Pointwise polarization tensor bounds, and applications
to voltage perturbations caused by thin
inhomogeneities.

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April 20, 2006

Abstract

The main result of this paper establishes optimal pointwise bounds for the polarization tensors that appear in representation formulae for the voltage perturbations caused by low volume fraction inhomogeneities. Furthermore it is demonstrated how these pointwise bounds may be used to derive a particularly simple version of the representation formulae in the case of thin inhomogeneities.

1. Introduction and statement of main result
2. Some preliminary results
3. Proof of Theorem 3
4. Applications of the pointwise trace bounds

1 Introduction and statement of main result

In earlier work [8] we have derived a very general representation formula for the voltage perturbation caused by volumetrically small inhomogeneities in an otherwise known conductor. The ingredients of this formula are (1) a limiting probability measure, (2) a “background” fundamental solution, and (3) an “effective” polarization tensor. In [9] we proved optimal average bounds on the very essential polarization tensor, and we used these bounds to obtain sharp estimates for the volume of the inhomogeneities in terms of a few boundary voltage measurements. The principal goal of this paper is to prove that these optimal average bounds actually hold in a pointwise sense. A second goal is to show how the optimal pointwise bounds may be used to establish a particular, very specific form of the representation formula for thin inhomogeneities. This “thin inhomogeneity

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representation formula” generalizes the one derived under much more restrictive smoothness assumptions in [5]; the proof given here also significantly simplifies the proof given in [4]. We now briefly recall the situation and the main result from [8].

We consider a conducting object that occupies a bounded, smooth domain $\Omega \subset \mathbb{R}^m$. For simplicity we take $\partial \Omega$ to be $C^\infty$, but this assumption could be considerably weakened. $\gamma_0(\cdot)$ denotes the smooth background conductivity, that is, the conductivity in the absence of any inhomogeneities. We suppose that

$$0 < c_0 \leq \gamma_0(x) \leq C_0 < \infty, \quad x \in \Omega$$

for some fixed constants $c_0$ and $C_0$. For simplicity, we assume that $\gamma_0$ is $C^\infty(\overline{\Omega})$, but this latter assumption could also be considerably weakened. The function $\psi$ denotes the imposed boundary current. It suffices that $\psi \in H^{1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi \, d\sigma = 0$. The background voltage potential, $U$, is the solution to the boundary value problem

$$\begin{align*}
\nabla \cdot (\gamma_0(x) \nabla U) &= 0 \quad \text{in } \Omega, \\
\gamma_0(x) \frac{\partial U}{\partial n} &= \psi \quad \text{on } \partial\Omega.
\end{align*}$$

Here $n$ denotes the unit outward normal to the domain $\Omega$.

Let $\omega_\epsilon$ denote a set of inhomogeneities inside $\Omega$. The geometric assumptions about the set of inhomogeneities are very simple: we suppose the set $\omega_\epsilon$ is measurable, and separated away from the boundary, (i.e., $\text{dist}(\omega_\epsilon, \partial\Omega) > d_0 > 0$). Most importantly, we suppose that $0 < |\omega_\epsilon|$ gets arbitrarily small, where $|\omega_\epsilon|$ denotes the Lebesgue measure of $\omega_\epsilon$. Let $\hat{\gamma}_\epsilon$ denote the conductivity profile in the presence of the inhomogeneities. The function $\hat{\gamma}_\epsilon$ is equal to $\gamma_0$, except on the set of inhomogeneities; on the set of inhomogeneities we suppose that $\hat{\gamma}_\epsilon$ equals the restriction of some other smooth function, $\gamma_1 \in C^\infty(\overline{\Omega})$, with

$$0 < c_1 \leq \gamma_1(x) \leq C_1 < \infty, \quad x \in \Omega.$$

In other words

$$\hat{\gamma}_\epsilon(x) = \begin{cases} 
\gamma_0(x), & x \in \Omega \setminus \omega_\epsilon \\
\gamma_1(x), & x \in \omega_\epsilon
\end{cases}$$

The voltage potential in the presence of the inhomogeneities is denoted $u_\epsilon(x)$. It is the solution to

$$\begin{align*}
\nabla \cdot (\hat{\gamma}_\epsilon(x) \nabla u_\epsilon) &= 0 \quad \text{in } \Omega, \\
\hat{\gamma}_\epsilon(x) \frac{\partial u_\epsilon}{\partial n} &= \psi \quad \text{on } \partial\Omega.
\end{align*}$$

We normalize both $U$ and $u_\epsilon$ by requiring that

$$\int_{\partial\Omega} U \, d\sigma = 0, \quad \text{and} \quad \int_{\partial\Omega} u_\epsilon \, d\sigma = 0.$$
We note that the individual voltages $U$ and $u_\varepsilon$ need not be smooth (or even continuous) on $\partial \Omega$, however, the difference $u_\varepsilon - U$ is smooth in a neighborhood of $\partial \Omega$, due to the regularity of $\gamma_0$, and the fact that $\omega_\varepsilon$ is strictly interior.

Before stating the main theorem from [8] we shall make some preliminary observations. Let $1_{\omega_\varepsilon}$ denote the characteristic function corresponding to the set $\omega_\varepsilon$, i.e., the function which takes the value 1 on the set and the value 0 outside. Since the family of functions $|\omega_\varepsilon|^{-1} 1_{\omega_\varepsilon}$ is bounded in $L^1(\Omega)$, it follows from a combination of the Banach-Alaoglu Theorem and the Riesz Representation Theorem that we may find a regular, positive Borel measure $\mu$, and a subsequence $\omega_{\varepsilon_n}$, with $|\omega_{\varepsilon_n}| \to 0$, such that

$$|\omega_{\varepsilon_n}|^{-1} 1_{\omega_{\varepsilon_n}} \, dx \to d\mu .$$

The convergence refers to the weak* topology of the dual of $C^0(\overline{\Omega})$. More precisely, for any $\phi \in C^0(\overline{\Omega})$

$$|\omega_{\varepsilon_n}|^{-1} \int_{\omega_{\varepsilon_n}} \phi \, dx \to \int_{\Omega} \phi \, d\mu .$$

The measure $\mu$ satisfies $\int_{\Omega} d\mu = 1$, so it is indeed a probability measure. Due to the fact that the sets $\omega_\varepsilon$ stay uniformly bounded away from the boundary, there exists a fixed compact set $K_0 \subset \Omega$ which strictly contains $\omega_\varepsilon$, in the sense that

$$\omega_\varepsilon \subset K_0 \subset \Omega , \quad \text{and} \quad \text{dist}(\omega_\varepsilon, \Omega \setminus K_0) > \delta_0 > 0 .$$

The support of $\mu$ lies inside the same compact set $K_0$. We shall need the so called Neumann function $N(x,y)$ for the operator $\nabla \cdot (\gamma_0 \nabla \cdot)$. For $y \in \Omega$, $N(\cdot,y)$ is the solution to the boundary value problem

$$\nabla_x \cdot (\gamma_0(x) \nabla_x N(x,y)) = \delta_y \quad \text{in} \ \Omega ,$$

$$\gamma_0(x) \frac{\partial N}{\partial n_x} = \frac{1}{|\partial \Omega|} \quad \text{on} \ \partial \Omega ,$$

normalized by $\int_{\partial \Omega} N(x,y) \, d\sigma_x = 0$. The function $N(x,y)$ may be extended by continuity to $y \in \overline{\Omega}$. For $y \in \partial \Omega$ the function $N(\cdot,y)$ may also be interpreted as the solution to the boundary value problem

$$\nabla_x \cdot (\gamma_0(x) \nabla_x N(x,y)) = 0 \quad \text{in} \ \Omega ,$$

$$\gamma_0(x) \frac{\partial N}{\partial n_x} = -\delta_y + \frac{1}{|\partial \Omega|} \quad \text{on} \ \partial \Omega .$$

The main result from [8] may now be formulated as follows.

**Theorem 1.** Let $\omega_{\varepsilon_n}$ be a sequence of measurable subsets, with $|\omega_{\varepsilon_n}| \to 0$, for which (3) and (4) hold. Given any $\psi \in H^{-1/2}(\partial \Omega)$, with $\int_{\partial \Omega} \psi \, d\sigma = 0$, let $U$ and
Let \( \omega \) denote the solutions to (1) and (2), respectively. There exists a subsequence, also denoted \( \omega \), and a matrix valued function \( M \in L^2(\Omega, d\mu) \) such that

\[
(u - U)(y) = |\omega| \int_\Omega (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial N_i}{\partial x_j} (x, y) d\mu(x)
\]

\[
\quad + o(|\omega|) \quad y \in \partial \Omega.
\]

(5)

The values of the function \( M(\cdot) \) are symmetric, positive definite matrices in the sense that

\[
M_{ij}(x) = M_{ji}(x), \quad \text{and} \quad \min\{1, \frac{\gamma_0(x)}{\gamma_1(x)}\} |\xi|^2 \leq M_{ij}(x) \xi_i \xi_j \leq \max\{1, \frac{\gamma_0(x)}{\gamma_1(x)}\} |\xi|^2,
\]

\( \xi \in \mathbb{R}^m, \quad \mu \text{ almost everywhere in the set } \{ x : \gamma_0(x) \neq \gamma_1(x) \} .
\]

(6)

The subsequence \( \omega \) and the matrix valued function \( M \in L^2(\Omega, d\mu) \) are independent of the boundary flux \( \psi \). The term \( o(|\omega|) \) is such that \( ||o(\omega)||_{L^\infty(\partial \Omega)} ||\omega|| \) converges to 0 for any fixed \( \psi \in H^{-1/2} \), and uniformly on \( \{ \psi : \int_{\partial \Omega} \psi d\sigma = 0, ||\psi||_{L^2(\partial \Omega)} \leq 1 \} \).

Given any \( \xi \in \mathbb{R}^m \) let \( V \) and \( \nu \) denote the solutions to

\[
\nabla \cdot (\gamma_0 \nabla V) = \nabla \cdot (\gamma_0 \xi) \text{ in } \Omega, \quad \gamma_0 \frac{\partial V}{\partial n} = \gamma_0 \xi \cdot n \text{ on } \partial \Omega,
\]

(7)

and

\[
\nabla \cdot (\tilde{\gamma} \nabla \nu) = \nabla \cdot (\gamma_0 \xi) \text{ in } \Omega, \quad \tilde{\gamma} \frac{\partial \nu}{\partial n} = \gamma_0 \xi \cdot n \text{ on } \partial \Omega,
\]

(8)

respectively. We note that \( V(x) \) has a very simple formula: \( V(x) = \xi \cdot x + \text{cst} \) – for convenience we take the constant to be zero. In [8] and [9] we proved the following result, which provides a fairly explicit characterization of the polarization tensor \( M \), whose existence is asserted by Theorem 1.

**Proposition 1.** Let \( \omega \) be a sequence of measurable subsets, with \( |\omega| \to 0 \), for which (3) and (4) hold. Let \( V \) and \( \nu \) denote solutions to (7) and (8), respectively. There exists a subsequence, also denoted \( \omega \), and a matrix valued function \( M \in L^2(\Omega, d\mu) \), characterized by

\[
\int_\Omega M_{ij} \xi_i \xi_j d\mu = \frac{1}{|\omega|} \int_{\omega} \nabla \nu \cdot \nabla \phi dx + o(1),
\]

(9)

for any \( \phi \in C^0(\overline{\Omega}) \) and any \( \xi \in \mathbb{R}^m \). The matrix valued function \( M(\cdot) \) and the subsequence \( \omega \) have the exact properties stated in Theorem 1, in particular the representation formula (5) holds for any boundary flux \( \psi \in H^{-1/2}(\partial \Omega) \), with \( \int_{\partial \Omega} \psi d\sigma = 0 \).
We refer the reader to [8] for remarks concerning variations and immediate generalizations of the representation formula (5) for $u_\varepsilon$. Explicit representation formulas of this type are of significant interest from an imaging point of view. For instance: if one has very detailed knowledge of the “voltage boundary signatures” of internal inhomogeneities, then it becomes possible to design very effective numerical methods to identify the volume of the inhomogeneities and their location. We refer the reader to [1], [2], [7], [9], and [10] for examples of numerical methods based on such formulas. The optimal estimates for the volume of $\omega_\varepsilon$ given in [9] and [10] rely on the following “average” characterization of the polarization tensor $M$.

**Theorem 2.** Suppose the conductivities $\gamma_0$ and $\gamma_1$ are constants, with $\gamma_0 \neq \gamma_1$. Let $\mu$ and $M_{ij}(\cdot) \in L^2(\Omega, d\mu)$ be the limiting probability measure and the polarization tensor from Proposition 1 (and Theorem 1), and let $a(s)$ denote the average of $m - 1$ copies of 1 and one copy of $s$, i.e.,

$$a(s) = \frac{1}{m}((m - 1) + s).$$

Then

$$\text{Trace } \left( \int_\Omega M \, d\mu \right) \leq m \, a\left( \frac{\gamma_0}{\gamma_1} \right),$$

and

$$\text{Trace } \left( \int_\Omega M \, d\mu \right)^{-1} \leq m \, a\left( \frac{\gamma_1}{\gamma_0} \right).$$

As stated earlier one goal of this paper is to prove that these bounds actually hold in a pointwise sense. We formulate this as the following theorem.

**Theorem 3.** Let $\gamma_0$ and $\gamma_1$ be $C^\infty$ conductivities. Let $\mu$ and $M_{ij}(\cdot) \in L^2(\Omega, d\mu)$ be the limiting probability measure and the polarization tensor from Proposition 1 (and Theorem 1), and let $a(s)$ denote the average of $m - 1$ copies of 1 and one copy of $s$, i.e.,

$$a(s) = \frac{1}{m}((m - 1) + s).$$

Then

$$\text{Trace } M(x) \leq m \, a\left( \frac{\gamma_0(x)}{\gamma_1(x)} \right),$$

and

$$\text{Trace } M^{-1}(x) \leq m \, a\left( \frac{\gamma_1(x)}{\gamma_0(x)} \right),$$

$\mu$ almost everywhere in the set $\{ x : \gamma_0(x) \neq \gamma_0(x) \}$. 

5
Some preliminary results

The purpose of this section is to derive alternate characterizations of the polarization tensor \( M \), alternate, that is, to the one already provided by Proposition 1. The principal results in that direction are the variational characterizations given in Lemma 3 and Lemma 4. We first note that we could equally well have defined \( M \), through the solution of auxiliary Dirichlet problems, and we also note that the resulting \( M \) is independent of the bounded domain \( \Omega \) (as long as \( \omega_\epsilon \subset K_0 \subset \Omega \)). These statements are an immediate consequence of the last identities in following lemma.

Lemma 1. Suppose \( \omega_\epsilon \subset K_0 \subset \Omega \subset \Omega_1 \). Suppose the bounded domains \( \Omega \) and \( \Omega_1 \) are either smooth \((C^2)\) or Lipschitz and convex, and suppose \( \gamma_0 \) and \( \gamma_1 \) have been extended to positive \( C^\infty \) functions on all of \( \overline{\Omega_1} \). For fixed \( \xi \in \mathbb{R}^m \) let \( V = \xi \cdot x \), and let \( v_\epsilon \) be a solution to (8). Let \( v_\epsilon^{(1)} \), \( \tilde{v}_\epsilon^{(1)} \) be solutions to

\[
\nabla \cdot \left( \gamma_\epsilon \nabla v_\epsilon^{(1)} \right) = \nabla \cdot (\gamma_0 \xi) \quad \text{in} \quad \Omega_1, \quad \gamma_\epsilon \frac{\partial v_\epsilon^{(1)}}{\partial n} = \gamma_0 \xi \cdot n \quad \text{on} \quad \partial \Omega_1 . \quad (10)
\]

and

\[
\nabla \cdot \left( \gamma_\epsilon \nabla \tilde{v}_\epsilon^{(1)} \right) = \nabla \cdot (\gamma_0 \xi) \quad \text{in} \quad \Omega_1, \quad \tilde{v}_\epsilon^{(1)} = V (= \xi \cdot x) \quad \text{on} \quad \partial \Omega_1 . \quad (11)
\]

Suppose \( v_\epsilon \) and \( v_\epsilon^{(1)} \), are “normalized” by

\[
\int_{\Omega} v_\epsilon \, dx = \int_{\Omega} V \, dx \quad \text{and} \quad \int_{\Omega_1} v_\epsilon^{(1)} \, dx = \int_{\Omega_1} V \, dx .
\]

Then

\[
\| v_\epsilon^{(1)} - V \|_{H^1(\Omega_1)} + \| v_\epsilon - V \|_{H^1(\Omega)} \leq C |\omega_\epsilon|^{\frac{1}{2}} ,
\]

and, for any \( \eta > 0 \), there exists a constant \( C_\eta \) such that

\[
\| \tilde{v}_\epsilon^{(1)} - V \|_{L^2(\Omega_1)} + \| v_\epsilon^{(1)} - V \|_{L^2(\Omega_1)} + \| v_\epsilon - V \|_{L^2(\Omega)} \leq C_\eta |\omega_\epsilon|^{\frac{1}{2} + \frac{1}{m^*} - \eta} ,
\]

with \( m^* = \max\{m, 2\} \) (\( m \) being the dimension of the ambient space). Furthermore

\[
\int_{\omega_\epsilon} \nabla v_\epsilon^{(1)} \cdot \nabla V \phi \, dx = \int_{\omega_\epsilon} \nabla v_\epsilon \cdot \nabla V \phi \, dx + o(|\omega_\epsilon|) = \int_{\omega_\epsilon} \nabla v_\epsilon \cdot \nabla V \phi \, dx + o(|\omega_\epsilon|) ,
\]

for any \( \phi \in C^0(\overline{\Omega}) \).

Proof. A proof of the inequalities

\[
\| v_\epsilon^{(1)} - V \|_{H^1(\Omega_1)} + \| v_\epsilon - V \|_{H^1(\Omega)} \leq C |\omega_\epsilon|^{\frac{1}{2}}
\]

and

\[
\| v_\epsilon^{(1)} - V \|_{L^2(\Omega_1)} + \| v_\epsilon - V \|_{L^2(\Omega)} \leq C_\eta |\omega_\epsilon|^{\frac{1}{2} + \frac{1}{m^*} - \eta}
\]

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is found in [8] (cf. Lemma 1 of [8]). Similarly a proof of the identity
\[ \int_{\omega} \nabla v^{(1)} \cdot \nabla V \phi \, dx = \int_{\omega} \nabla v \cdot \nabla V \phi \, dx + o(|\omega|) \]
is found in [9] (cf. Proposition 1 of [9]). The remaining inequalities
\[ \|v^{(1)} - V\|_{H^1(\Omega_1)} \leq C|\omega|^{1/2}, \quad \|v^{(1)} - V\|_{L^2(\Omega_1)} \leq C|\omega|^{1/2 + \frac{1}{m}} \], (12)
and the remaining identity
\[ \int_{\omega} \nabla \tilde{v}^{(1)} \cdot \nabla V \phi \, dx = \int_{\omega} \nabla v \cdot \nabla V \phi \, dx + o(|\omega|) \] (13)
is proved in ways that are very similar to those presented in [8] and [9], but for the sake of completeness we provide an outline of the arguments. From the variational formulations of (7) and (11) it follows directly that
\[ \left| \int_{\Omega_1} \gamma \nabla (\tilde{v}^{(1)} - V) \cdot \nabla w \, dx \right| = \left| \int_{\Omega_1} (\gamma_0 - \gamma) \nabla V \cdot \nabla w \, dx \right| \]
\[ \leq C|\omega|^{1/2} \|\nabla V\|_{L^\infty(\omega)} \|\nabla w\|_{L^2(\Omega_1)} \]
\[ \leq C|\omega|^{1/2} \|\nabla w\|_{L^2(\Omega_1)} \]
for any \( w \in H^1_0(\Omega_1) \). This verifies the first inequality of (12). We also have
\[ \int_{\Omega_1} \gamma_0 \nabla (\tilde{v}^{(1)} - V) \cdot \nabla w \, dx = \int_{\Omega_1} (\gamma_0 - \gamma) \nabla \tilde{v}^{(1)} \cdot \nabla w \, dx , \quad \forall w \in H^1_0(\Omega_1) . \] (14)
Select \( w_0 \) as the solution to
\[ \nabla \cdot (\gamma_0 \nabla w_0) = V - \tilde{v}^{(1)} \quad \text{in} \quad \Omega_1 , \quad w_0 = 0 \quad \text{on} \quad \partial \Omega_1 . \]
Elliptic estimates show that \( \|w_0\|_{H^2(\Omega_1)} \leq C\|V - \tilde{v}^{(1)}\|_{L^2(\Omega_1)} \). This is true if \( \Omega_1 \) is \( C^2 \), and it is also true if \( \Omega_1 \) is only Lipschitz, but convex (cf. [13] Theorem 3.2.1.2). A similar estimate holds for the corresponding Neumann problem for both \( C^2 \) and convex domains (cf. [13] Theorem 3.2.1.3). After insertion of \( w_0 \) into (14) we obtain
\[ \int_{\Omega_1} (\tilde{v}^{(1)} - V)^2 \, dx = \int_{\Omega_1} \gamma_0 \nabla (\tilde{v}^{(1)} - V) \cdot \nabla w_0 \, dx \]
\[ = \left| \int_{\Omega_1} (\gamma_0 - \gamma) \nabla \tilde{v}^{(1)} \cdot \nabla w_0 \, dx \right| \]
\[ \leq C \left( \int_{\omega} |\nabla \tilde{v}^{(1)}|^q \, dx \right)^{1/q} \left( \int_{\Omega_1} |\nabla w_0|^p \, dx \right)^{1/p} \]
\[ \leq C_q \left( \int_{\omega} |\nabla \tilde{v}^{(1)}|^q \, dx \right)^{1/q} \|w_0\|_{H^2(\Omega_1)} \]
\[ \leq C_q \left( \int_{\omega} |\nabla \tilde{v}^{(1)}|^q \, dx \right)^{1/q} \|V - \tilde{v}^{(1)}\|_{L^2(\Omega_1)} , \] (15)
provided \( p \) and \( q \) are related by \( \frac{1}{q} + \frac{1}{p} = 1 \), and provided we require that \( q > \frac{2m^*}{2m^* + 2} \) (so that \( 1 < p < \frac{2m^*}{m^* - 2} \), and therefore, by Sobolev’s Imbedding Theorem \( (\int_{\Omega_1} |\nabla w_0|^p \, dx)^{1/p} \leq C_p \|w_0\|_{H^1(\Omega_1)} \), cf. [12], p. 155). For any \( 1 < q < 2 \)

\[
\|\nabla \tilde{v}_e^{(1)}\|_{L^q(\omega_1)} \leq \|\nabla (\tilde{v}_e^{(1)} - V)\|_{L^q(\omega_1)} + \|\nabla V\|_{L^q(\omega_1)} \\
\leq \left( \int_{\omega_1} 1 \, dx \right)^s \|\nabla (\tilde{v}_e^{(1)} - V)\|_{L^2(\omega_1)} + |\xi| |\omega_1|^{1/q} \\
\leq C(|\omega_1|^{s+1/2} + |\omega_1|^{1/q}) ,
\]

(16)

with \( s = \frac{1}{q} - \frac{1}{2} \). A combination of (15) and (16) yields

\[
\|\tilde{v}_e^{(1)} - V\|_{L^2(\Omega_1)} \leq C_q \left( \int_{\omega_1} |\nabla (\tilde{v}_e^{(1)} - V)\|_{L^q(\omega_1)} \, dx \right)^{1/q} \leq C_q |\omega_1|^{1/q} ,
\]

for any \( \frac{2m^*}{m^* + 2} < q < 2 \). We note that \( \frac{1}{q} \) approaches \( \frac{m^* - 2}{2m^*} = \frac{1}{2} + \frac{1}{m^*} \) from below as \( q \) approaches \( \frac{2m^*}{m^* + 2} \) from above. The previous estimate now immediately implies that, given any \( \eta > 0 \), there exists a constant \( C_\eta \) such that

\[
\|\tilde{v}_e^{(1)} - V\|_{L^2(\Omega_1)} \leq C_\eta |\omega_1|^{\frac{1}{2} + \frac{1}{m^*} - \eta} ,
\]

exactly the second inequality in (12). It only remains to prove the identity (13). To that end we introduce \( D = \{ x \in \Omega_1 : \text{dist}(x, \partial \Omega) < \delta \} \), a sufficiently small neighborhood of \( \partial \Omega \). Since \( \tilde{v}_e^{(1)} - V \) is \( \gamma_0 \)-harmonic outside \( K_\delta \), elliptic estimates, in combination with the second inequality of (12), imply that

\[
\|\tilde{v}_e^{(1)} - V\|_{C^1(D)} \leq C\|\tilde{v}_e^{(1)} - V\|_{L^2(\Omega_1)} \leq C_\eta |\omega_1|^{\frac{1}{2} + \frac{1}{m^*} - \eta} .
\]

Hence

\[
\|\tilde{\gamma}_e \frac{\partial \tilde{v}_e^{(1)}}{\partial n} - \tilde{\gamma}_e \frac{\partial v_\epsilon}{\partial n}\|_{C^0(\partial \Omega)} = \|\tilde{\gamma}_e \frac{\partial \tilde{v}_e^{(1)}}{\partial n} - \tilde{\gamma}_e \frac{\partial v_\epsilon}{\partial n}\|_{C^0(\partial \Omega)} \leq C_\eta |\omega_1|^{\frac{1}{2} + \frac{1}{m^*} - \eta} ,
\]

and therefore, by energy estimates

\[
\|\nabla \tilde{v}_e^{(1)} - \nabla v_\epsilon\|_{L^2(\Omega)} \leq \|\tilde{\gamma}_e \frac{\partial \tilde{v}_e^{(1)}}{\partial n} - \tilde{\gamma}_e \frac{\partial v_\epsilon}{\partial n}\|_{H^{-1/2}(\partial \Omega)} \leq C_\eta |\omega_1|^{\frac{1}{2} + \frac{1}{m^*} - \eta} .
\]

The last inequality immediately implies that

\[
\left| \int_{\omega_1} (\nabla \tilde{v}_e^{(1)} - \nabla v_\epsilon) \cdot \nabla \phi \, dx \right| \leq C_{\eta, \phi} |\omega_1|^{1/2} \|\nabla \tilde{v}_e^{(1)} - \nabla v_\epsilon\|_{L^2(\Omega)} \leq C_{\eta, \phi} |\omega_1|^{1 + \frac{1}{m^*} - \eta} ,
\]

for any \( \phi \in C^0(\Omega) \), exactly as desired in (13).
We now introduce $w_\epsilon = v_\epsilon - V$, $w_\epsilon^{(1)} = v_\epsilon^{(1)} - V$, and $\hat{w}_\epsilon^{(1)} = \hat{v}_\epsilon^{(1)} - V$. With $\phi$ replaced by $\phi(\gamma_1 - \gamma_0)$ (9) takes the form

$$\int_\Omega (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu = \frac{1}{|\omega_{\epsilon n}|} \int_{\omega_{\epsilon n}} (\gamma_1 - \gamma_0) |\xi|^2 \phi \, dx \tag{17}$$

+ $\frac{1}{|\omega_{\epsilon n}|} \int_{\omega_{\epsilon n}} (\gamma_1 - \gamma_0) \nabla w_{\epsilon n} \cdot \xi \phi \, dx + o(1)$.

The functions $w_\epsilon^{(1)}$ and $\hat{w}_\epsilon^{(1)}$ are solutions to

$$\nabla \cdot \left( \hat{\gamma}_\epsilon \nabla w_\epsilon^{(1)} \right) = \nabla \cdot ((\gamma_0 - \gamma_1) 1_{\omega_\epsilon} \xi) \text{ in } \Omega_1, \quad \hat{\gamma}_\epsilon \frac{\partial w_\epsilon^{(1)}}{\partial n} = 0 \text{ on } \partial \Omega_1 \quad \tag{18}$$

and

$$\nabla \cdot \left( \gamma_\epsilon \nabla \hat{w}_\epsilon^{(1)} \right) = \nabla \cdot ((\gamma_0 - \gamma_1) 1_{\omega_\epsilon} \xi) \text{ in } \Omega_1, \quad \hat{w}_\epsilon^{(1)} = 0 \text{ on } \partial \Omega_1 \quad \tag{19}$$

respectively.

From now on, unless otherwise stated, we only consider smooth functions $\phi \in C^\infty(\bar{\Omega})$ that are uniformly positive, i.e., satisfy $\min_{x \in \Omega_1} \phi(x) > 0$. We introduce the functions $w_\epsilon^{(1)}_{\epsilon, \phi}$ and $\hat{w}_\epsilon^{(1)}_{\epsilon, \phi}$ as solutions to

$$\nabla \cdot \left( \hat{\gamma}_\epsilon \phi \nabla w_\epsilon^{(1)}_{\epsilon, \phi} \right) = \nabla \cdot ((\gamma_0 - \gamma_1) 1_{\omega_\epsilon} \xi \phi) \text{ in } \Omega_1, \quad \hat{\gamma}_\epsilon \phi \frac{\partial w_\epsilon^{(1)}_{\epsilon, \phi}}{\partial n} = 0 \text{ on } \partial \Omega_1 \quad \tag{20}$$

normalized by $\int_{\Omega_1} w_\epsilon^{(1)}_{\epsilon, \phi} \, dx = 0$, and

$$\nabla \cdot \left( \gamma_\epsilon \phi \nabla \hat{w}_\epsilon^{(1)}_{\epsilon, \phi} \right) = \nabla \cdot ((\gamma_0 - \gamma_1) 1_{\omega_\epsilon} \xi \phi) \text{ in } \Omega_1, \quad \hat{w}_\epsilon^{(1)}_{\epsilon, \phi} = 0 \text{ on } \partial \Omega_1 \quad \tag{21}$$

Lemma 1 asserts that

$$\| \nabla \hat{w}_\epsilon^{(1)}_{\epsilon, \phi} \|_{L^2(\Omega_1)} + \| \nabla w_\epsilon^{(1)}_{\epsilon, \phi} \|_{L^2(\Omega_1)} = O(|\omega_\epsilon|^{1/2}) \quad \tag{22}$$

and that

$$\| \hat{w}_\epsilon^{(1)} \|_{L^2(\Omega_1)} + \| w_\epsilon^{(1)} \|_{L^2(\Omega_1)} = o(|\omega_\epsilon|^{1/2}) \quad ,$$

and that

$$\int_{\omega_\epsilon} \nabla \hat{w}_\epsilon^{(1)} \cdot \xi (\gamma_1 - \gamma_0) \phi \, dx = \int_{\omega_\epsilon} \nabla w_\epsilon^{(1)} \cdot \xi (\gamma_1 - \gamma_0) \phi \, dx + o(|\omega_\epsilon|)$$

$$= \int_{\omega_\epsilon} \nabla w_\epsilon \cdot \xi (\gamma_1 - \gamma_0) \phi \, dx + o(|\omega_\epsilon|) \quad \tag{23}$$

Since $w_\epsilon^{(1)}_{\epsilon, \phi}$ and $\hat{w}_\epsilon^{(1)}_{\epsilon, \phi}$ are the same as $w_\epsilon^{(1)}$ and $\hat{w}_\epsilon^{(1)}$ in the case when $\gamma_0$ and $\gamma_1$ are replaced by $\phi \gamma_0$ and $\phi \gamma_1$, it follows immediately from Lemma 1 that

$$\| \nabla \hat{w}_\epsilon^{(1)}_{\epsilon, \phi} \|_{L^2(\Omega_1)} + \| \nabla w_\epsilon^{(1)}_{\epsilon, \phi} \|_{L^2(\Omega_1)} = O(|\omega_\epsilon|^{1/2}) \quad ,$$


\[ \| \tilde{w}_{e,\phi}^{(1)} \|_{L^2(\Omega_1)} + \| w_{e,\phi}^{(1)} \|_{L^2(\Omega_1)} = o(|\omega_e|^{1/2}) \quad , \] (24)

and that
\[ \int_{\omega_e} \nabla w_{e,\phi}^{(1)} \cdot \xi(\gamma_1 - \gamma_0) \, \phi \, dx = \int_{\omega_e} \nabla w_{e,\phi}^{(1)} : \xi(\gamma_1 - \gamma_0) \, \phi \, dx + o(|\omega_e|) \quad . \] (25)

Using the estimate (24), the weak formulation of (18), the estimate (22), and the weak formulation of (20), in that order, we calculate
\[ \int_{\omega_e} \nabla w_{e,\phi}^{(1)} : \xi(\gamma_1 - \gamma_0) \, \phi \, dx = \int_{\omega_e} \nabla \left( \phi w_{e,\phi}^{(1)} \right) : \xi(\gamma_1 - \gamma_0) \, dx + o(|\omega_e|) 
= - \int_{\Omega_1} \tilde{\gamma}_e \nabla \left( \phi w_{e,\phi}^{(1)} \right) \cdot \nabla w_e \, dx + o(|\omega_e|) 
= - \int_{\Omega_1} \tilde{\gamma}_e \phi \nabla w_{e,\phi}^{(1)} : \nabla w_e \, dx + o(|\omega_e|) 
= \int_{\omega_e} \nabla w_{e,\phi}^{(1)} : \xi(\gamma_1 - \gamma_0) \, \phi \, dx + o(|\omega_e|) \quad . \] (26)

For the last identity we used (23). Combining the weak formulations of the equations for \( w_{e,\phi}^{(1)} \) and \( \tilde{w}_{e,\phi}^{(1)} \) with (25) and (26) we now obtain the following lemma.

**Lemma 2.** Let \( \Omega_1 \), and \( w_e = v_e - V \) as in Lemma 1. Let \( w_{e,\phi}^{(1)} \) and \( \tilde{w}_{e,\phi}^{(1)} \) be as above, for some smooth, uniformly positive \( \phi \). Then
\[ \int_{\Omega_1} \tilde{\gamma}_e \phi \nabla w_{e,\phi}^{(1)} : \nabla w_{e,\phi}^{(1)} \, dx = - \int_{\Omega_1} \nabla w_{e,\phi}^{(1)} : (\gamma_1 - \gamma_0) 1_{\omega_e} \xi \, \phi \, dx 
= - \int_{\Omega} \nabla w_e : (\gamma_1 - \gamma_0) 1_{\omega_e} \xi \, \phi \, dx + o(|\omega_e|) \]

and
\[ \int_{\Omega_1} \tilde{\gamma}_e \phi \nabla \tilde{w}_{e,\phi}^{(1)} : \nabla \tilde{w}_{e,\phi}^{(1)} \, dx = - \int_{\Omega_1} \nabla \tilde{w}_{e,\phi}^{(1)} : (\gamma_1 - \gamma_0) 1_{\omega_e} \xi \, \phi \, dx 
= - \int_{\Omega} \nabla \tilde{w}_e : (\gamma_1 - \gamma_0) 1_{\omega_e} \xi \, \phi \, dx + o(|\omega_e|) \quad . \]

A simple computation, based on Lemma 2, gives
\[ \int_{\Omega_1} \tilde{\gamma}_e \left[ \nabla w_{e,\phi}^{(1)} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_e} \xi \right] \, \phi \, dx 
= \int_{\omega_e} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1} |\xi|^2 \, \phi \, dx + \int_{\Omega_1} \tilde{\gamma}_e \nabla w_{e,\phi}^{(1)} : \nabla w_{e,\phi}^{(1)} \, \phi \, dx 
+ 2 \int_{\Omega_1} \nabla w_{e,\phi}^{(1)} : (\gamma_1 - \gamma_0) 1_{\omega_e} \xi \, \phi \, dx 
= \int_{\omega_e} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1} |\xi|^2 \, \phi \, dx + \int_{\Omega} \nabla w_e : (\gamma_1 - \gamma_0) 1_{\omega_e} \xi \, \phi \, dx + o(|\omega_e|) \quad . \]
or, after rearrangement,

\[
\int_{\Omega} \nabla w \cdot (\gamma_1 - \gamma_0) 1_{\omega_{\epsilon}} \xi \phi \, dx
= - \int_{\omega_{\epsilon}} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1} |\xi|^2 \phi \, dx + \int_{\Omega_{\epsilon}} \tilde{\gamma}_{\epsilon} \left| \nabla w_{\epsilon,\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon}} \xi \right|^2 \phi \, dx + o(|\omega_{\epsilon}|) .
\]  

(27)

Insertion of (27) into the expression (17) now yields

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu
= \frac{1}{|\omega_{\epsilon_{n}}|} \int_{\omega_{\epsilon_{n}}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx
+ \frac{1}{|\omega_{\epsilon_{n}}|} \int_{\Omega_{\epsilon}} \tilde{\gamma}_{\epsilon_{n}} \left| \nabla w_{\epsilon_{n},\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_{n}}} \xi \right|^2 \phi \, dx + o(1) .
\]

A completely similar calculation gives

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu
= \frac{1}{|\omega_{\epsilon_{n}}|} \int_{\omega_{\epsilon_{n}}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx
+ \frac{1}{|\omega_{\epsilon_{n}}|} \int_{\Omega_{\epsilon}} \tilde{\gamma}_{\epsilon_{n}} \left| \nabla w_{\epsilon_{n},\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_{n}}} \xi \right|^2 \phi \, dx + o(1) .
\]

The equations (20) and (21) for \( w_{\epsilon,\phi}^{(1)} \) and \( \tilde{w}_{\epsilon,\phi}^{(1)} \) guarantee that these are the minimizers for the expression

\[
\int_{\Omega_{\epsilon}} \tilde{\gamma}_{\epsilon} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon}} \xi \right|^2 \phi \, dx
\]

in \( H^1(\Omega_1) \) and \( H^1_0(\Omega_1) \), respectively, and so we obtain the following lemma.

**Lemma 3.** Let \( \omega_{\epsilon_{n}} \) and \( M \) be a subsequence and a polarization tensor as introduced by Proposition 1 (and Theorem 1). Let \( \Omega_1 \) be as in Lemma 1, and suppose \( \phi \) is a uniformly positive, smooth function. Then

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu
= \frac{1}{|\omega_{\epsilon_{n}}|} \int_{\omega_{\epsilon_{n}}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx
+ \frac{1}{|\omega_{\epsilon_{n}}|} \min_{w \in H^1(\Omega_1)} \int_{\Omega_{\epsilon}} \tilde{\gamma}_{\epsilon_{n}} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_{n}}} \xi \right|^2 \phi \, dx + o(1) ,
\]

(28)
and
\[ \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu \]
\[ = \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx \]
\[ + \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H_0^1(\Omega_1)} \int_{\Omega_1} \tilde{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega_{\epsilon_n} \xi \right|^2 \phi \, dx + o(1) \]  

(29)

Remark 1. The identities (28) and (29) hold for any \((C^2\text{ or convex}) \Omega_1\) with \(\omega_{\epsilon_n} \subset K_0 \subset \Omega_1\), and not just any \(\Omega_1\) with \(\Omega \subset \subset \Omega_1\). To see this let \(\Omega_1\) be any, \(C^2\text{ or convex, bounded domain with}\) \(\omega_{\epsilon_n} \subset K_0 \subset \Omega_1\), and select \(\tilde{\Omega}\) (smooth) such that \(\omega_{\epsilon_n} \subset K_0 \subset \tilde{\Omega} \subset \subset \Omega_1\) and let \(M\) denote the polarization tensor associated to \(\tilde{\Omega}\) (and some subsequence of the sequence \(\{\epsilon_n\}\)). We already know that \(M d\mu = M d\mu\), for instance because, due to Lemma 3,
\[ \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu \]
\[ = \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx \]
\[ + \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H_0^1(\Omega_2)} \int_{\Omega_2} \tilde{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega_{\epsilon_n} \xi \right|^2 \phi \, dx + o(1) \]

for any uniformly positive, smooth \(\phi\), and any domain \(\Omega_2\), which compactly contains \(\Omega\) and \(\tilde{\Omega}\). Note that this identity also implies that \(M (= M\,\text{a.e.})\) is the “effective” polarization tensor associated to the entire sequence \(\{\epsilon_n\}\) and the domain \(\tilde{\Omega}\). Again, due to Lemma 3,
\[ \int_{\tilde{\Omega}} (\gamma_1 - \gamma_0) \tilde{M}_{ij} \xi_i \xi_j \phi d\mu \]
\[ = \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx \]
\[ + \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H_0^1(\Omega_1)} \int_{\Omega_1} \tilde{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega_{\epsilon_n} \xi \right|^2 \phi \, dx + o(1) \]

since \(\Omega_1\) compactly contains \(\tilde{\Omega}\). From the previous identity it now follows that
\[ \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu \]
\[ = \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx \]
\[ + \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H_0^1(\Omega_1)} \int_{\Omega_1} \tilde{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega_{\epsilon_n} \xi \right|^2 \phi \, dx + o(1) \]
as desired.

From the first of the formulae in this lemma we may immediately obtain the pointwise bounds of Theorem 1, as follows. Firstly, the right-hand side of the identity (28) is the sum of two non-negative terms, and thus

$$\int_\Omega (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu \geq \frac{1}{|\omega_n|} \int_{\omega_n} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \, \phi \, dx + o(1) \quad (30)$$

Secondly, by taking $\nabla w = 0$ in (28), we obtain

$$\int_\Omega (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu \leq \frac{1}{|\omega_n|} \int_{\omega_n} (\gamma_1 - \gamma_0) |\xi|^2 \, \phi \, dx + o(1) \quad (31)$$

In limit as $|\omega_n| \to 0$, (30)-(31) give

$$\int_\Omega (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \, \phi \, d\mu \leq \int_\Omega (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu \leq \int_\Omega (\gamma_1 - \gamma_0) |\xi|^2 \, \phi \, d\mu ,$$

for any smooth, uniformly positive $\phi$. A simple limiting argument insures that these inequalities hold for all non-negative, continuous functions $\phi$, and this in turn insures that

$$\min \left(1, \frac{\gamma_0(x)}{\gamma_1(x)}\right) |\xi|^2 \leq M(x) \xi \cdot \xi \leq \max \left(1, \frac{\gamma_0(x)}{\gamma_1(x)}\right) |\xi|^2$$

for all $\xi \in \mathbb{R}^m$, $\mu$ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$. In order to verify the stronger, optimal bounds of Theorem 3, it will be convenient to have the following characterization of $M$.

**Lemma 4.** Let $\omega_n$ and $M$ be a subsequence and a polarization tensor as introduced by Proposition 1 (and Theorem 1). Suppose $Q = (-L,L)^m$ is a $m$-dimensional cube, with $\omega_n \subset K_0 \subset Q$, and suppose $\phi$ is a uniformly positive, smooth function. Then

$$\int_\Omega (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu$$

$$= \frac{1}{|\omega_n|} \int_{\omega_n} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \, \phi \, dx$$

$$+ \frac{1}{|\omega_n|} \min_{w \in H^1_{\text{per}}(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_n} \xi \right|^2 \phi \, dx + o(1) ,$$

where $H^1_{\text{per}}(Q)$ denotes the set of $Q$-periodic elements of $H^1(Q)$.

**Proof.** Since $H^1_0(Q) \subset H^1_{\text{per}}(Q) \subset H^1(Q)$

$$\min_{w \in H^1(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_n} \xi \right|^2 \phi \, dx$$

$$\leq \min_{w \in H^1_{\text{per}}(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_n} \xi \right|^2 \phi \, dx$$

$$\leq \min_{w \in H^1_0(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_n} \xi \right|^2 \phi \, dx .$$

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Since $Q$ is convex with $\omega_{\epsilon_n} \subset K_0 \subset Q$ a combination of these inequalities with Lemma 3 (and Remark 1) now gives

$$\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu + o(1)$$

$$\leq \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx$$

$$+ \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H^1_{per}(Q)} \int_Q \nabla w \cdot [\nabla \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_n}} \xi]^2 \phi dx$$

$$\leq \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu + o(1),$$

as desired.

3 Proof of Theorem 3

In the course of proving this theorem the following lemma will be very useful.

**Lemma 5.** Let $Q = (-L, L)^m$ be an $m$-dimensional cube with $\omega_{\epsilon} \subset K_0 \subset Q$. Let $a$ be a uniformly positive, smooth function on $Q$, and let $f$ be a continuous function on $Q$. For any $\xi \in \mathbb{R}^m$, let $u_\xi$ be the $Q$-periodic solution to

$$-\nabla \cdot (a \nabla u_\xi) = \nabla \cdot (f(x)1_{\omega_{\epsilon}}(x) \xi) \text{ in } \mathbb{R}^m,$$

(32)

with $\int_Q u_\xi \, dx = 0$. Here we have also used $a$, $f$, and $1_{\omega_{\epsilon}}$ to denote the $Q$-periodic extensions of these same functions. The symmetric matrix $S_\epsilon$, defined by

$$S_\epsilon \xi : \xi = -\int_Q a \nabla u_\xi \cdot \nabla u_\xi \, dx$$

$$= \int_Q f(x)1_{\omega_{\epsilon}}(x) \xi \cdot \nabla u_\xi \, dx \text{ for } \xi \in \mathbb{R}^m,$$

is negative, and satisfies

$$\text{Trace}(S_\epsilon) = -\int_{\omega_{\epsilon}} \frac{f^2(x)}{a(x)} \, dx \; + \; o(|\omega_{\epsilon}|).$$

**Proof.** Since $a$ is positive it follows that $S_\epsilon \leq 0$. Multiplication of (32) by $u_\xi$, and integration by parts immediately yield

$$\|\nabla u_\xi\|_{L^2(Q)} \leq C |\xi| |\omega_{\epsilon}|^{\frac{1}{2}}.$$

(33)

Let $v_\xi$ denote the $Q$-periodic solution to the following auxiliary problem

$$-\Delta v_\xi = \nabla \cdot (f(x)1_{\omega_{\epsilon}}(x) \xi) \text{ in } \mathbb{R}^m,$$

(34)
with \( \int_Q v^\xi dx = 0 \). A variant of the duality argument from the proof of Lemma 1 will give that
\[
\|v^\xi\|_{L^2(Q)} \leq C_\eta |\xi| \left| \omega_\epsilon \right|^{\frac{1}{2} + \frac{1}{m^*} - \eta},
\] (35)
for any \( \eta > 0 \). The argument goes as follows. Let \( w^\xi \) denote the \( Q \)-periodic solution to
\[\Delta w^\xi = v^\xi \text{ in } \mathbb{R}^m,\]
“normalized” by \( \int_Q w^\xi dx = 0 \). We have that
\[
\|w^\xi\|_{H^2(Q)} \leq C \|v^\xi\|_{L^2(Q)},
\]
and so
\[
\int_Q (v^\xi)^2 dx = -\int_Q \nabla w^\xi \cdot \nabla v^\xi dx
\]
\[
= \int_Q f(x) 1_{\omega_\epsilon}(x) \xi \cdot \nabla w^\xi dx
\]
\[
\leq C|\xi| |\omega_\epsilon|^{1/q} \left( \int_Q |\nabla w^\xi|^p dx \right)^{1/p},
\] (36)
provided \( 1 < q < \infty \), and \( 1/p + 1/q = 1 \). If we select \( q > \frac{2m^*}{m^*+2} \), \( m^* = \max\{2, m\} \) (so that \( 1 < p < \frac{2m^*}{m^*+2} \), and therefore, by Sobolev’s Imbedding Theorem \( (\int_Q |\nabla w^\xi|^p dx)^{1/p} \leq C_p \|w^\xi\|_{H^2(Q)} \)) it immediately follows that
\[
\|v^\xi\|^2_{L^2(Q)} \leq C_\eta |\xi| |\omega_\epsilon|^{\frac{1}{q}} \|v^\xi\|_{L^2(Q)}.
\]
By taking \( q \) sufficiently close to \( \frac{2m^*}{m^*+2} \) we obtain (35). We now compute
\[
S_\epsilon \xi \cdot \xi = \int_Q f(x) 1_{\omega_\epsilon}(x) \xi \cdot \nabla u^\xi dx
\]
\[
= -\int_Q \nabla v^\xi \cdot \nabla u^\xi dx
\]
\[
= -\int_Q a \nabla \left( \frac{v^\xi}{a} \right) \cdot \nabla u^\xi dx + \int_Q av^\xi \nabla \left( \frac{1}{a} \right) \cdot \nabla u^\xi dx
\]
\[
= -\int_Q a \nabla \left( \frac{v^\xi}{a} \right) \cdot \nabla u^\xi dx + o(|\omega_\epsilon|),
\]
thanks to (33) and (35). From the weak form of (32) we deduce that
\[
S_\epsilon \xi \cdot \xi = \int_Q f(x) 1_{\omega_\epsilon}(x) \xi \cdot \nabla \left( \frac{v^\xi}{a} \right) dx + o(|\omega_\epsilon|)
\]
\[
= \int_Q \frac{f(x)}{a(x)} 1_{\omega_\epsilon}(x) \xi \cdot \nabla v^\xi dx + o(|\omega_\epsilon|),
\]
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where, for the last identity, we again used (35). As a consequence,

\[
\text{Trace}(S_\epsilon) = \int_Q \frac{f(x)}{a(x)} 1_{\omega_\epsilon}(x) \sum_{i=1}^m \frac{\partial v^i}{\partial x_i} \, dx + o(|\omega_\epsilon|) ,
\]

\(v^i\) being the \(Q\)-periodic solution to

\[-\Delta v^i = \frac{\partial}{\partial x_j} \left( f(x) 1_{\omega_\epsilon}(x) \right) \quad \text{in } \mathbb{R}^m ,
\]
with \(\int_Q v^i \, dx = 0\). Note that, because of the periodic boundary conditions,

\[
\frac{\partial}{\partial x_j} v^i = - \frac{\partial^2}{\partial x_j \partial x_i} \Delta^{-1} \left( f(x) 1_{\omega_\epsilon}(x) - \frac{1}{|Q|} \int_{\omega_\epsilon} f(x) \, dx \right) ,
\]
for all \(1 \leq i, j \leq m\), and therefore

\[
\sum_{i=1}^m \frac{\partial v^i}{\partial x_i} = - f(x) 1_{\omega_\epsilon}(x) + O(|\omega_\epsilon|) .
\]

It follows immediately that

\[
\text{Trace}(S_\epsilon) = - \int_Q \frac{f^2(x)}{a(x)} 1_{\omega_\epsilon}(x) \, dx + o(|\omega_\epsilon|) .
\]

We are now ready for the

Proof of Theorem 3. We use the Hashin-Shtrikman variational technique as described in detail by Kohn and Milton (cf. [14]). For related polarization tensor bounds see also [3] and [15]. To simplify the notations we introduce \(p_\epsilon = \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon, \xi} \). According to Lemma 4 we have

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu = \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon_n} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx \quad \text{(37)}
\]

\[
+ \frac{1}{|\omega_\epsilon|} \min_{w \in H^1_{\text{per}}(Q)} \int_Q \hat{\gamma}_\epsilon \cdot |\nabla w + p_\epsilon|^2 \phi \, dx + o(1) .
\]

After introduction of a smooth reference medium, \(c\), with \(0 < c < \min\{\gamma_0, \gamma_1\}\), we may write

\[
\int_Q \hat{\gamma}_\epsilon |\nabla w + p_\epsilon|^2 \phi \, dx = \sup_{\eta \in L^2(Q)} - \int_{\Omega} \frac{1}{(\hat{\gamma}_\epsilon - c)} |\eta|^2 \phi \, dx + 2 \int_Q (\nabla w + p_\epsilon) \cdot \eta \phi \, dx
\]

\[
+ \int_Q c |\nabla w + p_\epsilon|^2 \phi \, dx
\]

\[
= \sup_{\eta \in L^2(Q)} - \int_{\Omega} \left( \frac{1}{(\hat{\gamma}_\epsilon - c)} + \frac{1}{c} \right) |\eta|^2 \phi \, dx
\]

\[
+ \int_Q c |\nabla w + p_\epsilon + \frac{1}{c} \eta|^2 \phi \, dx .
\]

\[
\text{Proof of Theorem 3. We use the Hashin-Shtrikman variational technique as described in detail by Kohn and Milton (cf. [14]). For related polarization tensor bounds see also [3] and [15]. To simplify the notations we introduce } p_\epsilon = \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon, \xi} . \text{ According to Lemma 4 we have}
\]

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi \, d\mu = \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon_n} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx \quad \text{(37)}
\]

\[
+ \frac{1}{|\omega_\epsilon|} \min_{w \in H^1_{\text{per}}(Q)} \int_Q \hat{\gamma}_\epsilon \cdot |\nabla w + p_\epsilon|^2 \phi \, dx + o(1) .
\]

\[
\text{After introduction of a smooth reference medium, } c, \text{ with } 0 < c < \min\{\gamma_0, \gamma_1\}, \text{ we may write}
\]

\[
\int_Q \hat{\gamma}_\epsilon |\nabla w + p_\epsilon|^2 \phi \, dx = \sup_{\eta \in L^2(Q)} - \int_{\Omega} \frac{1}{(\hat{\gamma}_\epsilon - c)} |\eta|^2 \phi \, dx + 2 \int_Q (\nabla w + p_\epsilon) \cdot \eta \phi \, dx
\]

\[
+ \int_Q c |\nabla w + p_\epsilon|^2 \phi \, dx
\]

\[
= \sup_{\eta \in L^2(Q)} - \int_{\Omega} \left( \frac{1}{(\hat{\gamma}_\epsilon - c)} + \frac{1}{c} \right) |\eta|^2 \phi \, dx
\]

\[
+ \int_Q c |\nabla w + p_\epsilon + \frac{1}{c} \eta|^2 \phi \, dx .
\]
Consequently, by using the inequality “\( \min \sup \geq \sup \min \)”, we obtain

\[
\min_{w \in H^{1}_{\text{per}}(Q)} \int_{Q} \hat{\gamma} \left| \nabla w + p_{e} \right|^{2} \phi \, dx \\
\geq \sup_{\eta \in L^{2}(Q)} \left( - \int_{Q} \left( \frac{1}{\hat{\gamma}} + \frac{1}{c} \right) |\eta|^{2} \phi \, dx \right) \\
+ \min_{w \in H^{1}_{\text{per}}(Q)} \int_{Q} c \left| \nabla w + p_{e} + \frac{1}{c} \eta \right|^{2} \phi \, dx .
\]

The minimum of the last right hand side is attained at the \( Q \)-periodic solution of

\[-\nabla \cdot (c \phi \nabla w) = \nabla \cdot (\phi (cp_{e} + \eta)) \text{ in } \mathbb{R}^{m} ,
\]

i.e., at

\[w_{0} = (-\Delta_{c\phi})^{-1} \nabla \cdot (\phi (cp_{e} + \eta)) ,
\]

where \( \Delta_{c\phi} \) denotes the operator \( \nabla \cdot (c \phi \nabla ) \) operating on \( Q \)-periodic functions (with average zero). For that \( w_{0} ,
\]

\[-\int_{Q} \left( \frac{1}{\hat{\gamma}} + \frac{1}{c} \right) |\eta|^{2} \phi \, dx + \int_{Q} c \left| \nabla w_{0} + p_{e} + \frac{1}{c} \eta \right|^{2} \phi \, dx
\]

\[= - \int_{Q} \left( \frac{1}{\hat{\gamma}} + \frac{1}{c} \right) |\eta|^{2} \phi \, dx + \int_{Q} c \left| p_{e} + \frac{1}{c} \eta \right|^{2} \phi \, dx
\]

\[-\int_{Q} c \left| \nabla w_{0} \right|^{2} \phi \, dx
\]

\[= \int_{Q} c |p_{e}|^{2} \phi \, dx - \int_{Q} \frac{1}{\hat{\gamma}} |\eta|^{2} \phi \, dx + 2 \int_{Q} \eta \cdot p_{e} \phi \, dx
\]

\[+ \int_{Q} (cp_{e} + \eta) \phi \cdot L_{c\phi} ((cp_{e} + \eta) \phi) \, dx .
\]

(40)

where \( L_{c\phi} \) is the linear operator on \( L^{2}_{\text{per}}(Q)^{m} \) \( (L^{2} \text{ vector fields on } Q \text{ that have been periodically extended to all of } \mathbb{R}^{m}) \) defined by

\[L_{c\phi} \eta = \nabla (-\Delta_{c\phi})^{-1} \nabla \cdot \eta .
\]

If we introduce

\[G_{\epsilon, \phi}(\eta) = - \int_{Q} \frac{1}{\hat{\gamma}} |\eta|^{2} \phi \, dx + 2 \int_{Q} \frac{\hat{\gamma}}{\hat{\gamma}} p_{e} \cdot \eta \phi \, dx + \int_{Q} \eta \cdot L_{c\phi}(\eta \phi) \, dx ,
\]

then a simple calculation yields

\[G_{\epsilon, \phi}(cp_{e} + \eta) = \int_{Q} \left( \frac{c}{\hat{\gamma}} + \frac{1}{c} \right) |p_{e}|^{2} \phi \, dx - \int_{Q} \frac{1}{\hat{\gamma}} |\eta|^{2} \phi \, dx + 2 \int_{Q} \eta \cdot p_{e} \phi \, dx
\]

\[+ \int_{Q} (cp_{e} + \eta) \phi \cdot L_{c\phi} ((cp_{e} + \eta) \phi) \, dx .
\]

(41)
Therefore, by a combination of (37), (39), (40), and (41), and use of the formula
\[p_c = \frac{\gamma - \gamma_1}{\gamma_1} \omega, \xi,\] we obtain
\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{i\xi} \xi_j \phi \, d\mu \\
\geq \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi \, dx - \frac{1}{|\omega_{\epsilon_n}|} \int_{Q} (\gamma_1 - c) |p_c|^2 \phi \, dx \\
+ \frac{1}{|\omega_{\epsilon_n}|} \sup_{\eta \in L^2(Q)} G_{c,\epsilon_n}(cp_{\epsilon_n} + \eta) + o(1),
\]
for any continuous function \(\lambda\) for \(c > 0\) and \(c < \min(\gamma_0, \gamma_1)\). The inequality in (42) may actually be shown to be an equality (i.e., in deriving (39) and forward) we have that “\(\min \sup = \sup \min\)” but for our purpose (to prove bounds) we do not need this fact. In the case of a smooth reference medium \(c > \max(\gamma_0, \gamma_1)\) we may proceed identically, except that the “\(\sup\)” (in (38) and forward) gets replaced by an “\(\inf\)”. In that case the interchange of the order of the “\(\inf\)” and the “\(\min\)” is automatically associated with an equality, and so we end up with
\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{i\xi} \xi_j \phi \, d\mu \\
= \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \frac{\gamma_0 - c}{\gamma_1 - c} |\xi|^2 \phi \, dx + \frac{1}{|\omega_{\epsilon_n}|} \inf_{\sigma \in L^2(Q)} G_{c,\epsilon_n}(\sigma) + o(1),
\]
for \(c > \max(\gamma_0, \gamma_1)\). A simple calculation gives,
\[
G_{c,\epsilon}(\lambda \gamma_1 p_c) + \int_{\omega_{\epsilon}} (\gamma_1 - \gamma_0) \frac{\gamma_0 - c}{\gamma_1 - c} \phi \, dx |\xi|^2 \\
= G_{c,\epsilon}(\lambda (\gamma_1 - \gamma_0) 1_{\omega_{\epsilon}} \xi) + \int_{\omega_{\epsilon}} (\gamma_1 - \gamma_0) \phi \, dx |\xi|^2 - \int_{\omega_{\epsilon}} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1 - c} \phi \, dx |\xi|^2 \\
= \int_{\omega_{\epsilon}} (\gamma_1 - \gamma_0) \phi \, dx |\xi|^2 - \int_{\omega_{\epsilon}} (\lambda - 1)^2 \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1 - c} \phi \, dx |\xi|^2 \\
+ \int_{Q} (\gamma_1 - \gamma_0) \lambda \phi 1_{\omega_{\epsilon}} \xi \cdot L_{c\phi} ((\gamma_1 - \gamma_0) \lambda \phi 1_{\omega_{\epsilon}} \xi) \, dx,
\]
for any continuous function \(\lambda\). We specifically choose \(\lambda_c = 1 + |c - \gamma_1|\), and note that whenever \(c\) tends to \(\gamma_1\), \(\lambda_c\) tends to 1. With this choice of \(\lambda\)
\[
\int_{\omega_{\epsilon}} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1 - c} (\lambda_c - 1)^2 \phi \, dx = \int_{\omega_{\epsilon}} (\gamma_1 - \gamma_0)^2 (\gamma_1 - c) \phi \, dx.
\]
If we only use test functions of the form \(\sigma = \lambda_c \gamma_1 p_c\), then (42) gives
\[
\int_{\Omega} (\gamma_1 - \gamma_0) M \xi \cdot \xi \, d\mu \geq \left( \int_{\Omega} (\gamma_1 - \gamma_0) \phi \, d\mu - \int_{\Omega} (\gamma_1 - \gamma_0)^2 (\gamma_1 - c) \phi \, d\mu \right) |\xi|^2 \\
+ \frac{1}{|\omega_{\epsilon_n}|} A_{\epsilon_n} \xi \cdot \xi + o(1),
\]
(45)
for $0 < c < \min(\gamma_1, \gamma_0)$. Here the negative definite, symmetric matrix $A_c$ is given by

$$A_c \xi \cdot \xi = \int_Q (\gamma_1 - \gamma_0) \lambda_c \phi_1 \omega_c \xi \cdot L_c \phi ((\gamma_1 - \gamma_0) \lambda_c \phi_1 \omega_c \xi) \, dx$$

According to Lemma 5 (with $a = c \phi$ and $f = (\gamma_1 - \gamma_0) \lambda_c \phi$) we have that

$$\text{Trace}(A_c) = - \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\lambda^2}{c} \phi_1 \omega_c \, dx + o(|\omega_c|) \, .$$

(46)

A combination of (45) and (46), and passage to the limit $|\omega_{c_n}| \to 0$ gives

$$\int_{\Omega} (\gamma_1 - \gamma_0) \text{Trace}(M) \, \phi \, d\mu \geq m \int_{\Omega} (\gamma_1 - \gamma_0) \phi \, d\mu - \int_{\Omega} (\gamma_1 - \gamma_0)^2 \left( m(\gamma_1 - c) + \frac{\lambda^2}{c} \right) \phi \, d\mu \, ,$$

for $0 < c < \min(\gamma_0, \gamma_1)$. Letting $c$ tend to $\min(\gamma_0, \gamma_1)$ we now obtain

$$\int_{\Omega} (\gamma_1 - \gamma_0) \text{Trace}(M) \, \phi \, d\mu \geq \int_{\Omega} (\gamma_1 - \gamma_0) \left( m - 1 + \frac{\gamma_0}{\gamma_1} \right) \chi_0(x) \phi \, d\mu$$

$$+ \int_{\Omega} (\gamma_1 - \gamma_0) \left( m - m(\gamma_1 - \gamma_0)^2 - \frac{\gamma_1 - \gamma_0}{\gamma_0} \lambda^2 \right) \left( 1 - \chi_0(x) \right) \phi \, d\mu \, ,$$

(47)

where $\chi_0$ is the indicator function of the set $\{x : \gamma_1(x) < \gamma_0(x)\}$. Similarly if, for $c > \max(\gamma_0, \gamma_1)$, we only use test functions of the form $\sigma = \lambda_c \gamma_1 p_c$ in (43) then a passage to the limit $|\omega_{c_n}| \to 0$, followed by a passage to the limit $c \to \max(\gamma_0, \gamma_1)$ gives

$$\int_{\Omega} (\gamma_1 - \gamma_0) \text{Trace}(M) \, \phi \, d\mu \leq \int_{\Omega} (\gamma_1 - \gamma_0) \left( m - 1 + \frac{\gamma_0}{\gamma_1} \right) \chi_1(x) \phi \, d\mu$$

$$+ \int_{\Omega} (\gamma_1 - \gamma_0) \left( m - m(\gamma_1 - \gamma_0)^2 - \frac{\gamma_1 - \gamma_0}{\gamma_0} \lambda^2 \right) \left( 1 - \chi_1(x) \right) \phi \, d\mu \, ,$$

(48)

where $\chi_1$ is the indicator function of the set $\{x : \gamma_0(x) < \gamma_1(x)\}$. We recall that $\phi$ is an arbitrary, smooth, uniformly positive function. By a limiting argument the inequalities (47)-(48) are seen to hold for an arbitrary, continuous $\phi \geq 0$. Using (47) for $\phi$ whose support lies inside the support of $\chi_0$ (i.e., inside $\{x : \gamma_1(x) < \gamma_0(x)\}$) and (48) for $\phi$ whose support lies inside the support of $\chi_1$ (i.e., inside $\{x : \gamma_0(x) < \gamma_1(x)\}$) we arrive at the first of the bounds of Theorem 3.
Let $\eta$ be a constant vector. By a simple calculation

$$G_{c,\omega}(1, \omega, \eta) = \int_{\omega} \frac{1}{\gamma_1 - c} \phi \ dx \ |\eta|^2 + 2 \int_{\omega} \frac{\gamma_1 - \gamma_0}{\gamma_1 - c} \phi \ dx \ \xi \cdot \eta + B_c \eta \cdot \eta,$$

where $B_c$ is the negative definite matrix given by

$$B_{c,\omega} \eta \cdot \eta = \int_{Q} \phi(1, \omega, \eta) \cdot L_{c,\omega}(\phi) \ dx.$$

According to Lemma 5,

$$\text{Trace}(B_c) = -\int_{\omega} \frac{1}{c} \phi \ dx + o(\lambda) \ (49).$$

If we only use test functions of the form $\sigma = 1, \omega, \eta$, then (42) yields

$$\int_{\Omega} (\gamma_1 - \gamma_0) M \xi \cdot \xi \phi d\mu - \frac{1}{|\omega|} \int_{\omega} \frac{\gamma_1 - \gamma_0}{\gamma_1 - c} \phi \ dx \ \xi \cdot \eta$$

$$\geq \frac{1}{|\omega|} \int_{\omega} (\gamma_1 - \gamma_0) \frac{\gamma_0 - c}{\gamma_1 - c} \phi \ dx + \frac{1}{|\omega|} \int_{\omega} \frac{1}{\gamma_1 - c} \phi \ dx \ |\eta|^2$$

$$+ \frac{1}{\lambda} B_c \eta \cdot \eta + o(1),$$

for any $\xi, \eta \in \mathbb{R}^m$, provided $0 < c < \min(\gamma_0, \gamma_1)$. By taking $\eta = e_k$ (the $k'th$ standard basis vector) and $\xi = \xi_k$, for $k = 1, \ldots, m$, then summing these inequalities, and passing to the limit as $|\omega| \to 0$, we arrive at

$$\int_{\Omega} (\gamma_1 - \gamma_0) \sum_{k=1}^{m} M \xi_k \cdot \xi_k \phi d\mu - 2 \int_{\Omega} \frac{\gamma_1 - \gamma_0}{\gamma_1 - c} \phi \ d\mu \sum_{k=1}^{m} \xi_k \cdot e_k$$

$$\geq \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\gamma_0 - c}{\gamma_1 - c} \phi \ d\mu \sum_{k=1}^{m} |\xi_k|^2 - m \int_{\Omega} \frac{1}{\gamma_1 - c} \phi \ d\mu$$

$$- \int_{\Omega} \frac{1}{c} \phi \ d\mu, \quad (50)$$

provided $0 < c < \min(\gamma_0, \gamma_1)$. Similarly, starting from (43) we obtain

$$\int_{\Omega} (\gamma_1 - \gamma_0) \sum_{k=1}^{m} M \xi_k \cdot \xi_k \phi d\mu - 2 \int_{\Omega} \frac{\gamma_1 - \gamma_0}{\gamma_1 - c} \phi \ d\mu \sum_{k=1}^{m} \xi_k \cdot e_k$$

$$\leq \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\gamma_0 - c}{\gamma_1 - c} \phi \ d\mu \sum_{k=1}^{m} |\xi_k|^2 - m \int_{\Omega} \frac{1}{\gamma_1 - c} \phi \ d\mu$$

$$- \int_{\Omega} \frac{1}{c} \phi \ d\mu, \quad (51)$$

provided $c > \max(\gamma_0, \gamma_1)$. By a limiting argument the inequalities (50)- (51) hold for all continuous $\phi \geq 0$. 

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Using (50) for \( \phi \) whose support lies inside \( \{ x : \gamma_0(x) < \gamma_1(x) \} \) we see that for any smooth function \( c < \min(\gamma_0, \gamma_1) \) there exists a set \( E_c \subset \{ x : \gamma_0(x) < \gamma_1(x) \} \), with \( \mu(E_c) = 0 \), such that

\[
\sum_{k=1}^{m} M(x) \xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\
\geq \frac{\gamma_0(x) - c(x)}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} |\xi_k|^2 \\
- \frac{1}{\gamma_1(x) - \gamma_0(x)} \left( \frac{m}{\gamma_1(x) - c(x)} + \frac{1}{c(x)} \right) ,
\]

for all \( x \in \{ x : \gamma_0(x) < \gamma_1(x) \} \setminus E_c \) and all \( \xi_k \in \mathbb{R}^m, 1 \leq k \leq m \). By picking a sequence of smooth functions, \( c \), that tend pointwise to \( \min(\gamma_0, \gamma_1) \) we conclude that

\[
\mu \text{ almost everywhere in the set } \{ x : \gamma_0(x) < \gamma_1(x) \} .
\]

Using (51) for \( \phi \) whose support lies inside \( \{ x : \gamma_1(x) < \gamma_0(x) \} \) we see that for any smooth function \( c > \max(\gamma_0, \gamma_1) \) there exists a set \( \tilde{E}_c \subset \{ x : \gamma_1(x) < \gamma_0(x) \} \), with \( \mu(\tilde{E}_c) = 0 \), such that

\[
\sum_{k=1}^{m} M(x) \xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - \gamma_0(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\
\geq \frac{\gamma_0(x) - c(x)}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} |\xi_k|^2 \\
- \frac{1}{\gamma_1(x) - \gamma_0(x)} \left( \frac{m}{\gamma_1(x) - c(x)} + \frac{1}{c(x)} \right) ,
\]

for all \( x \in \{ x : \gamma_0(x) < \gamma_1(x) \} \setminus \tilde{E}_c \) and all \( \xi_k \in \mathbb{R}^m, 1 \leq k \leq m \). By picking a sequence of smooth functions, \( c \), that tend pointwise to \( \max(\gamma_0, \gamma_1) \) we conclude that

\[
\mu \text{ almost everywhere in the set } \{ x : \gamma_1(x) < \gamma_0(x) \} .
\]

A combination of (52) and
shows that there exists a set $E$, with $\mu(E) = 0$, such that
\[
\sum_{k=1}^{m} M(x) \xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - \gamma_0(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \geq \frac{-1}{\gamma_1(x) - \gamma_0(x)} \left( \frac{m}{\gamma_1(x) - \gamma_0(x)} + \frac{1}{\gamma_0(x)} \right),
\]
for all $x$ in $\{ x : \gamma_0(x) \neq \gamma_0(x) \} \setminus E$, and all $\xi_k \in \mathbb{R}^m$, $1 \leq k \leq m$. For a fixed $x \in \{ x : \gamma_0(x) \neq \gamma_0(x) \} \setminus E$ we now take
\[
\xi_k = \frac{1}{\gamma_1(x) - \gamma_0(x)} M(x)^{-1} e_k,
\]
and insert it into (54), to obtain
\[
\text{Trace} \left( M(x)^{-1} \right) \leq m + \frac{\gamma_1(x) - \gamma_0(x)}{\gamma_0(x)} = m - 1 + \frac{\gamma_1(x)}{\gamma_0(x)},
\]
for all $x$ in $\{ x : \gamma_0(x) \neq \gamma_0(x) \} \setminus E$, which is exactly the second inequality of Theorem 3.

4 Applications of the pointwise trace bounds

In this section, we will show that the pointwise trace bounds of Theorem 3 allow us to give an explicit formula for the polarization tensor corresponding to asymptotically thin domains. After combination with Theorem 1 we thus quite simply obtain an explicit representation formula for the voltage perturbation $u_\varepsilon - U$. This representation formula was previously studied by Beretta et. al. [4, 5]. The analysis in those papers rely on fine regularity estimates for the gradient of solutions of linear elliptic problems. This analysis in fact provides precise information on the asymptotic rate of the error of the representation formula. Here we simply verify its asymptotic correctness, however, due to the fact that we do not rely on the aforementioned elliptic estimates, we are able to deal with “thin” sets $\omega_\varepsilon$ with very “rapidly oscillating” boundaries. The key observation is the following.

Lemma 6. Let $\omega_\varepsilon$, $M$ and $\mu$ be as in Proposition 1 (and Theorem 1), and suppose $\gamma_0$ and $\gamma_1$ are smooth conductivities. Let $\Omega'$ be a smooth domain with $\omega_\varepsilon \subset K_0 \subset \Omega' \subset \Omega$. Suppose there exists a sequence of functions $w_\varepsilon \in H^1(\Omega')$ such that
\[
\lim_{|\omega_\varepsilon| \to 0} \frac{1}{|\omega_\varepsilon|} \left\| \nabla w_\varepsilon - 1_{\omega_\varepsilon} \nabla x_m \right\|^2_{L^2(\Omega')} = 0,
\]
where $x_m$ is the last coordinate of $x$ in the orthonormal base $e_1, \ldots, e_m$. Then
\[
M(x) = \sum_{j=1}^{m-1} e_j \otimes e_j + \frac{\gamma_0(x)}{\gamma_1(x)} e_m \otimes e_m,
\]
$\mu$ almost everywhere in $\Omega' \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$ (and $\Omega \setminus \{ x : \gamma_0(x) \neq \gamma_1(x) \}$).
The trace bounds from Theorem 3 now assert that the other eigenvalues for any smooth, uniformly positive function $\phi$. Therefore, in the limit as $|\omega_{\epsilon_n}| \to 0$. From our hypothesis we have that

$$\int_\Omega (\gamma_1 - \gamma_0) M_{mm} \phi \, d\mu = \int_\Omega (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} \phi \, d\mu$$

for any smooth, uniformly positive function $\phi$. By a limiting argument we get that the same identity holds for any continuous $\phi \geq 0$. It follows immediately that $M(x) e_m \cdot e_m = M_{mm}(x) = \gamma_0(x)/\gamma_1(x)$, $\mu$ almost everywhere in $\Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$. We already know that $\min\{1, \gamma_0(x)/\gamma_1(x)\} \leq M(x) \leq \max\{1, \gamma_0(x)/\gamma_1(x)\}$, $\mu$ almost everywhere in $\Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$, i.e.,

$$\min\{1, \gamma_0(x)/\gamma_1(x)\} \leq \min_{|\xi|=1} M(x) \xi \cdot \xi \leq \max_{|\xi|=1} M(x) \xi \cdot \xi \leq \max\{1, \gamma_0(x)/\gamma_1(x)\},$$

and we may therefore conclude that, $\mu$ almost everywhere in $\Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$, either

$$M(x) e_m \cdot e_m = \min_{|\xi|=1} M(x) \xi \cdot \xi = \gamma_0(x)/\gamma_1(x) \quad (\text{and } \gamma_0(x) < \gamma_1(x)),$$

or

$$M(x) e_m \cdot e_m = \max_{|\xi|=1} M(x) \xi \cdot \xi = \gamma_0(x)/\gamma_1(x) \quad (\text{and } \gamma_0(x) > \gamma_1(x)).$$

In any event we have that $e_m$ is an eigenvector of $M(x)$ corresponding to the eigenvalue $\lambda_m = \gamma_0(x)/\gamma_1(x)$, $\mu$ almost everywhere in $\Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$. The trace bounds from Theorem 3 now assert that the other $m-1$ eigenvalues $\lambda_j(x)$, $1 \leq j \leq m-1$, of $M(x)$ satisfy

$$\sum_{j=1}^{m-1} \lambda_j(x) \leq m-1, \quad \text{and} \quad \sum_{j=1}^{m-1} \lambda_j^{-1}(x) \leq m-1,$$
\( \mu \) almost everywhere in \( \Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \} \). Since we also have that
\[
\min\{1, \gamma_0(x)/\gamma_1(x)\} \leq \lambda_j(x) \leq \max\{1, \gamma_0(x)/\gamma_1(x)\} ,
\]
(due to the fact that \( \min\{1, \gamma_0(x)/\gamma_1(x)\} \leq M(x) \leq \max\{1, \gamma_0(x)/\gamma_1(x)\} \)) we conclude that the \( m-1 \) eigenvalues \( \lambda_j(x) \), \( 1 \leq j \leq m-1 \), all are equal to one, \( \mu \) almost everywhere in \( \Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \} \). On the hyperplane orthogonal to \( \varepsilon_m \), the polarization tensor is thus, \( \mu \) almost everywhere in \( \Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \} \), the identity matrix, and so any orthogonal family in that plane is a base of eigenvectors, corresponding to the eigenvalue 1.

The analysis presented in this section would apply to more general “thin” domains, but for simplicity from now on we only consider symmetric ones with a flat mid-surface. We select a coordinate system \( x = (x', x_m) \) so that the mid-surface \( S_0 \) lies in the hyperplane \( x_m = 0 \). In other words, we consider \( \omega_\varepsilon \) of the form
\[
\omega_\varepsilon = \{(x', x_m) : x' \in S_0 , \ |x_m| < c\varepsilon(x')\} \subset \subset \Omega , \quad 0 < \varepsilon << 1 , \quad (55)
\]
where \( S_0 \) is a bounded smooth subdomain of \( \mathbb{R}^{m-1} \), and \( h_\varepsilon \in C^1(\overline{S}_0) \) satisfies
\[
0 < c_0 \leq \min_{x' \in S_0} h_\varepsilon(x') \leq \max_{x' \in S_0} h_\varepsilon(x') < C_0 , \quad \|h_\varepsilon\|_{C^1(\overline{S}_0)} \leq C \varepsilon^{-a} , \quad (56)
\]
for some positive constants \( c_0, C_0, C, \) and \( a < 1 \), all independent of \( \varepsilon \). From the uniform bound on \( h_\varepsilon \) it follows immediately that we may find a subsequence that converges weak* in \( L^\infty \) to some limit \( h_0 \in L^\infty(S_0) \). We shall without loss of generality therefore assume that
\[
\int_{S_0} h_\varepsilon(x')g(x') \, dx' \to \int_{S_0} h_0(x')g(x') \, dx' \quad \forall g \in L^1(S_0) , \quad (57)
\]
as \( \varepsilon \to 0 \).

**Proposition 2.** Let \( \omega_\varepsilon \) be as in (55) with \( h_\varepsilon \) satisfying (56) and (57). Suppose \( \gamma_0 \) and \( \gamma_1 \) are smooth conductivities. Given any \( \psi \in H^{-1/2}(\partial\Omega) \), with \( \int_{\partial\Omega} \psi \, d\sigma = 0 \), let \( U \) and \( u_\varepsilon \) denote the solutions to (1) and (2), respectively. Then
\[
(u_\varepsilon - U)(y) = |\omega_\varepsilon| \int_\Omega (\gamma_1 - \gamma_0)M \nabla U \cdot \nabla N(x,y) \, d\mu_x + o(|\omega_\varepsilon|) , \quad (58)
\]
for \( y \in \partial\Omega \), with
\[
d\mu = \frac{1}{\int_{S_0} h_0(x') \, dx'} h_0(x') \, dx'_{S_0} \quad \text{and} \quad M(x) = \sum_{j=1}^{m-1} e_j \otimes e_j + \frac{\gamma_0(x)}{\gamma_1(x)} e_m \otimes e_m .
\]

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Proof. Due to the assumptions (56) and (57) about $h_\epsilon$ it is straightforward to show that for any $\phi \in C^0(\Omega)$
\[
\frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \phi \, dx \to \int_{S_0} \frac{1}{h_0(x')} \int_{S_0} \phi(x',0) h_0(x') \, dx'.
\]
as $|\omega_\epsilon| \to 0$, in other words
\[
\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \to \frac{1}{\int_{S_0} h_0(x') \, dx'} h_0(x') \, dx' S_0 \quad (59)
\]
in the sense of measures, as $|\omega_\epsilon| \to 0$. Let $\phi_\epsilon(x')$ be a cutoff function on $\mathbb{R}^{m-1}$, with $\phi_\epsilon \equiv 1$ in $S_0$, $\text{supp}(\phi_\epsilon) \subset S_0^c = \{x' : \text{dist}(x', S_0) < \epsilon\}$, and $\|\phi_\epsilon\|_{C^1} \leq C\epsilon^{-1}$.

Similarly let $\psi_\epsilon(\cdot)$ be a cutoff function on $\mathbb{R}$, with $\psi_\epsilon \equiv 1$ in $(-\epsilon C_0, \epsilon C_0)$, $\text{supp}(\psi_\epsilon) \subset (-\epsilon^a C_0, \epsilon^a C_0)$, and $\|\psi_\epsilon\|_{C^1} \leq C\epsilon^{-a}$, $C_0$ and $a < 1$ being the constants from (56). For $\epsilon$ sufficiently small, we extend $h_\epsilon$ to $S_0^c$, in such a way that
\[
0 < \frac{1}{2} c_0 \leq \min_{x' \in \overline{S_0}} h_\epsilon(x') \leq \max_{x' \in \overline{S_0}} h_\epsilon(x') < 2C_0, \quad \|h_\epsilon\|_{C^1(S_0^c)} \leq C\epsilon^{-a}.
\]

We now define
\[
v_{m,\epsilon}(x) = \begin{cases} 
  e h_\epsilon(x') & \text{for } e h_\epsilon(x') < x_m \\
  x_m & \text{for } -e h_\epsilon(x') < x_m < e h_\epsilon(x') \\
  -e h_\epsilon(x') & \text{for } x_m < -e h_\epsilon(x')
\end{cases}
\]
and
\[
w_\epsilon(x) = \phi_\epsilon(x') \psi_\epsilon(x_m) v_{m,\epsilon}(x)
\]
for all $x \in S_0^c \times \mathbb{R}$. We set $w_\epsilon(x) = 0$ for all $x \in \Omega$, for which $x'$ lies outside $S_0^c$. A simple calculation gives
\[
\int_{\Omega} |\nabla w_\epsilon - 1_{\omega_\epsilon} \nabla x_m|^2 \, dx = \int_{\Omega \setminus \omega_\epsilon} |\nabla w_\epsilon|^2 \, dx
\]
\[
\leq \int_{(S_0 \setminus S_0) \times (-e^a C_0, e^a C_0)} |\nabla \phi_\epsilon(x') \psi_\epsilon(x_m)| v_{m,\epsilon}|^2 \, dx
\]
\[
+ \int_{(S_0 \setminus S_0) \times (-e^a C_0, e^a C_0)} \phi_\epsilon(x') \psi_\epsilon(x_m) \nabla v_{m,\epsilon}|^2 \, dx
\]
\[
+ \int_{(S_0 \setminus (-e^a C_0, e^a C_0)) \setminus \omega_\epsilon} |\nabla (\phi_\epsilon(x') \psi_\epsilon(x_m)) v_{m,\epsilon}|^2 \, dx
\]
\[
+ \int_{(S_0 \setminus (-e^a C_0, e^a C_0)) \setminus \omega_\epsilon} \phi_\epsilon(x') \psi_\epsilon(x_m) \nabla v_{m,\epsilon}|^2 \, dx
\]
\[
\leq C_1 \epsilon^{1+a} + C_2 \epsilon^{1+a} + C_3 C_2 \epsilon^{2-a} + C_4 \epsilon^{2-a}.
\]
As a consequence
\[
\frac{1}{|\omega_e|} \| \nabla w - 1_{\omega_e} \nabla x_m \|_{L^2(\Omega)}^2 = O(|\omega_e|^a + |\omega_e|^{1-a}) \to 0 ,
\] (60)
as $|\omega_e| \to 0$. From Theorem 1 we know that, after extraction of a subsequence, the representation formula (58) holds with $\mu$ being the limit (in the sense of measures) of $\frac{1}{|\omega_e|} 1_{\omega_e}$, i.e., according to (59)
\[
d\mu = \frac{1}{\int_{S_0} h_0(x') dx'} h_0(x') dx'_{S_0}.
\]
Due to the estimate (60) we may apply Lemma 6 to conclude that for this particular subsequence
\[
M(x) = \sum_{j=1}^{m-1} e_j \otimes e_j + \frac{\gamma_0(x)}{\gamma_1(x)} e_m \otimes e_m ,
\]
$\mu$ almost everywhere in $\{ x : \gamma_0(x) \neq \gamma_1(x) \}$. Conditioned on the extraction of a subsequence we have now established the representation formula (58), but since all elements of the formula are explicit, and independent of the particular subsequence it follows that the formula holds, provided only $|\omega_e| \to 0$, as stated in this proposition.

In contrast to [4, 5], we have allowed the surfaces of the thin domains to exhibit oscillations of amplitude $\epsilon$, on a scale of order $\epsilon^a$, $a < 1$. Some regularity of $h_\epsilon$ is essential for the result of Proposition 2 to hold; assuming just that $h_\epsilon$ is measurable and bounded will not suffice. We are also convinced that the condition $a < 1$ is essential – if we permit $a \geq 1$ we expect the emergence of other polarization tensors $M$ than that in Proposition 2.

We complete this section by providing a simple, two-dimensional example of a family of multiply connected domains, the limiting measure, $\lim \frac{1}{|\omega_e|} 1_{\omega_e}$, of which is the surface measure on a flat hyper-surface, but for which the “effective” polarization tensor is in general different from that of Proposition 2. By appropriately connecting the individual components of these domains with very thin “sheets”, and regularizing the surfaces, our construction would give rise to a family of the type $a = 1$ (referring to the notation of Proposition 2).

**Proposition 3.** Assume that the conductivities $\gamma_0$ and $\gamma_1$ are constants, with $\gamma_0 \neq \gamma_1$, and assume that the domain $\Omega$ contains the square $[-1,1] \times [-1,1]$. Let $\delta$ be a fixed parameter, $0 < \delta < \infty$, and define $\delta^* = \delta + \sqrt{\delta}$. For any $0 < \epsilon < \min\{\delta, 1/\delta^*\}$ let $\omega_\epsilon^\delta \subset [-1,1] \times [-1,1]$ denote the open set
\[
\omega_\epsilon^\delta = \bigcup_{k=0}^{N_{\epsilon,\delta}} \left( Q_\epsilon^\delta + \left( k\epsilon \delta^* \right) \right)
\]

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Figure 1: Two adjacent “cells” of $\omega_\varepsilon$, for $\varepsilon = \frac{1}{2}$ on the left, and $\varepsilon = 2$ on the right.

with

$$N_{\varepsilon, \delta} = \left[ \frac{1}{\varepsilon \delta^2} \right] - 1 , \quad \text{and} \quad Q_\varepsilon = \left( -\frac{\varepsilon\delta}{2}, -\frac{\varepsilon}{2\delta^2} - \frac{\varepsilon}{2\delta} \right).$$

Then

$$\frac{\varepsilon}{\delta} - \varepsilon^2 \leq |\omega_\varepsilon^\delta| \leq \frac{\varepsilon}{\delta^2} ,$$

and

$$\lim_{\varepsilon \to 0} \int_{\Omega} \frac{1}{|\omega_\varepsilon^\delta(x)|} \omega_\varepsilon^\delta(x) \phi(x) \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{\delta^*}{\varepsilon} 1_{\omega_\varepsilon^\delta(x)} \phi(x) \, dx = \int_{0}^{1} \phi(x_1, 0) \, dx_1 ,$$

for all $\phi \in C^0(\Omega)$. Let $\omega_\varepsilon^\delta_n, M_\varepsilon^\delta(\cdot) \in L^2((0,1), dx)$ be a subsequence and a polarization tensor, the existence of which are guaranteed by Proposition 1 (and Theorem 1). The polarization tensor necessarily has the form

$$M_\varepsilon^\delta = \begin{bmatrix} m_1^\delta & 0 \\ 0 & m_2^\delta \end{bmatrix} ,$$

with

$$\lim_{\delta \to 0} m_1^\delta = \frac{\gamma_0}{\gamma_1} , \quad \lim_{\delta \to 0} m_2^\delta = 1 ,$$

and

$$\lim_{\delta \to \infty} m_1^\delta = 1 , \quad \lim_{\delta \to \infty} m_2^\delta = \frac{\gamma_0}{\gamma_1} ,$$

the limits being, e.g., in $L^2((0,1), dx)$.

Proof. The verification of the bounds on the Lebesgue measure of $\omega_\varepsilon^\delta$, and the proof of the limiting identity

$$\lim_{\varepsilon \to 0} \frac{1}{|\omega_\varepsilon^\delta|} \omega_\varepsilon^\delta \, dx = \lim_{\varepsilon \to 0} \frac{\delta^*}{\varepsilon} 1_{\omega_\varepsilon^\delta} \, dx = dx_1|_{(0,1)}$$

are quite straightforward, and are left to the reader. The fact that $M_\varepsilon^\delta$ has eigendirections that are parallel (and orthogonal) to the $x_1$ axis, follows immediately from (9)-(8) and natural solution symmetries, if we enlarge the domain $\Omega$ to
be symmetric in the $x_1$ axis (and note that the polarization tensor is independent of $\Omega$). It thus only remains to verify the limiting identities concerning $m_1^\delta$ and $m_2^\delta$. We note that it suffices to verify that

$$\lim_{\delta \to 0} m_1^\delta = \frac{\gamma_0}{\gamma_1}, \quad \text{and} \quad \lim_{\delta \to \infty} m_2^\delta = \frac{\gamma_0}{\gamma_1} \quad \text{in } L^2((0,1), dx). \quad (61)$$

If for instance we have verified the first limiting statement in (61), then it follows immediately (since $\min\{1, \frac{\gamma_0}{\gamma_1}\} \leq m_1^\delta \leq \max\{1, \frac{\gamma_0}{\gamma_1}\}$) that

$$\lim_{\delta \to 0} \frac{1}{m_1^\delta} = \frac{\gamma_1}{\gamma_0} \quad \text{in } L^2((0,1), dx).$$

From integration of the inequalities of Theorem 3 we now get

$$\limsup_{\delta \to 0} \int_0^1 m_2^\delta(x) \, dx \leq 1, \quad \text{and} \quad \limsup_{\delta \to -0} \int_0^1 \frac{1}{m_2^\delta(x)} \, dx \leq 1.$$

Use of these inequalities and Cauchy-Schwarz’s inequality yields

$$1 = (\int_0^1 1 \, dx)^2 \leq \liminf_{\delta \to 0} \int_0^1 m_2^\delta(x) \, dx \cdot \limsup_{\delta \to -0} \int_0^1 \frac{1}{m_2^\delta(x)} \, dx \leq 1,$$

based on which we immediately conclude that

$$\lim_{\delta \to 0} \int_0^1 m_2^\delta(x) \, dx = \liminf_{\delta \to 0} \int_0^1 m_2^\delta(x) \, dx = \limsup_{\delta \to -0} \int_0^1 m_2^\delta(x) \, dx = 1.$$

Since $m_2^\delta - 1$ is either a.e. positive, or a.e. negative, it follows that

$$\lim_{\delta \to 0} \int_0^1 |m_2^\delta(x) - 1| \, dx = 0,$$

and so

$$\int_0^1 |m_2^\delta(x) - 1|^2 \, dx \leq \left(\max(1, \frac{\gamma_0}{\gamma_1}) + 1\right) \int_0^1 |m_2^\delta(x) - 1| \, dx \to 0 \quad \text{as } \delta \to 0,$$

i.e., $\lim_{\delta \to 0} m_2^\delta = 1$ in $L^2((0,1), dx)$. Similarly, the second limiting statement in (61) implies that $\lim_{\delta \to \infty} m_1^\delta = 1$ in $L^2((0,1), dx)$. We now proceed to verify the two limiting statements of (61). In that regard it suffices to verify that

$$\lim_{\delta \to 0} \int_0^1 m_1^\delta \, dx = \frac{\gamma_0}{\gamma_1}, \quad \text{and} \quad \lim_{\delta \to \infty} \int_0^1 m_2^\delta \, dx = \frac{\gamma_0}{\gamma_1}. \quad (62)$$

The expressions $m_1^\delta - \frac{\gamma_0}{\gamma_1}$ and $m_2^\delta - \frac{\gamma_0}{\gamma_1}$ are namely either a.e. positive, or a.e. negative, and (62) thus implies

$$\lim_{\delta \to 0} \int_0^1 |m_1^\delta(x) - \frac{\gamma_0}{\gamma_1}| \, dx = 0, \quad \text{and} \quad \lim_{\delta \to \infty} \int_0^1 |m_2^\delta(x) - \frac{\gamma_0}{\gamma_1}| \, dx = 0.$$
Due to the a.e. boundedness of the expressions $m_1^\delta - \frac{\gamma_0}{\gamma_1}$ and $m_2^\delta - \frac{\gamma_0}{\gamma_1}$ the $L^2$ convergence, as stated in (61), follows. From Lemma 3 (and Remark 1) we know that

$$
\int_0^1 m_2^\delta(x) \, dx = \frac{\gamma_0}{\gamma_1} + \frac{1}{\gamma_1 - \gamma_0} \lim_{\epsilon \to 0} \min_{w \in H^1(\Omega)} \frac{\delta^*}{\epsilon_n} \int_\Omega \left[ \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega^\delta} \nabla x_i \right]^2 \, dx.
$$

In order to arrive at (62) it is thus sufficient to find two families of functions $w_{i,\epsilon} \in H^1(\Omega)$, $0 < \epsilon < \min\{\delta, 1/\delta^*\}$, $i = 1, 2$, so that

$$
\frac{\delta^*}{\epsilon} \int_\Omega \left[ \nabla w_{i,\epsilon} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega^\delta} \nabla x_i \right]^2 \, dx \leq g_i(\delta),
$$

for $\delta$ sufficiently close to zero (for $i = 1$), and $\delta$ sufficiently large (for $i = 2$), where $g_1$ is such that $\lim_{\delta \to 0} g_1(\delta) = 0$, and $g_2$ is such that $\lim_{\delta \to \infty} g_2(\delta) = 0$.

We first consider $\delta < 1$. We define a function $f_\epsilon$ with compact support in $[-(1/\delta^* + d)\epsilon, (1/\delta^* + d)\epsilon] \times [-1/(2\delta) + d\epsilon, (1/(2\delta) + d)\epsilon]$, $d$ being a positive constant, a precise value of which will be chosen later. The function $f_\epsilon$ is constructed in the first quadrant as follows.

For $0 \leq x_2 \leq \epsilon/(2\delta)$, we choose

$$
\begin{cases}
f_\epsilon(x) = x_1 & \text{for } x_1 \in \left[0, \frac{\delta^*}{2}\right], \\
f_\epsilon(x) = \frac{\delta^*}{2} \left(1 + \frac{\delta^* - 2x_2}{2d\epsilon}\right) & \text{for } x_1 \in \left[\frac{\delta^*}{2}, \frac{\delta^*}{2} + d\epsilon\right].
\end{cases}
$$

For $\epsilon/(2\delta) \leq x_2 \leq \epsilon/(2\delta) + d\epsilon$, and $0 \leq x_1 \leq \frac{\delta^*}{2}$, we choose

$$
f_\epsilon(x) = x_1 \left(1 + \frac{\epsilon/\delta - 2x_2}{2d\epsilon}\right).
$$

Finally, for $x_2 \geq \epsilon/(2\delta)$, $x_1 \geq \delta\epsilon/2$ and $\left(\frac{\delta\epsilon - 2x_1}{2d\epsilon}\right)^2 + \left(\frac{\epsilon/\delta - 2x_2}{2d\epsilon}\right)^2 \leq 1$, we choose

$$
f_\epsilon(x) = \frac{\delta^*}{2} \left(1 - \sqrt{\left(\frac{\delta\epsilon - 2x_1}{2d\epsilon}\right)^2 + \left(\frac{\epsilon/\delta - 2x_2}{2d\epsilon}\right)^2}\right).
$$

The function $f_\epsilon$ is then extended by zero elsewhere and to the other quadrants by $f_\epsilon(-x_1, x_2) = -f_\epsilon(x_1, x_2)$, $f_\epsilon(x_1, -x_2) = f_\epsilon(x_1, x_2)$. A straightforward computation shows that

$$
\int_\Omega \left| \nabla f_\epsilon - 1_{Q_\epsilon^*} \nabla x_1 \right|^2 \, dx = c^2 \left(\frac{\delta}{2d} + \frac{\delta^3}{6d} + \frac{2d\delta}{3} + \frac{\pi}{4} \delta^2\right).
$$
Now select \( d = \sqrt{\delta}/2 \). Then \( f_\epsilon \) is compactly supported in \([-\frac{\delta^*}{2}, \frac{\delta^*}{2}] \times [-\left(\frac{\delta}{2 \sqrt{\delta}} + \frac{\sqrt{\delta}}{2}\right) \epsilon, \left(\frac{\delta}{2 \sqrt{\delta}} + \frac{\sqrt{\delta}}{2}\right) \epsilon] \subset [-1, 1] \times [-1, 1]. Therefore, introducing

\[
w_{1,\epsilon}(x) = -\sum_{k=0}^{N_{\epsilon,\delta}} \frac{\gamma_1 - \gamma_0}{\gamma_1} f_\epsilon(x_1 + k\epsilon \delta^*, x_2)
\]

we have

\[
\frac{\delta^*}{\epsilon} \int_{\Omega} \hat{\gamma}_\epsilon \left| \nabla w_{1,\epsilon} + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega^\delta_1 \nabla x_1 \right|^2 dx \\
\leq \gamma_0 \left( \frac{\gamma_1 - \gamma_0}{\gamma_1} \right)^2 \left( \frac{\sqrt{\delta}}{3} + \frac{\delta^{3/2}}{3} + \frac{\pi}{4} \delta^2 \right) .
\]

Since the right-hand side tends to zero with \( \delta \), the proof of (63) is complete in the case \( \delta \to 0 \).

We now turn the attention to \( \delta > 1 \). By exchanging the roles of \( x_1 \) and \( x_2 \), and replacing \( \delta \) by \( 1/\delta \) in the previous construction of \( f_\epsilon \), we obtain a function \( \tilde{f}_\epsilon(x) \) with compact support in \([-\left(\frac{\delta}{2} + d\right) \epsilon, \left(\frac{\delta}{2} + d\right) \epsilon] \times [-\left(\frac{\delta}{2} + d\right) \epsilon, \left(\frac{\delta}{2} + d\right) \epsilon], \]

such that

\[
\int_{\Omega} \left| \nabla \tilde{f}_\epsilon - 1_{Q_{\epsilon}^\delta} \nabla x_2 \right|^2 dx = \epsilon^2 \left( \frac{1}{2d \delta} + \frac{1}{6d \delta^3} + \frac{2d}{3 \delta} + \frac{\pi}{4 \delta^2} \right) .
\]

Now select \( d = 1/2\sqrt{\delta} \). Then the function \( \tilde{f}_\epsilon \) is compactly supported in \([-\frac{\delta^*}{2}, \frac{\delta^*}{2}] \times [-\left(\frac{\delta}{2 \sqrt{\delta}} + \frac{\sqrt{\delta}}{2}\right) \epsilon, \left(\frac{\delta}{2 \sqrt{\delta}} + \frac{\sqrt{\delta}}{2}\right) \epsilon] \subset [-1, 1,] \times [-1, 1]. Therefore, introducing

\[
w_{2,\epsilon}(x) = -\sum_{k=0}^{N_{\epsilon,\delta}} \frac{\gamma_1 - \gamma_0}{\gamma_1} \tilde{f}_\epsilon(x_1 + k\epsilon \delta^*, x_2)
\]

we have

\[
\frac{\delta^*}{\epsilon} \int_{\Omega} \hat{\gamma}_\epsilon \left| \nabla w_{2,\epsilon} + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega^\delta \nabla x_2 \right|^2 dx \\
\leq \gamma_0 \left( \frac{\gamma_1 - \gamma_0}{\gamma_1} \right)^2 \left( \frac{1}{\sqrt{\delta}} + \frac{1}{3 \delta^{3/2}} + \frac{1}{3 \delta^{3/2}} + \frac{\pi}{4 \delta^2} \right) .
\]

Since the right-hand side tends to zero as \( \delta \to \infty \), the proof of (63) is complete in the case \( \delta \to \infty \). This completes the proof of Proposition 3.

\[\square\]

**Remark 2.** Due to the construction of \( \omega^\delta_\epsilon \) we expect the matrices \( M^\delta \) to be constant along the interval \((0, 1)\). In that case the \( L^2 \) convergence in Proposition 3 is really the convergence of constants. Proposition 3 asserts that the (set of all possible) polarization tensors \( M^\delta \) converge to

\[
\begin{pmatrix}
\gamma_0/\gamma_1 & 0 \\
0 & 1
\end{pmatrix}
\]

(64)
as $\delta \to 0$, and converge to
\begin{equation}
\begin{pmatrix}
1 & 0 \\
0 & \gamma_0/\gamma_1
\end{pmatrix}
\end{equation}
(65)
as $\delta \to \infty$. It is thus clear that for $\delta$ sufficiently close to zero $M^\delta$ will be different from the polarization tensor of Proposition 2. Let us also note that we do expect the matrix $M^\delta$ to be independent of the particular subsequence $\epsilon_n$, i.e., we do expect a unique polarization tensor $M^\delta$ (as $\epsilon \to 0$ for fixed $\delta$). Since we also expect the matrices $M^\delta$ to vary continuously with $\delta$, there should exist a nonempty open interval of $\delta$'s for which we have “effective” polarization tensors $M^\delta$ that are different from both the matrix (64) and the matrix (65).

References


**Acknowledgments**

The research of Y. Capdeboscq was partially supported by “Action Concertée Incitative” 991ZVA24. The research of M.S. Vogelius was partially supported by NSF grants DMS–0307119 and INT–0003788.