CHARACTERIZING NONLINEAR SPATIO-TEMPORAL SYSTEMS IN THE FREQUENCY DOMAIN

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In this paper a new concept, spatio-temporal generalized frequency response functions (STGFRF), is introduced for the first time to characterize nonlinear spatio-temporal dynamical systems in the frequency domain. A probing method is developed to calculate the STGFRFs recursively for both continuous and discrete spatio-temporal systems. The algorithm is computationally compact and exposes the explicit relationship between the continuous and discrete models and the elements of the generalized frequency response functions.

Keywords: Spatio-temporal system; frequency response; spatio-temporal generalized frequency response function; probing method.

1. Introduction

Linear spectral analysis for temporal systems has greatly matured and is widely used in almost every branch of science and engineering. The Volterra series based nonlinear spectral analysis methods, namely generalized frequency response functions (GFRF), inheriting the merits of linear spectral analysis, have also been widely used in both analysis and design of nonlinear systems. In the past several decades, many important techniques for nonlinear spectral analysis have been developed such as the calculation of generalized transfer functions [Billings & Peyton Jones, 1990; Billings & Tsang, 1989a, 1989b; Chua & Ng, 1979; Lang et al., 2007; Peyton Jones & Billings, 1989], the determination of the output frequency range [Lang & Billings, 1997], the characteristics of nonlinear frequency response functions [Yue et al., 2005], the design of energy transfer filters [Billings & Lang, 2002], and the estimation of the output frequency response function [Lang et al., 2007]. One important method to identify the generalized frequency response functions (GFRF) was introduced by Billings and co-workers [Billings & Peyton Jones, 1990; Billings & Tsang, 1989a, 1989b], which consists of estimating a NARMAX (Nonlinear Auto-Regressive Moving Average with exogenous inputs) model description of the system and then computing the generalized frequency response functions directly from the estimated model using a probing method. Frequency domain analysis has also been widely used in the analysis of spatial systems, especially in image processing [Mathews & Sicuranza, 2000; Sicuranza, 1992]. Although Volterra series and the GFRF’s have been widely used for the investigation of nonlinear purely temporal systems, there are virtually no results about the application to dynamic spatio-temporal systems.

Spatio-temporal dynamical systems are an enormous class of highly complex systems which evolve over both time and space. Real life examples of spatio-temporal systems exist in almost all disciplines as diverse as ecology, biology, physics, chemistry, engineering, and the social sciences. Several
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authors have investigated distributed parameter systems in the frequency domain but all the results are focused on linear systems [Billings & Wei, 2007; Curtain & Morris, 2009; Rabenstein & Trautmann, 2002]. In this paper, Volterra series and associated frequency domain methods will be extended to spatio-temporal systems. Only translation-invariant spatio-temporal systems will be studied in the current investigation, where the properties and behaviors of the systems remain invariant with space and time translations.

In order to calculate and analyse the frequency response functions, spatio-temporal systems with external inputs will be considered. An example of a spatio-temporal system with an external input is introduced in Sec. 2. Section 3 introduces the probing method for the calculation of generalized frequency response functions for continuous and discrete temporal systems, which forms the background required for later sections. Section 4 extends this method to spatio-temporal dynamical systems. Two examples are considered in Sec. 5 to graphically show the spatio-temporal generalized frequency response functions. Conclusions are finally given in Sec. 6.

2. Spatio-Temporal Systems with External Inputs

In many cases, spatio-temporal systems are autonomous systems, that is, all the systems evolve from an initial condition and only depend on the initial conditions and the dynamic characteristics of the model. The frequency response of a system to a sinusoidal input signal is defined as the steady-state response of the system. The magnitude and phase of the output signal are functions of the input frequency. In order to calculate and analyse these kinds of frequency response functions, in this paper, spatio-temporal systems with external inputs are considered. It will be shown that spatio-temporal dynamical systems have many similar features to those observed in purely temporal dynamical systems.

As an example, consider a linear one-dimensional spatio-temporal system described by a partial differential equation with an external input

\[
\frac{\partial^2 y}{\partial t^2} + \xi_1 \frac{\partial y}{\partial t} + \xi_2 \left( \frac{\partial y}{\partial t} \right)^2 + \omega_0^2 y = \frac{\partial^2 y}{\partial x^2} + bu \quad (1)
\]

where \(y\) and \(u\) are functions of both the spatial coordinate \(x\) and the temporal coordinate \(t\), that is, both the output and input signals are spatio-temporal patterns denoted as \(y(t, x)\) and \(u(t, x)\) respectively.

Define the input signal as

\[
u(t, x) = \sin(\omega_0 t) \sin(\omega_1 x) \quad (2)\]

where \(\omega_0 = 2\pi \text{ (rad/s)}\) and \(\omega_1 = 2\pi \text{ (rad/s)}\) represent the spatial and temporal frequencies of the input separately. The input pattern is shown in Fig. 1(a).

Fig. 1. Simulation of spatio-temporal system (1). (a) The input pattern, (b) the output pattern.
Setting the parameters of system (1) as $\xi_1 = 4$, $\xi_2 = 0.5$, $\omega_0 = 1$, $c = 0.01$, $b = 1$ and simulating the system on a $1024 \times 1024$ lattice gave the steady-state output in Fig. 1(b).

Except for a slightly smaller magnitude and phase delay, the output pattern looks almost the same as the input pattern. However, the frequency domain analysis can discover more than these initial visual spatio-temporal appearances appear to show. Calculating the two-dimensional Fast Fourier transforms of the input and output yields the approximate frequency spectra of the input and output patterns shown in Fig. 2. The spectrum of the input only has peaks at $(\omega_0, \omega_0)$, $(-2\pi, -2\pi)$, $(+2\pi, -2\pi)$ and $(-2\pi, +2\pi)$ separately, which correspond to the temporal frequency $\omega_0$ and the spatial frequency $\omega_0$ in the input signal. However, the output spectrum is much richer than the input spectrum. The output pattern has peaks at all points $(\omega_p, \omega_q)$ where $p, q \in \mathbb{Z}$.

This is because of the effects of the nonlinear term $\xi_2(\partial y / \partial t)^2$.

3. The Probing Method

Billings and co-workers [Billings & Peyton Jones, 1990; Billings & Tsang, 1989a; Peyton Jones & Billings, 1989] showed that the harmonic input or probing method can be used to determine the $n$th order Generalized Frequency Response Function (GFRF) of a nonlinear system by equating the coefficients of the system output for an input defined as

Fig. 2. Frequency spectra of input and output patterns: (a) and (c) input spectrum, (b) and (d) output spectrum.
Then the second derivative of \( H(t) \) is of the form \( H(t) = \sum_{k=1}^{n} \frac{H_{k}(\omega_{k1}, \ldots, \omega_{kn})}{\omega_{k1}^{2} + \xi_{k1} \omega_{k1} + \omega_{0}^{2}} \) is the nth order output when \( A_{k} = 1 \) for all \( k = 1, 2, \ldots, n \). \( H_{k}(\omega_{k1}, \ldots, \omega_{kn}) \) is the nth order GFRF.

The procedure to calculate the generalized frequency response functions is briefly reviewed by considering a continuous and a discrete temporal example. For more details of the probing method, refer to the papers [Billings & Peyton Jones, 1990; Billings & Tsang, 1989a, 1989b; Peyton Jones & Billings, 1989].

### 3.1. Calculation of generalized frequency response functions for continuous temporal systems

Consider a purely temporal continuous dynamical system described by the differential equation

\[
\frac{d^{2}y}{dt^{2}} + \xi \frac{dy}{dt} + \omega_{0}^{2}y(t) = bu(t)
\]

The procedure begins by defining the probing input as

\[
u(t) = e^{i\omega_{0}t}
\]

and the corresponding output as

\[y(t) = H_{1}(\omega_{1})e^{i\omega_{0}t}
\]

The derivative of \( y(t) \) can then be written as

\[
\frac{dy}{dt} = j\omega_{1}H_{1}(\omega_{1})e^{i\omega_{0}t}
\]

Then the second derivative of \( y(t) \) is

\[
\frac{d^{2}y}{dt^{2}} = -\omega_{1}^{2}H_{1}(\omega_{1})e^{i\omega_{0}t}
\]

Substituting (4)–(7) into Eq. (3) yields

\[
-\omega_{1}^{2}H_{1}(\omega_{1})e^{i\omega_{0}t} + \xi j\omega_{1}H_{1}(\omega_{1})e^{i\omega_{0}t}
\]

\[
+ \xi(\omega_{1})^{2}H_{2}(\omega_{1})e^{2i\omega_{0}t}
\]

\[
+ \omega_{0}^{2}H_{1}(\omega_{1})e^{i\omega_{0}t} = \{e^{i\omega_{0}t}\}
\]

Equating the coefficients of \( e^{i\omega_{0}t} \) on both sides yields

\[
H_{1}(\omega_{1}) = \frac{b}{(\omega_{1})^{2} + \xi \omega_{1} + \omega_{0}^{2}}
\]

\( H_{1}(\omega_{1}) \) is the first order generalized frequency response function which characterizes the linear portion of the response. In this example, the first order generalized frequency response is a typical second order linear system with the natural frequency \( \omega_{0} \) and the damping ratio \( \xi/2\omega_{0} \).

Probing with an input with two different frequencies which is given as

\[
u(t) = e^{i\omega_{1}t} + e^{i\omega_{2}t}
\]

the corresponding response of the system is

\[y(t) = H_{1}(\omega_{1})e^{i\omega_{0}t} + H_{1}(\omega_{2})e^{i\omega_{0}t}
\]

\[+ H_{2}(\omega_{1}, \omega_{1})e^{2i\omega_{0}t}
\]

\[+ H_{2}(\omega_{2}, \omega_{2})e^{2i\omega_{0}t}
\]

\[+ 2H_{2}(\omega_{1}, \omega_{2})e^{i(\omega_{1} + \omega_{2})t}
\]

and the first and second order derivatives are

\[
\frac{dy}{dt} = j\omega_{1}H_{1}(\omega_{1})e^{i\omega_{0}t} + j\omega_{2}H_{1}(\omega_{2})e^{i\omega_{0}t}
\]

\[+ j2\omega_{1}H_{2}(\omega_{1}, \omega_{1})e^{2i\omega_{0}t}
\]

\[+ j2\omega_{2}H_{2}(\omega_{2}, \omega_{2})e^{2i\omega_{0}t}
\]

\[+ 2j(\omega_{1} + \omega_{2})H_{2}(\omega_{1}, \omega_{2})e^{i(\omega_{1} + \omega_{2})t}
\]

Substituting Eqs. (10)–(13) into (3) and equating coefficients of \( e^{i(\omega_{1} + \omega_{2})t} \) yields

\[
H_{2}(\omega_{1}, \omega_{2}) = \frac{\xi \omega_{1}\omega_{2}H_{1}(\omega_{1})H_{1}(\omega_{2})}{(\omega_{1} + \omega_{2})^{2} + \xi(\omega_{1} + \omega_{2}) + \omega_{0}^{2}}
\]

\( H_{2}(\omega_{1}, \omega_{2}) \) is the second order generalized frequency response function which characterizes the quadratic contribution to the response. An example of \( H_{2}(\omega_{1}, \omega_{2}) \) is given in Fig. 3.

Following this procedure all the higher order generalized frequency response functions can be calculated recursively.
3.2. Calculation of generalized frequency response functions for discrete temporal systems

The probing method can also be applied to calculating the generalized frequency response functions for a discrete nonlinear model.

System (3) can be discretised into a discrete model given in (15) by the forward finite difference method with sample interval $\Delta t$.

\[
g(k + 2) = a_1 g(k + 1) + a_2 g(k) + a_3 g^2(k + 1) + a_4 g(k) g(k + 1) + a_5 g^2(k) + b_1 u(k)
\]

where

\[
\begin{align*}
a_1 &= 2 - \xi_1 \Delta t \\ a_2 &= -1 + \xi_1 \Delta t - \omega_0^2 (\Delta t)^2 \\ a_3 &= -\xi_2 \\ a_4 &= 2 \xi_2 \\ a_5 &= -\xi_2 \\ b_1 &= b (\Delta t)^2
\end{align*}
\]

System (15) will now be used to illustrate the probing method for the calculation of the generalized frequency response function of discrete systems.
 Define the discrete probing input
\[ u(k) = e^{j\omega t_k} \]  
and the corresponding output as
\[ y(k) = H_{d1}(j\omega t_k)e^{j\omega t_k\Delta t} \]  
where \( H_{d1}(j\omega t) \) is the first order generalized frequency response function of the discrete system (15).

The one step ahead and two step ahead outputs are
\[ y(k + 1) = H_{d1}(j\omega t_k)e^{j\omega t_k(k+1)\Delta t} \]  
and
\[ y(k + 2) = H_{d1}(j\omega t_k)e^{j\omega t_k(k+2)\Delta t} \]  
Substituting (17)–(20) into (15) yields
\[ H_{d1}(j\omega t_k)e^{j\omega t_k(k+2)\Delta t} = a_1 H_{d1}(j\omega t_k)e^{j\omega t_k(k+1)\Delta t} + a_2 H_{d1}(j\omega t_k)e^{j\omega t_k\Delta t} + a_3 (H_{d1}(j\omega t_k)e^{j\omega t_k}\Delta t)^2 + a_4 (H_{d1}(j\omega t_k)e^{j\omega t_k\Delta t})^2 + b_1 e^{j\omega t_k\Delta t} \]

Fig. 4. The second order generalized frequency response function of the discrete system (15) with \( a_1 = 1.994, a_2 = -0.9941, a_3 = -1, a_4 = 2, a_5 = -1, b_1 = 0.0001, \Delta t = 0.01 \); (a) and (c) show the magnitude, (b) and (d) show the phase.
Equating the coefficient of $e^{j\omega_1 k \Delta t}$ yields

$$H_d(j\omega_1) = \frac{b_1}{e^{j\omega_1 k \Delta t} - a_1 e^{j\omega_2 k \Delta t} - a_2} \quad (22)$$

$H_d(j\omega_1)$ is the first order generalized frequency response function which describes the same system characteristics as the $H_d(j\omega_2)$ does in Sec. 3.

Probing the discrete system with an input with two different frequencies $\omega_1$ and $\omega_2$

$$u(k) = e^{j\omega_1 k \Delta t} + e^{j\omega_2 k \Delta t} \quad (23)$$

The corresponding output can be defined as

$$y(k) = H_d(j\omega_1) e^{j\omega_1 k \Delta t} + H_d(j\omega_2) e^{j\omega_2 k \Delta t}$$

$$+ H_d(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2) k \Delta t} + 2H_d(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2) k \Delta t} \quad (24)$$

where $H_d(j\omega_1)$ is the second order generalized frequency response function of the discrete system (15).

$H_d(j\omega_1, j\omega_2)$ is the second order generalized frequency response function which characterizes the same features of the system as the $H_d(j\omega_1, j\omega_2)$ does in Sec. 3.1. An example of $H_d(j\omega_1, j\omega_2)$ is given in Fig. 4 which is almost exactly the same as the second order generalized frequency response function shown in Fig. 3.

Formalizing this approach for the class of general nonlinear differential equations and nonlinear discrete time models produces a set of recursive equations to compute all orders of generalized frequency response functions [Peyton Jones & Billings, 1989].

4. Calculation of STGFRF for Spatio-Temporal Dynamical Systems

For a time-invariant nonlinear dynamical system with certain restrictions [Boyd & Chua, 1985; Frechet, 1910; Rugh, 1981], the relation between the output and the input can be described using a Volterra series form. Now the ideas of the Volterra series expansion and the probing method will be extended to nonlinear spatio-temporal dynamical systems.

The time advanced outputs are

$$y(k + 1) = H_d(j\omega_1) e^{j\omega_1 (k+1) \Delta t} + H_d(j\omega_2) e^{j\omega_2 (k+1) \Delta t} + H_d(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2) (k+1) \Delta t} + 2H_d(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2) (k+1) \Delta t} \quad (25)$$

and

$$y(k + 2) = H_d(j\omega_1) e^{j\omega_1 (k+2) \Delta t} + H_d(j\omega_2) e^{j\omega_2 (k+2) \Delta t} + H_d(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2) (k+2) \Delta t} + 2H_d(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2) (k+2) \Delta t} \quad (26)$$

Substituting (23)–(26) into (15) and equating the coefficients of $e^{j(\omega_1 + \omega_2) k \Delta t}$ on both sides yields

$$H_d(j\omega_1, j\omega_2) = \frac{(2a_1 e^{j\omega_1 k \Delta t} + a_4 e^{j\omega_2 k \Delta t} + e^{j(\omega_1 + \omega_2) k \Delta t}) + 2b_2 H_d(j\omega_1) H_d(j\omega_2)}{2e^{j(\omega_1 + \omega_2) k \Delta t} - 2a_1 e^{j(\omega_1 + \omega_2) k \Delta t} - 2a_2} \quad (27)$$

For a linear spatio-temporal systems described by a constant coefficient inhomogeneous partial differential equation, the solution of the partial differential equation can be expressed as a convolution of the fundamental solution and the input pattern according to the Ehrenpreis-Malgrange theorem [Ehrenpreis, 1954, 1955; Malgrange, 1956], that is

$$y(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - \xi, t - \tau) u(\xi, \tau) d\xi d\tau \quad (28)$$

where $u(\xi, \tau)$ represents the input of the spatio-temporal system and $h(x - \xi, t - \tau)$ represents the fundamental solution which is the response of the partial differential equation to the two-dimensional Dirac delta function $\delta(x - \xi, t - \tau)$. The fundamental solution corresponds to the impulse response in traditional linear system theory and the first order Volterra kernel.
systems which can be expanded as a series of generalised Volterra convolutions as

$$y(x, t) = \sum_{n=1}^{\infty} y_n(x, t)$$

(29)

where $y_n(x, t)$ is the $n$th order output of the system defined as

$$y_n(x, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\xi_1, \tau_1, \ldots, \xi_n, \tau_n) \times \prod_{j=1}^{n} u(x - \xi_j, t - \tau_j) \prod_{j=1}^{n} d\xi_j d\tau_j$$

(30)

Here $h_n(\xi_1, \tau_1, \ldots, \xi_n, \tau_n)$ is named as the $n$th order generalised Volterra kernel of the spatio-temporal system, in which there are $n$ pairs of dummy variable $(\xi_i, \tau_i)$ representing the space and time coordinates. This class of nonlinear spatio-temporal systems widely exist. An intuitive example is a nonlinear spatio-temporal system of the Hammerstein structure where a static nonlinearity is connected in cascade with a linear dynamical spatio-temporal model.

There are $n!$ different asymmetric $n$th order kernels denoted as $h_n^{\text{sym}}(\xi_1, \tau_1, \ldots, \xi_n, \tau_n)$ by reordering the dummy variable pairs $(\xi_i, \tau_i)$. It is often difficult to determine all these $n!$ kernels. However, a symmetric $n$th order kernel denoted as $h_n^{\text{sym}}(\xi_1, \ldots, \xi_n, \tau_n)$ can be uniquely defined as

$$h_n^{\text{sym}}(\xi_1, \ldots, \xi_n, \tau_n) = \frac{1}{n!} \sum_{\pi(\xi_1, \ldots, \xi_n, \tau_n)} h_n^{\text{sym}}(\xi_{\pi(1)}, \ldots, \xi_{\pi(n)}, \tau_n)$$

(31)

where $\pi(\xi_1, \ldots, \xi_n, \tau_n)$ represents all the permutations of the $n$ dummy variable pairs.

Applying the $n$-dimensional Fourier transform to the symmetric Volterra kernel produces the spatio-temporal generalised frequency response functions as

$$H^{\text{ST}}_n(j\omega_1, \ldots, j\omega_n, j\omega_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_n^{\text{sym}}(\xi_1, \ldots, \xi_n, \tau_n) \times \prod_{j=1}^{n} j\omega_j e^{-j\omega_j \xi_j} \prod_{j=1}^{n} d\xi_j d\tau_j$$

(32)

The probing method will now be developed to calculate the generalised frequency response functions of a class of spatio-temporal systems. For a bounded input bounded output stable spatio-temporal system, define the probing input as $u(t, x) = \sum_{k=1}^{\infty} e^{j\omega_k t + j\omega_k x}$, where $\omega_k$ are the temporal and spatial frequencies separately.

The steady-state output of a spatio-temporal system can then be defined as $y(t, x) = \sum_{n=1}^{\infty} y_n(t, x)$, where $y_n(t, x)$ is the $n$th order output which is of the form $y_n(t, x) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} H^{\text{ST}}_n(\omega_{k_1}, \ldots, \omega_{k_n} + j\omega_1, \ldots, j\omega_n)$, and $H^{\text{ST}}_n(\omega_{k_1}, \ldots, \omega_{k_n}, \omega_1, \ldots, \omega_n)$ is the $n$th order Spatio-Temporal Generalised Frequency Response Function.

4.1. Calculation of STGFRF for continuous nonlinear spatio-temporal systems

Now consider a nonlinear spatio-temporal system given as

$$\frac{\partial^2 y}{\partial t^2} + \xi_1 \frac{\partial y}{\partial t} + \xi_1 \left( \frac{\partial y}{\partial x} \right)^2 + \omega_0 y(x, t)$$

$$= c \frac{\partial^2 y}{\partial x^2} + \omega_0 y(x, t)$$

(33)

Probing the system with an input defined as

$$u(x, t) = e^{j\omega_1 x + j\omega_1 t}$$

(34)

the output can then be defined as

$$y(t, x) = H^{\text{ST}}_1(j\omega_1, j\omega_1) e^{j\omega_1 t + j\omega_1 x}$$

(35)

Accordingly the temporal and spatial derivatives of the output are

$$\frac{\partial y}{\partial t} = j\omega_1 H^{\text{ST}}_1(j\omega_1, j\omega_1) e^{j\omega_1 t + j\omega_1 x}$$

(36)

$$\frac{\partial^2 y}{\partial t^2} = -\omega_0^2 H^{\text{ST}}_1(j\omega_1, j\omega_1) e^{j\omega_1 t + j\omega_1 x}$$

(37)

and

$$\frac{\partial^2 y}{\partial x^2} = -\omega_0^2 H^{\text{ST}}_1(j\omega_1, j\omega_1) e^{j\omega_1 t + j\omega_1 x}$$

(38)

Substituting Eqs. (34)–(38) into system (33) yields

$$-\omega_0^2 H^{\text{ST}}_1(j\omega_1, j\omega_1) e^{j\omega_1 t + j\omega_1 x} + \xi_1 j\omega_1 H^{\text{ST}}_1(j\omega_1, j\omega_1) e^{j\omega_1 t + j\omega_1 x} + \xi_1 \left( \frac{\partial y}{\partial x} \right)^2 + \omega_0 y(x, t)$$

and

$$c \frac{\partial^2 y}{\partial x^2} + \omega_0 y(x, t) = 0$$

(39)
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\[ f(t, x) = H^ST_{1}(j\omega_1, j\omega_2)e^{i\omega_1 t + j\omega_1 x} + H^ST_{2}(j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x} \]

\[ + 2H^ST_{12}(j\omega_1, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_2 x + j\omega_1 x} \]

\[ + 2\omega_1 H^ST_{2}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x + j\omega_2 x} \]

\[ + 2\omega_2 H^ST_{1}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_1 x + j\omega_2 x} \] (42)

and the spatial and temporal derivatives are

\[ \frac{\partial f}{\partial t} = j\omega_1 H^ST_{1}(j\omega_1, j\omega_1)e^{i\omega_1 t + j\omega_1 x} + j\omega_2 H^ST_{2}(j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x} \]

\[ + j(\omega_1 + \omega_2)2H^ST_{12}(j\omega_1, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_2 x + j\omega_1 x} \]

\[ + 2\omega_1 H^ST_{2}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x + j\omega_2 x} \]

\[ + 2\omega_2 H^ST_{1}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_1 x + j\omega_2 x} \] (43)

\[ \frac{\partial^2 f}{\partial t^2} = -\omega_1^2 H^ST_{1}(j\omega_1, j\omega_1)e^{i\omega_1 t + j\omega_1 x} - \omega_2^2 H^ST_{2}(j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x} \]

\[ - (\omega_1 + \omega_2)^2 H^ST_{12}(j\omega_1, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_2 x + j\omega_1 x} \]

\[ - 4\omega_1 H^ST_{2}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x + j\omega_2 x} \]

\[ - 4\omega_2 H^ST_{1}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_1 x + j\omega_2 x} \] (44)

and

\[ \frac{\partial^2 f}{\partial x^2} = -\omega_1^2 H^ST_{1}(j\omega_1, j\omega_1)e^{i\omega_1 t + j\omega_1 x} - \omega_2^2 H^ST_{2}(j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x} \]

\[ - (\omega_1 + \omega_2)^2 H^ST_{12}(j\omega_1, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_2 x + j\omega_1 x} \]

\[ - 4\omega_1 H^ST_{2}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_2 t + j\omega_2 x + j\omega_2 x} \]

\[ - 4\omega_2 H^ST_{1}(j\omega_2, j\omega_2, j\omega_2, j\omega_2)e^{i\omega_1 t + j\omega_1 x + j\omega_2 x} \] (45)

Substituting (41)–(45) into (33) and the second order spatio-temporal generalized frequency response function is

\[ H^ST_{2}(j\omega_1, j\omega_2, j\omega_2, j\omega_2) = \frac{\xi_2\omega_1\omega_2 H_{1}(j\omega_1, j\omega_2)H_{2}(j\omega_2, j\omega_2)}{(\omega_1 + \omega_2)^2 + \xi_1\omega_1 + \omega_2^2 + \omega_1\omega_2^2} \]

(46)
4.2. Calculation of STGFRF for discrete nonlinear spatio-temporal systems

Now the probing method will be developed to calculate the STGFRF for discrete nonlinear spatio-temporal systems. Discretising system (33) using a forward-time-centred-space finite difference method yields a discrete spatio-temporal system

\[ y(k + 2, h) = a_1 y(k + 1, h) + a_2 y(k, h) + a_3 y^2(k + 1, h) + \cdots + a_n y^n(k, h) + b_1 u(k, h) \]

where

\[ a_1 = 2 - \xi_1 \Delta t \]
\[ a_2 = -1 + \xi_1 \Delta t - \omega_0^2(\Delta t)^2 - \frac{2c(\Delta t)^2}{(\Delta x)^2} \]
\[ a_3 = -\xi_2 \]
\[ a_4 = 2\xi_2 \]

and the corresponding output as

\[ u(k, h) = e^{j \omega_1 k \Delta t + j \omega_2 h \Delta x} \]

Firstly, define the discrete probing input as

\[ u(k, h) = e^{j \omega_1 k \Delta t + j \omega_2 h \Delta x} \]

and the corresponding output as

\[ y(k, h) = H_{ST}^D(j \omega_1, j \omega_2) e^{j \omega_1 k \Delta t + j \omega_2 h \Delta x} \]

where

\[ H_{ST}^D(j \omega_1, j \omega_2) = \frac{b_1}{e^{j \omega_1 k \Delta t} - a_1 e^{j \omega_1 k \Delta t} - a_2 - d_1 e^{j \omega_2 h \Delta x} - d_2 e^{-j \omega_2 h \Delta x}} \]

Probing with inputs consisting of two different frequencies

\[ u(k, h) = e^{j \omega_1 k \Delta t + j \omega_2 h \Delta x} + e^{j \omega_1 k \Delta t + j \omega_3 h \Delta x} \]

the output can be defined as

\[ y(k, h) = H_{ST}^D(j \omega_1, j \omega_2) e^{j \omega_1 k \Delta t + j \omega_2 h \Delta x} + H_{ST}^D(j \omega_1, j \omega_3) e^{j \omega_1 k \Delta t + j \omega_3 h \Delta x} \]

Substituting Eqs. (49)–(51) into (47) and equating the coefficients of \( e^{j \omega_1 k \Delta t + j \omega_2 h \Delta x} \) on both sides yields the first order spatio-temporal generalized frequency response function

\[ \frac{H_{ST}^D(j \omega_1, j \omega_2)}{H_{ST}^D(j \omega_1, j \omega_3)} = \frac{(2a_2 e^{j \omega_1 k \Delta t} + a_1 e^{j \omega_1 k \Delta t} + e^{j \omega_1 k \Delta t}) + 2a_1 H_{ST}^D(j \omega_1, j \omega_3) H_{ST}^D(j \omega_1, j \omega_3) H_{ST}^D(j \omega_2, j \omega_2)}{2a_1 e^{j \omega_1 k \Delta t} - 2a_1 e^{j \omega_1 k \Delta t} + 2a_3 - 2d_1 e^{j \omega_1 k \Delta t} - 2d_2 e^{-j \omega_2 h \Delta x}} \]

Following this idea, all higher-order spatio-temporal generalized frequency response functions can be calculated recursively.
5. Illustrative Examples

5.1. STGFRF of a continuous spatio-temporal system

In this section some spatio-temporal systems will be analysed using the spatio-temporal generalized frequency response functions obtained in Sec. 4.

Consider the continuous spatio-temporal system (33) in Sec. 4.1. Set the system parameters as $\xi_1 = 0.6, \xi_2 = 1, \omega_0 = 1, b = 1$. The graph of the first order STGFRF $H_{ST}^{1}(j\omega_t, j\omega_x)$ is given in Fig. 5 which graphically describes the magnitude and phase versus the spatial and the temporal frequencies.

Figure 5 shows that when $\omega_x$ is fixed, the system behaves as a typical under-damped second order system over the temporal frequency $\omega_t$. However, the STGFRF $H_{ST}^{1}(j\omega_t, j\omega_x)$ depends on both the spatial frequency $\omega_x$ and the temporal frequency $\omega_t$. When the spatial frequency $\omega_x$ increases, the resonant frequency of the second order system increases and the peak of the magnitude gets thinner. Simulations show that the larger the diffusion coefficient is, the greater the effect of $\omega_x$ becomes.

Fig. 5. The first order STGFRF $H_{ST}^{1}(j\omega_t, j\omega_x)$ of the spatio-temporal system (33) with $\xi_1 = 0.6, \xi_2 = 1, \omega_0 = 1, b = 1$. (a) and (c) show the magnitude; (b) and (d) show the phase.
Given fixed spatial frequencies $\omega_{x1}$ and $\omega_{x2}$, the second order STGFRF $H_{ST2}^{ST}(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2})$ over the temporal frequencies $\omega_{t1}$ and $\omega_{t2}$ is graphically shown in Fig. 6. The STGFRF $H_{ST1}^{ST}(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2})$ has a similar shape with the temporal second order GFRF $H_{2}(j\omega_{t1}, j\omega_{t2})$ in Fig. 2.

5.2. **STGFRF of a discrete spatio-temporal system**

Discretising the continuous system yields the parameters of the discrete system: $a_1 = 1.994$, $a_2 = -1.0141, a_3 = -1, a_4 = 2, a_5 = -1, d_1 = d_2 = 0.01, b_1 = 0.0001$, where the spatial and the temporal interval are $\Delta x = 0.01, \Delta t = 0.01$.

The 3-D graph and the contour graph of the discrete first order STGFRF $H_{ST1}^{ST}(j\omega_{t1}, j\omega_{x1})$ and the second order STGFRF $H_{ST2}^{ST}(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2})$ are given in Figs. 7 and 8, respectively. Obviously, we obtain magnitudes and phases which are very close to the continuous case. However in the discrete version, parameters $a_1 \sim a_5$ represent the effects of the cell states in past time while parameters $d_1$ and $d_2$ show the effects of the left and right neighbors. The discrete STGFRFs not only depend on
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The first order STGFRF $H_{ST}^1(j\omega_1, j\omega_2)$ of the discrete spatio-temporal system (47) with $a_1 = 1.994$, $a_2 = -1.041$, $a_3 = -1$, $a_4 = 2$, $a_5 = -1$, $d_1 = d_2 = 0.01$, $b_1 = 0.0001$, (a) and (c) show the magnitude, (b) and (d) show the phase.

6. Conclusions

A new description of nonlinear spatio-temporal systems in the frequency domain, the spatio-temporal generalized frequency response functions, has been presented for the first time. A probing method has been developed to determine the generalized frequency response functions from both continuous PDE models and discrete CML models. Several examples were employed to demonstrate that the new results are correct.

Although only one-dimensional spatio-temporal systems are considered in this paper, the methods can easily be extended to arbitrary $n$-dimensional spatio-temporal systems. The spatio-temporal generalized frequency response functions open a new avenue for the study of spatio-temporal systems. Combined with the identification methods proposed both for continuous spatio-temporal systems...
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Fig. 8. The second order generalized frequency response function $H_{ST}^{a_2} \left(j\omega_1, j\omega_2, j\omega_3 \right)$ of the discrete spatio-temporal system (47) with $a_1 = 1.994, a_2 = -1.0141, a_3 = -1, a_4 = 2, a_5 = -1, d_1 = d_2 = 0.01, b_1 = 0.0001, \Delta t = \Delta x = 0.01, \omega_{x1} = \omega_{x2} = 10$. (a) and (c) show the magnitude, (b) and (d) show the phase.

and for discrete spatio-temporal systems [Billings & Coca, 2002; Billings et al., 2006; Coca & Billings, 2001; Guo & Billings, 2006; Pan & Billings, 2008], the spatio-temporal generalized frequency response functions can be a powerful tool in the analysis of spatio-temporal systems.

Note that in this paper, the study has focused on the frequency domain aspect of a class of translation invariant spatio-temporal systems driven by some external inputs. Clearly, there is a need to investigate nontranslation invariant systems, such as time varying spatio-temporal systems from a frequency domain point of view. Other problems that require further study, from a frequency domain perspective include investigating how spatio-temporal dynamics evolve over both time and space from a specific initial state, and how they are influenced by different boundary conditions. One of the rational for studying these problems lies in the control strategies widely adopted in control engineering for spatio-temporal dynamical systems, that is distributed control, point-wise control, and boundary control. The methodology for addressing the above problems, and interpreting,
analysing, and computing those newly defined generalized frequency response functions for linear and nonlinear spatio-temporal dynamical systems is under investigation.

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References


