graph $G$ its line graph $L(G)$ is the graph whose vertices are the edges of $G$. Two edges of $G$ incident to the same vertex are connected by an edge in $L(G)$. The line graph, although itself not a subdivision graph, plays a significant role in the definitions of subdivision graphs and in presenting our results in a more compact way.

Now we introduce the four types of graphs resulting from edge subdivision. Two of them, the subdivision graph and the total graph, belong to the folklore, while the other two were introduced in [2] and further investigated in [16]. We refer the reader to the latter reference for figures illustrating the subdivision graphs.

The subdivision graph (or simply subdivision) of a graph $G$ is a graph obtained by replacing each edge of $G$ by a path of length 2. Equivalently, an additional vertex is inserted in each edge of $G$. We denote the subdivision of $G$ by $S(G)$.

The total graph $T(G)$ of a graph $G$ is a graph on the vertex set $V(G) \cup E(G)$, and adjacency in $T(G)$ is inherited from both adjacency and incidence in $G$.

The triangle parallel graph of a graph $G$ is denoted by $R(G)$ and obtained by replacing each edge of $G$ by a triangle. Its vertex set is the union of $V(G)$ and $E(G)$, and its edge set is the union of the respective edge sets of $G$ and $S(G)$.

Finally, the line superposition graph $Q(G)$ of a graph $G$ is defined as a graph on the same vertex set as $S(G)$ whose edge set is the union of the edge sets of $S(G)$ and $L(G)$.

The following formulas for distances in subdivision graphs have been es-
tablished in [16].

**Lemma A.** For any \( v, v' \in V(G) \),

\[
\frac{1}{2} d_{S(G)}(v, v') = d_{T(G)}(v, v') = d_{R(G)}(v, v') = d_{Q(G)}(v, v') - 1 = d_G(v, v').
\]

**Lemma B.** For any \( e, e' \in E(G) \),

\[
\frac{1}{2} d_{S(G)}(e, e') = d_{T(G)}(e, e') = d_{R(G)}(e, e') = d_{Q(G)}(e, e') = d_{L(G)}(e, e').
\]

### 3 Main Results

In this section we present bounds and, when possible, exact formulas for the eccentric connectivity indices of subdivision graphs. We proceed roughly in the order of increased complexity, instead in the order in which the graphs were introduced.

#### 3.1 Subdivision

**Theorem 3.1** Let \( G \) be a connected graph. Then

(i) For each \( v \in V(G) \), \( \varepsilon_{S(G)}(v) = 2\varepsilon_G(v) \),

(ii) For each \( e \in E(G) \), \( 2\varepsilon_{L(G)}(e) \leq \varepsilon_{S(G)}(e) \leq 2\varepsilon_{L(G)}(e) + 1 \).

**Proof.** The first claim is obvious. To prove the second claim, we start from the fact that the distance between \( e \) and \( e' \) in \( S(G) \) is twice the distance between the same vertices in \( L(G) \) (Lemma B). Now for \( e' = uv \) we have

\[
\max\{d_{S(G)}(e, u), d_{S(G)}(e, v)\} \leq d_{S(G)}(e, e') + 1.
\]

On the other hand, for each \( v \in V(G) \) and \( e' = uv \), by the above inequality, \( d_{S(G)}(e, v) \leq d_{S(G)}(e, e') + 1 \). Therefore for \( e \in E(G) \), \( 2\varepsilon_{L(G)}(e) \leq \varepsilon_{S(G)}(e) \leq 2\varepsilon_{L(G)}(e) + 1 \). \( \square \)

By combining the above result with the fact that all “new” vertices in \( S(G) \) have degree 2, we obtain compact formulas for the bounds on \( \xi(S(G)) \).

**Corollary 3.2**

\[
2\xi(G) + 4\xi(L(G)) \leq \xi(S(G)) \leq 2\xi(G) + 4\xi(L(G)) + 2|E(G)|.
\]

**Proof.** By definition, \( \xi(S(G)) = 2\xi(G) + 2\sum_{e \in E(G)} \varepsilon_{S(G)}(e) \). By Theorem 1, the sum on the right hand side is bounded from below by \( 2\sum_{e \in E(G)} \varepsilon_{L(G)}(e) = 2\xi(L(G)) \) and from above by \( 2\sum_{e \in E(G)} \varepsilon_{L(G)}(e) + |E(G)| = 2\xi(L(G)) + |E(G)| \). \( \square \)
Crucial for the above result was the fact that the eccentricity of all vertices is attained at vertices of degree one. That enables us to obtain explicit formulas also for a wider class of graphs that share that property with trees. A vertex of degree one in a graph is usually called a pendant vertex. The proof follows by the same reasoning as above and we omit it.

**Corollary 3.3** Let $G$ be a connected graph such that eccentricity of each vertex is attained just at a pendant vertex. Then
\[
\xi(S(G)) = 2\xi(G) + 4\zeta(L(G)) + 2|E(G)|.
\]

### 3.2 Triangle parallel graph

**Theorem 3.4** Let $G$ be a connected graph. Then

(i) For any $v \in V(G)$, $\varepsilon_G(v) \leq \varepsilon_{R(G)}(v) \leq \varepsilon_G(v) + 1$,

(ii) For any $e \in E(G)$, $\varepsilon_{R(G)}(e) = \varepsilon_{L(G)}(e) + 1$.

**Proof.** (i) By Lemma A, for each $v, v' \in V(G)$,
\[
d_{R(G)}(v, v') = d_G(v, v') \leq \varepsilon_G(v) + 1,
\]
and for $v \in V(G)$, $e = uu' \in E(G)$,
\[
d_{R(G)}(v, e) \leq \max\{d_{R(G)}(v, u), d_{R(G)}(v, u')\} + 1 \leq \varepsilon_G(v) + 1.
\]
Then $\varepsilon_{R(G)}(v) \leq \varepsilon_G(v) + 1$, and by the above equation, we conclude that $\varepsilon_G(v) \leq \varepsilon_{R(G)}(v) + 1$.

(ii) By Lemma B, for $e, e' \in E(G)$, $d_{R(G)}(e, e') = d_{L(G)}(e, e') + 1$. Then $\varepsilon_{R(G)}(e) \geq \varepsilon_{L(G)}(e) + 1$. On the other hand, for $v \in V(G)$ and $e' = uv$ we have $d_{R(G)}(e, v) \leq d_{R(G)}(e, e')$. From the above equations we conclude that $\varepsilon_{R(G)}(e) = \varepsilon_{L(G)}(e) + 1$. \(\square\)

As in the previous subsection, we obtain the bounds on $\xi(R(G))$ in terms of $\xi(G)$ and $\zeta(L(G))$.

**Corollary 3.5** Let $G$ be a connected graph. Then
\[
2\xi(G) + 2\zeta(L(G)) + 2|E(G)| \leq \xi(R(G))
\]
\[
\leq 2\xi(G) + 2\zeta(L(G)) + 2|V(G)| + 2|E(G)|.
\]

The right inequality from the first claim of Theorem 3.5 becomes equality for some vertex of $G$ if and only if $G$ contains odd cycles. As a consequence, in bipartite graphs we have $\varepsilon_{R(G)}(v) = \varepsilon_G(v)$. That enables us to obtain an explicit formula for the eccentric connectivity index of triangle parallel graphs.
of bipartite graphs.

**Corollary 3.6** Let $G$ be a connected bipartite graph. Then
\[ \xi(R(G)) = 2\xi(G) + 2\xi(L(G)) + 2|E(G)|. \]

**Proof.** It is sufficient to prove that for $v \in V(G)$, $\varepsilon_{R(G)}(v) = \varepsilon_G(v)$. Suppose that there exists $v \in V(G)$ such that $\varepsilon_{R(G)}(v) > \varepsilon_G(v)$. Then by Theorem 3.5, $\varepsilon_{R(G)}(v) = \varepsilon_G(v) + 1$. It implies that the eccentricity of $v$ is attained in a vertex of $R(G)$ corresponding to an edge $e$ of $G$, say $e = uv$ and $\varepsilon_{R(G)}(v) = d_G(v, e) = \varepsilon_G(v) + 1$. Let $P_{uv}$ be a shortest path between $v$ and $e$ in $R(G)$. It is easy to see that at least one of $u$ or $u'$ belongs to $P_{uv}$. Suppose $u \in P_{uv}$. So $d_{R(G)}(v, e) = d_G(v, u) + 1 = \varepsilon_G(v) + 1$ and hence $d_G(v, u) = \varepsilon_G(v)$. Since $u \in P_{uv}$, $d_G(v, u') \geq d_G(v, u)$. So $d_G(v, u') = \varepsilon_G(v)$. Therefore, there exists a path of length $\varepsilon_G(v)$ in $G$ between $v$ and $u$, there exists a path of length 1 in $G$ between $u$ and $u'$ and there exists a path of length $\varepsilon_G(v)$ in $G$ between $u'$ and $v$ and so there exists an odd closed walk in $G$ containing $v$. Therefore $G$ is not bipartite and this is a contradiction.
\[ \square \]

**Corollary 3.7** Let $T$ be a tree with $n$ vertices. Then
\[ \xi(R(T)) = 2(\xi(T) + \zeta(L(T)) + (n - 1)). \]

Again, the result can be strengthened to all graphs where all vertices achieve their eccentricity only at a pendant vertex.

**Corollary 3.8** Let $G$ be a connected graph such that eccentricity of each vertex is attained only at a pendant vertex. Then
\[ \xi(R(G)) = 2\xi(G) + 2\xi(L(G)) + 2|E(G)|. \]

### 3.3 Superposition

**Theorem 3.9** Let $G$ be a connected graph. Then
(i) For each $v \in V(G)$, $\varepsilon_{Q(G)}(v) = \varepsilon_G(v) + 1$,
(ii) For each $e \in E(G)$, $\varepsilon_{L(G)}(e) \leq \varepsilon_{Q(G)}(e) \leq \varepsilon_G(e) + 1$.

**Proof.** (i) It is easy to see that, by Lemma A, for all $v, v' \in V(G)$, $d_{Q(G)}(v, v') = d_G(v, v') + 1$. Then for $v \in V(G)$, $\varepsilon_{Q(G)}(v) \geq \varepsilon_G(v) + 1$. Now suppose that $v \in V(G)$ and $e = uu' \in E(G)$. Then
\[
\begin{align*}
d_{Q(G)}(v, e) &\leq \max\{d_{Q(G)}(v, u), d_{Q(G)}(v, u')\} \\
&= \max\{d_G(v, u), d_G(v, u')\} + 1 \\
&\leq \varepsilon_G(v) + 1.
\end{align*}
\]
Therefore, $\varepsilon_{Q(G)}(v) \leq \varepsilon_G(v) + 1$, and this completes the proof of (i).

(ii) For all $e, e' \in E(G)$ we have, by Lemma B, $d_{Q(G)}(e, e') = d_{L(G)}(e, e')$. Then for each $e \in E(G)$, $\varepsilon_{Q(G)}(e) \geq \varepsilon_{L(G)}(e)$. On the other hand, for $e \in E(G)$ and $v \in V(G)$ such that $e' = uv \in E(G)$,

$$d_{Q(G)}(e, v) \leq d_{Q(G)}(e, e') + d_{Q(G)}(e', v) = d_{L(G)}(e, e') + 1 \leq \varepsilon_{L(G)}(e) + 1,$$

and hence $\varepsilon_{L(G)} \leq \varepsilon_{Q(G)} \leq \varepsilon_{L(G)} + 1$. 

By applying the above result we obtain bounds for $\xi(Q(G))$ including both the eccentric connectivity index and the total eccentricity of $L(G)$.

**Corollary 3.10** Let $G$ be a connected graph. Then

(i) $\xi(Q(G)) \geq \xi(G) + 2|E(G)| + \xi(L(G)) + 2\zeta(L(G))$,

(ii) $\xi(Q(G)) \leq \xi(G) + 4|E(G)| + \xi(L(G)) + 2|E(L(G))| + 2\zeta(L(G)).$

Again, if all vertices of a bipartite graph $G$ attain their eccentricity only at pendant vertices, the second inequality of the above Corollary can be strengthened to an equality.

**Corollary 3.11** Let $G$ be a bipartite connected graph such that the eccentricity of each vertex is attained only at a pendant vertex. Then

$$\xi(Q(G)) = \xi(G) + 4|E(G)| + \xi(L(G)) + 2|E(L(G))| + 2\zeta(L(G)).$$

**Proof.** By Theorem 3.12 and Corollary 3.13, it is sufficient to show that for any $e \in E(G)$, $\varepsilon_{Q(G)}(e) = \varepsilon_{L(G)}(e) + 1$. Suppose that there exists $e \in E(G)$ such that $\varepsilon_{Q(G)}(e) = \varepsilon_{L(G)}(e)$. Then there exists $e' \in E(G)$ such that $k = \varepsilon_{Q(G)}(e) = d_{Q(G)}(e, e') = d_{L(G)}(e, e')$. Let $e = uw$ and $e' = u'v'$,

$$e = e_1 \cdots e_k = e'.
$$

Hence $\varepsilon_{Q(G)}(e) = d_{Q(G)}(e, e') = d_{Q(G)}(u, u')$. It is easy to see that $u'$ is not a pendant vertex. Then there exists a pendant vertex $u''$, such that the eccentricity of $u$ is attained at $u''$, and then $k = d_{Q(G)}(u, u') < d_{Q(G)}(u, u'') = \varepsilon_{Q(G)}(u)$.

On the other hand,

$$k - 1 = d_{Q(G)}(u, u') - 1 < d_{Q(G)}(u, u'') - 1$$

$$= d_{Q(G)}(u, u'') - d_{Q(G)}(u, e)$$

$$\leq d_{Q(G)}(e, u'') \leq d_{Q(G)}(e, e')$$

$$= \varepsilon_{Q(G)}(e) = k.$$

Hence, $k = d_{Q(G)}(e, e') < d_{Q(G)}(e, e') + 1$ and $d_{Q(G)}(u, u'') = k + 1$. Now suppose that $e''$ is the pendant edge containing $u''$. Then

$$e = e_1' e_2' e_3' \cdots e_{k-1}' e_k = e'' \longrightarrow u'' = v_k,$$
\[ e_u = u_1'' \overrightarrow{u_2''} \overrightarrow{u_3''} \ldots \overrightarrow{u_k''} = e'' \overrightarrow{u_{k+1}} = u''. \]

Hence \( u = u_1 - v = v_1 - v_2 - \ldots - v_k = u'' = u_{k+1} - u_k - u_{k-1} - \ldots - u_1 \) is an odd closed walk in \( G \) and since \( G \) is bipartite, this is a contradiction. \( \square \)

**Corollary 3.12** Let \( T \) be a tree. Then

\[ \xi(Q(T)) = \xi(T) + 4|E(T)| + \xi(L(T)) + 2|E(L(T))| + 2\zeta(L(T)). \]

### 3.4 Total graph

The case of total graph is the most complicated, since neither for vertices nor for edges of \( G \) we can express their eccentricity in \( T(G) \) in form of an equality.

**Theorem 3.13** Let \( G \) be a connected graph. Then

(i) For \( v \in V(G) \), \( \varepsilon_G(v) = \varepsilon_T(G)(v) \leq \varepsilon_G(v) + 1 \),

(ii) For \( e \in E(G) \), \( \varepsilon_L(G)(e) \leq \varepsilon_T(G)(e) \leq \varepsilon_L(G)(e) + 1 \).

**Proof.** (i) By Lemma A, for all \( v, v' \in V(G) \), we have \( d_{T(G)}(v, v') = d_G(v, v') \).

Then \( \varepsilon_T(G)(v) \geq \varepsilon_G(v) \). On the other hand, for \( v \in V(G) \) and \( e = uu' \in E(G) \), we have

\[ d_{T(G)}(v, e) = \max\{d_{T(G)}(v, u), d_{T(G)}(v, u')\} + 1 \leq \varepsilon_G(v) + 1. \]

Hence \( \varepsilon_T(G)(v) \leq \varepsilon_G(v) + 1. \)

(ii) By Lemma B, for all \( e, e' \in E(G) \), we have \( d_{T(G)}(e, e') = d_L(G)(e, e') \).

Then \( \varepsilon_T(G)(e) \geq \varepsilon_L(G)(e) \). On the other side, for a pair \( v \in V(G) \) and \( e \in E(G) \), take an edge \( e' = vv' \). Then

\[ d_{T(G)}(v, e) \leq d_{T(G)}(e, e') + d_{T(G)}(e', v) = d_L(G)(e, e') + 1 \leq \varepsilon_L(G)(e) + 1. \]

Hence \( \varepsilon_T(G)(e) \leq \varepsilon_L(G)(e) + 1. \) \( \square \)

**Corollary 3.14** Let \( G \) be a connected graph. Then the following inequalities hold.

(i) \( \xi(T(G)) \geq 2\xi(G) + \xi(L(G)) + 2\zeta(L(G)) \),

(ii) \( \xi(T(G)) \leq 2\xi(G) + 6|E(G)| + \xi(L(G)) + 2|E(L(G))| + 2\zeta(L(G)). \)

Once more, we can express \( \xi(T(G)) \) for trees in form of an equality.

**Corollary 3.15** Let \( T \) be a tree. Then

\[ \xi(T(T)) = 2\xi(T) + \xi(L(T)) + 2|E(T)| + 2\zeta(L(T)). \]
Proof. Since $T$ is a tree, for any $v \in V(T)$, $\varepsilon_{T(T)}(v) = \varepsilon_T(v)$ and for $e \in E(T)$, $\varepsilon_{T(T)}(e) = \varepsilon_T(e) + 1$. By Theorem 3.16, the proof is complete.

Corollary 3.16 The following inequalities hold for a connected bipartite graph $G$.

(i) $\xi(T(G)) \geq 2\xi(G) + \xi(L(G)) + 2\zeta(L(G))$,
(ii) $\xi(T(G)) \leq 2\xi(G) + 2|E(G)| + \xi(L(G)) + 2|E(L(G))| + 2\zeta(L(G))$.

Proof. We claim that for $v \in V(G)$, $\varepsilon_{T(G)}(v) = \varepsilon_G(v)$. If this is not the case, then there exists $v \in V(G)$ such that $\varepsilon_{T(G)}(v) = \varepsilon_G(v) + 1$. So there exists an edge $e = uu' \in E(G)$ such that $\varepsilon_{T(G)}(v) = d_{T(G)}(v,e)$. Since $d_{T(G)}(u,v) + 1 \leq \varepsilon_G(v) + 1 = \varepsilon_{T(G)}(v)$, and $d_{T(G)}(v,e) = d_{T(G)}(v,u) + 1$, we have $d_{T(G)}(v,u) + 1 = \varepsilon_G(v) + 1$, and by a similar argument, $d_{T(G)}(v,u') + 1 = \varepsilon_G(v) + 1$. Also, $d_{T(G)}(v,u) = d_G(v,u)$ and $d_{T(G)}(v,u') = d_G(v,u')$, so $d_G(v,u') = d_G(v,u) = \varepsilon_G(v)$. Then there exists an odd closed walk of length $2\varepsilon_G(v) + 1$ in $G$. Hence $G$ is not bipartite and this is a contradiction. Hence for $v \in V(G)$, $\varepsilon_{T(G)}(v) = \varepsilon_G(v)$. The claim now follows by Theorem 3.16 and Corollary 3.17.

References


