A SIMPLE SCALAR COUPLED MAP LATTICE MODEL FOR EXCITABLE MEDIA

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A simple scalar coupled map lattice (sCML) model for excitable media is derived in this paper. The new model, which has a simple structure, is shown to be closely related to the observed phenomena in excitable media. Properties of the sCML model are also investigated. Illustrative examples show that this kind of model is capable of reproducing the behavior of excitable media and of generating complex spatiotemporal patterns.

Keywords: Scalar coupled map lattice model; excitable media; pattern generation.

1. Introduction

Excitable media are a very important class of nonlinear spatio-temporal dynamic systems which widely exist in biological, physical, chemical and ecological systems. A wide variety of patterns have been observed in excitable media including solitary patterns, target-like patterns, spiral waves, and so on. Excitable media have been widely and intensively studied including theoretical analysis, numerical simulations, experiments and system identification.

Cells in excitable media can be characterized by three states: resting, excited and refractory. A cell in a resting state is stable for small perturbations while a perturbation with strength greater than a certain threshold can cause this cell to undergo a large excursion. Usually, the shape of the response does not depend on the perturbation strength, as long as the perturbation exceeds the threshold. After this strong response, the system returns to its initial resting state. A subsequent excitation can be generated after a suitable length of time, called the refractory period, has passed. This property of excitable media is commonly called excitability [Zykov, 2008].

Many studies have shown that an enormous and complex range of macroscopic behaviors can be generated using relatively simple microscopic models. Therefore, a good model is vital for the investigation of excitable media, which should be as simple as possible provided that it provides an accurate description of the system. The model can be derived by employing first principles or by system identification [Billings & Coca, 2002; Guo et al., 2010; Pan et al., 2008; Zhao et al., 2007]. Several kinds of models have been proposed to describe excitable media including partial differential equations (PDE), coupled map lattices (CML), cellular automata (CA), cellular neural networks (CNN), hybrid automata (HA), etc.

The most commonly used models are reaction–diffusion equations where the local dynamics interact with the diffusive transportation to generate complex patterns. Excitable media are most naturally represented as partial differential equations, where the evolution of cells in the excitable media is modeled by coupled differential equations of a reaction–diffusion structure. Examples of partial differential equation models include the FHN model [FitzHugh, 1952] the Oragonator model [Field et al., 1972], the predator-prey models, the Barkley model [Barkley, 1991] and so on. The advantages of PDE models lie in the close connection with real systems.
However, PDE models for excitable media are always nonlinear and can be very complex. Furthermore, the number of differential equations required to represent a cell in the system may be large. For example, the Difrancesco–Noble model of Purkinje fibres has 14 dimensions and over a hundred parameters [Difrancesco & Noble, 1985]. Therefore both simulation and identification of partial differential equation models can often be a difficult task.

Coupled map lattice models defined on a discrete time and space lattice can be referred to as a discrete version of partial differential models. Owing to the computational efficiency and richness of dynamical behavior, coupled map lattice models have been widely used for the description, simulation, and identification of excitable media [Kawasaki et al., 1990]. A CML model for excitable media usually needs more than one variable and some of these variables may not be measurable in practical systems. Consequently, some of the difficulties of PDE models in identification of real systems also exist in CML models.

Cellular automata defined on a discrete lattice can simplify the dynamic description of a system by mapping the system behavior onto a few discrete states. In cellular automata models, the continuous effects of diffusion are mapped to simple rules based on neighborhood interactions. Typical cellular automata models include the Greenberg–Hasting model (GHM) [Greenberg & Hastings, 1978], Hodgodge Machine Model (HMM), Cyclic Cellular Automata model, and the Gerhardt–Schnuster–Tyson Model [Gerhardt et al., 1990]. On the one hand, cellular automata models are simple for the simulation of excitable media, on the other hand only a very limited number of parameters can be controlled when the neighborhood is determined.

Cellular neural networks which share the best features of both neural networks and cellular automata are another important class of model for excitable media. A CNN is made up of a massive collection of regularly spaced circuit items which communicate with each other through nearest neighbors. CNNs are large-scale nonlinear analogue circuits which process signals in real time [Chua & Yang, 1988]. Simulation applications of CNNs have been developed into a wide range of disciplines [Fortuna et al., 2001]. CNNs also provide a link between nonlinear differential equations and discrete cellular automata [Chua, 2007]. Cellular neural networks have the ability to model any arbitrarily complex nonlinear relationships and are therefore very useful in many applications. However this kind of model, which is of a "black box" nature, cannot easily be written down and so cannot easily be used to understand the fundamental behaviors in terms of the system characteristics in excitable media.

Hybrid automata models which combine discrete transition graphs with continuous dynamics have also been designed to describe excitable media [Ye et al., 2005; Ye et al., 2008]. The discrete variables define the automaton's modes of behavior while the continuous variables give more details. In Ye et al.'s model, the response of the system was divided into four discrete phases: resting and final repolarization, stimulated, upstroke, and plateau and early repolarization. Each of the phases was fitted using a linear differential equation. Hybrid automata models greatly increase the efficiency of the computation without losing the essential system features. Nevertheless, further work is needed to relate the hybrid automata models to the physical meaning and behavior of excitable media.

A new scalar coupled map lattice model (sCML) was recently proposed [Guo et al., 2010] which has the merits of both coupled map lattice and cellular automata models. In [Guo et al., 2010] the form of a scalar coupled map lattice model was deduced by studying and discretizing reaction–diffusion PDE models. An orthogonal least squares algorithm was then adopted to identify the terms and coefficients in the models for both simulated patterns and real images recorded from experiments on a Belousov–Zhabotinsky reaction. The scalar coupled map lattice model is essentially a delayed state reconstruction of the multivariable coupled map lattice model, which describes the explicit relationship between the measurement variable and the associated delayed states. Using a continuous variable to present the dynamic state in excitable media, the new sCML model inherits the advantages of CML models in description but is of a simpler structure. The new sCML model, in which only one measurement variable is needed, avoids the complex theoretical details and is computationally efficient. Since it is based on the observed spatio-temporal patterns the sCML model closely relates to the observed phenomena and can be identified from real experimental data.

In this study the simple sCML model, introduced in [Guo et al., 2010] will be studied in more detail. A decomposition of the model and
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2. General Form of the Scalar Coupled Map Lattice Model

A lattice dynamical system (LDS) is a spatially extended dynamical system composed of a finite or infinite number of interacting dynamical systems, each assigned to a node, named as a cell, of a one- or multi-dimensional lattice of integers representing a discretization of the physical space. The dynamics of a LDS can be viewed as a combination of local dynamics, involving the local state-space variables assigned to every cell and spatial interactions [Billings & Coca, 2002].

An autonomous lattice dynamical system can be represented using a general CML model as

\[ y_i(t) = f[p^	ext{pr} q^	ext{ps} y_i(t)] \]

where \( y_i(t) \) represents the state-space variables assigned to the cell with a spatial coordinate \( i \in \mathbb{Z}^n \) of an \( n \)-dimensional lattice; \( p^\text{pr} \) is a temporal shift operator and \( q^\text{ps} \) is a multivalued spatial shift operator.

As a simplification of the general CML model, the lattice equations can be assumed to be evolving on a uniform lattice, that is, the CML model is spatially invariant over the entire lattice. Therefore, the macroscopic phenomena of a LDS can be investigated by studying the microscopic model of one specified cell. For simplicity, denote \( y_i(t) \) as \( y(t) \) by dropping the spatial index \( i \).

In general, each cell can be coupled with all the other cells in the lattice. This represents a globally coupled LDS. Very often, however, the spatial interactions are restricted to only a finite set of neighboring cells. In a reaction–diffusion model, these interaction effects are represented by the diffusion terms denoted as \( \nabla^2 \), where \( \nabla^2 \) is the Laplace operator defined as \( \nabla^2 y(x, t) \equiv \sum_{i=1}^{n} \frac{\partial^2 y}{\partial x_i^2} \). The Laplace operator in two dimensions can be approximated using a finite difference method as

\[ \nabla^2 y(t) = \frac{y_{i+1,j}^{(t)} + y_{i-1,j}^{(t)} + y_{i,j+1}^{(t)} + y_{i,j-1}^{(t)} - 4y_{i,j}^{(t)}}{(d)^2} \]  

There are very often more than one dynamical variable \( y(x,t) \) involved in an excitable media system, which represents the state of different components. In many cases, however, only one dynamical variable can be accessed for the measurement, that is, only a series of spatio-temporal patterns are observed. Therefore the method of time-delay embedding coordinates may be needed to get a large enough set of dynamical variables to describe the systems.

It is natural to consider a measurement system for the excitable media system. The measurement subsequent analysis reveals that the model can be related to physically occurring spatio-temporal behaviors in a transparent way. This shows how different features in the model produce different behaviors and it is shown that this helps in understanding complex excitable media phenomena. Both excitability and bifurcations involving fixed points are studied and explained in relation to the model properties. These results are then used to inform the design of simple scalar coupled map lattice models that can replicate the behaviors observed in real excitable media experiments. Finally, it is shown how the models introduced are able to explain most of the important features of excitable media, and illustrative examples are used to demonstrate the design of sCML models. The models introduced are considerably simpler and easier to relate to the underlying behavior of excitable media than previous models, and this means that the models can be used to understand the basic core underlying behaviors and properties of excitable media in a much more transparent way.

The paper will be organized as follows. Section 2 introduces the general form of the sCML model. The sCML models are analyzed in Sec. 3. Section 4 shows how to design a sCML model based on several simple rules. Illustrative examples are given in Sec. 5 to demonstrate the application of the new results. Conclusions are finally given in Sec. 6.
model has been employed to identify excitable media [Guo et al., 2010]. Two examples of the sCML model one from a real Belousov–Zhabotinsky experiment are analyzed in detail in Appendix A.

3. Analysis of the sCML Model

In this section, a sCML model of a specific structure is intensively studied. It will be shown that although the sCML model is of a very simple structure, it is able to explain most of the behavior observed in excitable media. This is important because this provides the potential to relate specific spatio-temporal behaviors to specific physical and experimental conditions and hence aids the understanding of this apparently complex phenomenon.

3.1. Decomposition of the sCML model

Assume the general sCML model (3) is of a reaction-diffusion structure, that is, the delayed states \( p^{m_i} \) and time-delayed diffusion terms \( p^{m_i}(\nabla^2 z) \) affect the evolution of the state separately. Therefore the sCML model (3) can be rewritten of the form

\[
z(t) = R(p^{m_i} z) + D(p^{m_i}(\nabla^2 z))
\]

(4)

where function \( R(p^{m_i} z) \) represents the reaction part and function \( D(p^{m_i}(\nabla^2 z)) \) represents the diffusion part of the sCML model. Obviously, both model (A.1) and (A.2) given in the appendix are of the reaction-diffusion form in (4).

For convenience of analysis, define a \( G \)-function of the sCML model (4) as

\[
g(p^{m_i} z) = R(p^{m_i} z) - z(t - 1).
\]

(5)

Then model (4) can be rewritten as

\[
z(t) - z(t - 1) = g(p^{m_i} z) + D(p^{m_i}(\nabla^2 z)).
\]

(6)

The left-hand side of Eq. (6) represents the change of component \( z(t) \) over the sampling time; \( g(p^{m_i} z) \) is the \( G \)-function which represents the contribution of the local dynamics to the evolution of excitable media. Term \( D(p^{m_i}(\nabla^2 z)) \) represents the contribution of the diffusion part.

Define \( F \)-functions of model (4) as the partial derivatives of the \( G \)-function \( g(p^{m_i} z) \) with respect to \( z(t - i) \) \( i = 1, 2, \ldots, m_i \), that is

\[
f_i(p^{m_i} z) = \frac{\partial g}{\partial z(t - i)}.
\]

(7)

The \( F \)-functions describe how the previous states \( z(t - i) \) affect the evolution of an excitable media. The \( G \)-function and \( F \)-functions will be used in the following analysis.

The analysis in Appendix A shows that although the two sCML models are identified from two completely different excitable media both \( F \)-functions share some common properties, that is, \( f_1(z(t - 1)) \) and \( f_2(z(t - 2)) \) are of a similar shape but opposite sign. These commonly existing properties are abstracted as in Fig. 1.

In order to reveal more details, the \( F \)-functions (which are studied in more detail later in Figs. 9 and 11) are enlarged and shown in Fig. 1, where \( a \) and \( d \) are the points of intersection for \( f_1 \) and the real axis; \( b \) and \( c \) are the points of intersection for \( f_2 \) and the real axis satisfying \( a < b < c < d \). Notice that in practical systems \( a \) and \( b \) are usually less than zero in order to guarantee that \( f_1(0) > 0 \) and \( f_2(0) < 0 \). Based on Fig. 1, the sCML model will be analyzed in two different aspects: the excitability and the contribution of variables.

3.2. Analysis of the contribution of variables

According to the analysis in Appendix A, the two intersections of function \( f_1(z(t - 1)) \) and the real axis determine a special interval which is shown as \( [b, c] \) in Fig. 1. In this interval \( f_1(z(t - 1)) \) always takes a positive value while \( f_2(z(t - 2)) \) takes a negative value. Following the definition of the \( F \)-function in (7), this means that \( z(t) \) increases with \( z(t - 1) \) increasing, but decreases with \( z(t - 2) \) increasing.
intersections of Fig. 1, it is easy to see that the interval between the takes values can be roughly estimated from a specific response. Comparing Fig. 9 (and Fig. 11) with Fig. 1, it is easy to see that the interval between the intersections of \( f_2 \) and the real axis is close to the state space \( S \). That is, roughly speaking, \( z(t - 1) \) acts as an activator and \( z(t - 2) \) acts as an inhibitor over the whole range of state space \( S \).

### 3.3. Analysis of excitability

The shapes of \( f_1 \) and \( f_2 \) guarantee the excitable behavior of the spatio-temporal system. This coincides with the excitability property. However, when the perturbation is large enough, the system will leave the fixed point and will undergo a long excursion to return to the fixed point. Assuming \( z(t - 1) = z(t - 2) + dx \), denote the point as \( e \) at which \( f_1(z(t - 1)) \) and \( -f_2(z(t - 2)) \) cross, that is \( f_1(e) = -f_2(e) \). Point \( e \) divides the interval \([b, c]\) into two parts: \([b, e]\) and \([e, c]\). When \( dx \) is large enough, \( f_1(z(t - 1)) \) is \(-f_2(z(t - 2)) \) in the region \([b, e]\) while \( f_1(z(t - 1)) < -f_2(z(t - 2)) \) in the region \([e, c]\). This means \( z(t) \) will increase and then decrease back to the equilibrium state and this corresponds to a relatively longer journey compared with the case when the process is disturbed by a small disturbance.

### 3.4. Bifurcation involving fixed points

From the above analysis, the \( F \)-functions can be any functions of the shapes in Fig. 1. However, in the following analysis only the simplest case will be considered where both \( f_1(z(t - 1)) \) and \( f_2(z(t - 2)) \) are of a second order polynomial form.

Assume that the sCM model are of the form

\[
z(t) - z(t - 1) = g(p^n z) + D(p^m \nabla^2 z) = g_1(z(t - 1)) + g_2(z(t - 2)) + d_1 \nabla^2 z(t - 1) + d_2 \nabla^2 z(t - 2)
\]

where

\[
\begin{align*}
g_1(z(t - 1)) &= \alpha_1 z^3(t - 1) + \alpha_2 z^2(t - 1) + \alpha_3 z(t - 1) \\
g_2(z(t - 2)) &= \beta_1 z^3(t - 2) + \beta_2 z^2(t - 2) + \beta_3 z(t - 2)
\end{align*}
\]

(9)

For convenience of the analysis, rewrite the parameters in \( g_2(z(t - 2)) \) as

\[
\begin{align*}
-\alpha_1 z^3(t - 2) - \alpha_2 z^2(t - 2) - \alpha_3 z(t - 2) + \gamma z(t - 2)(\mu - (z(t - 2) - \kappa)^2).
\end{align*}
\]

(10)

Parameters \( \mu \), \( \kappa \), and \( \alpha_3 \) will be used as the bifurcation parameters in the following analysis.

Defining \( z_1(t) = z(t), z_2(t) = z(t - 1) \) yields the state-space equations of the local dynamics in the sCM model.

\[
\begin{align*}
z_1(t) &= z_1(t - 1) + g_1(z_1(t - 1), z_2(t - 1)) \\
z_2(t) &= z_1(t - 1).
\end{align*}
\]

(11)

The fixed point of (11) can then be calculated by solving the equation \( \mathbf{x}'(t) - \mathbf{x}(t - 1) = 0 \).

\[
\begin{align*}
g_1(z_1(t - 1), z_2(t - 1)) &= 0 \\
z_2(t - 1) - z_1(t - 1) &= 0.
\end{align*}
\]

(12)

According to the second equation in (12), denote \( z_2(t) = z_1(t) = v(t) \). Substituting \( v(t) \) into the first equation of (11), the state-space equations collapse to a single variable dynamical system

\[
v(t) - v(t - 1) = g(v(t)).
\]

(13)

Recalling the structure of the \( G \)-function, Eq. (13) can explicitly be written as

\[
g(v(t)) = \mu - (v(t) - \kappa)^2 = 0.
\]

(14)

Notice that the fixed points of system (14) uniquely corresponds to the fixed points of system (11). Namely, for a fixed point \( v^*(t) \), the corresponding fixed points of system (11) are \( x^*(t) = [v^*(t), v^*(t)] \). The number of fixed points of the system (11) will change with the changing of the number of fixed points of system (13). In order to study the bifurcation in system (11), the bifurcation of system (13) will be analyzed where \( \mu \) and \( \kappa \) will be considered as the bifurcation parameters.

The fixed points of (13), that is, the roots of Eq. (14) can be given as

\[
\begin{align*}
v_1 &= 0 \\
v_2 &= \kappa + \sqrt{\mu} \\
v_3 &= \kappa - \sqrt{\mu}.
\end{align*}
\]

(15)
Table 1. Fixed points of system (13).

<table>
<thead>
<tr>
<th>$\mu &lt; 0$</th>
<th>$\mu = 0$</th>
<th>$\kappa &gt; \mu &gt; 0$</th>
<th>$\mu = \kappa^2$</th>
<th>$\mu &gt; \kappa^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_0^1 = 0$</td>
<td>$v_0^2 = 0$</td>
<td>$v_0^3 = 0$</td>
<td>$v_0^3 = +v_0^2$</td>
<td>$v_0^3 = 0$</td>
</tr>
<tr>
<td>$v_1^1 = \kappa - j\sqrt{\mu}$</td>
<td>$v_1^2 = \kappa + j\sqrt{\mu}$</td>
<td>$v_1^3 = \kappa - \sqrt{\mu} &gt; 0$</td>
<td>$v_1^3 = \kappa + \sqrt{\mu} &gt; 0$</td>
<td>$v_1^3 = 0$</td>
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<tr>
<td>$v_2^1 = 0$</td>
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<tr>
<td>$v_3^1 = \kappa - j\sqrt{\mu}$</td>
<td>$v_3^2 = \kappa + j\sqrt{\mu}$</td>
<td>$v_3^3 = \kappa - \sqrt{\mu} &gt; 0$</td>
<td>$v_3^3 = \kappa + \sqrt{\mu} &gt; 0$</td>
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<td>$v_4^1 = 0$</td>
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<tr>
<td>$v_0^1 = \kappa - j\sqrt{\mu}$</td>
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<td>$v_0^3 = \kappa - \sqrt{\mu} &gt; 0$</td>
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<tr>
<td>$v_1^1 = 0$</td>
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<td>$v_2^2 = \kappa + j\sqrt{\mu}$</td>
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<td>$v_2^3 = \kappa + \sqrt{\mu} &gt; 0$</td>
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<td>$v_3^1 = 0$</td>
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</tbody>
</table>

The roots $v_2$ and $v_3$ are complex numbers when $\mu < 0$. There are totally three fixed points in the system, however only the ones of a real value are physically meaningful. Both the number of fixed points and the stabilities of these fixed points change with a change of the bifurcation parameters. The fixed points of system (13) are summarized in Table 1.

According to Table 1, the parameter space is divided into five regions by the three curves, $\mu = 0$, $\mu = \kappa^2$ and $\kappa = 0$, which are marked in Fig. 2. System (13) has three fixed points in parametric regions I–IV and only one fixed point in parametric region V. Fixing parameter $\kappa = 0$ and changing parameter $\mu$, a pitchfork bifurcation occurs when $\mu$ crosses the horizontal axis $\mu = 0$ [shown in Fig. 3(c)]. For the cases of $\kappa \neq 0$, a fold bifurcation occurs when $\mu$ crosses zero and a transition bifurcation occurs when $\mu$ crosses $\mu = \kappa^2$ [shown in Figs. 3(a) and 3(b)]. Fixing the parameter $\mu > 0$ and changing the parameter $\kappa$, two transition bifurcations occur respectively when $\kappa$ crosses $\kappa = -\sqrt{\mu}$ and $\kappa = \sqrt{\mu}$ [shown in Fig. 3(d)]. The bifurcation diagrams are summarized as Fig. 3.

It is easy to observe that the dynamics of the sCML model always has the same number of fixed points as system (13) does. However, the stability of these fixed points could be different, which depends on the eigenvalues of the Jacobian matrix at these fixed points. For the model in this study, only the case of $\mu < 0$ is considered to produce the excitable media patterns where only the fixed point at $[0,0]$ exists. The stability of the fixed point at $[0,0]$ is analyzed in the next subsection, where a Hopf bifurcation may occur when the parameter $\alpha_3$ increases.

3.5. The Hopf bifurcation

Consider the local dynamics of the sCML model. The Jacobian matrix of (11) at $[0,0]$ can be calculated as

$$J = \begin{bmatrix} 1 + \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 + \alpha_3 & -\alpha_3 + \gamma(\mu - \kappa^2) \\ 1 & 0 \end{bmatrix}. \quad (16)$$

The corresponding characteristic equation is

$$\begin{vmatrix} \lambda - (1 + \alpha_3) & -\alpha_3 + \gamma(\mu - \kappa^2) \\ -1 & \lambda \end{vmatrix} = 0 \quad (17)$$

and

$$\begin{vmatrix} \lambda - (1 + \alpha_3) & -\alpha_3 + \gamma(\mu - \kappa^2) \\ -1 & \lambda \end{vmatrix} = 0 \quad (17)$$

and

$$\begin{vmatrix} \lambda - (1 + \alpha_3) & -\alpha_3 + \gamma(\mu - \kappa^2) \\ -1 & \lambda \end{vmatrix} = 0 \quad (17)$$

and

$$\begin{vmatrix} \lambda - (1 + \alpha_3) & -\alpha_3 + \gamma(\mu - \kappa^2) \\ -1 & \lambda \end{vmatrix} = 0 \quad (17)$$
that is,  
\[ \lambda(\lambda - (1 + \alpha_3)) + \alpha_3 - \gamma(\mu - \kappa^2) = 0. \]  
(18)

The discriminant of the quadratic equation is  
\[ \Delta = (1 - \alpha_3)^2 + 4\gamma(\mu - \kappa^2) \]  
(19)

where \( \gamma > 0 \) and \( \mu < 0 < \kappa^2 \) according to the previous assumptions. The system has two conjugate complex roots when \( \Delta < 0 \), that is  
\[ -2\sqrt{\gamma(\kappa^2 - \mu)} < 1 - \alpha_3 < 2\sqrt{\gamma(\kappa^2 - \mu)}. \]  
(20)

The corresponding roots are  
\[ \lambda_{1,2} = \frac{1 + \alpha_3 \pm j\sqrt{-(1 - \alpha_3)^2 - 4\gamma(\mu - \kappa^2)}}{2}. \]  
(21)

According to the relationship between the roots and the coefficients of a quadratic equation, the modulus of the roots can be calculated as  
\[ |\lambda_1| = |\lambda_2| = \sqrt{\lambda_1\lambda_2} = \sqrt{\alpha_3 - \gamma(\mu - \kappa^2)}. \]  
(22)

The fixed point is stable when \( |\lambda_1| < 1 \), namely  
\[ \alpha_3 < 1 + \gamma(\mu - \kappa^2) \]  
(23)

while the fixed point loses stability when \( \alpha_3 \) increases over the critical value \( 1 + \gamma(\mu - \kappa^2) \). This means a Hopf bifurcation occurs where a pair of complex conjugate eigenvalues of the linearization around the fixed point crosses the unit circle in the \( Z \)-plane. The bifurcation diagram is shown in Fig. 4. The locus of the two eigenvalues show...
that the two real eigenvalues change into a pair of conjugate complex numbers when
\[-2\sqrt{\gamma(\kappa^2 - \mu)} < 1 - \alpha_3 \text{ and return to real values when } 1 - \alpha_3 > 2\sqrt{\gamma(\kappa^2 - \mu)}.\]

This Hopf bifurcation is important for excitable media. Different patterns would be observed if the model is simulated before and after the Hopf-like bifurcation. For example, spiral patterns will be generated before the Hopf-like bifurcation, while target-like patterns will be generated after the bifurcation. If \(z(k)\) represents the concentration of a component in an excitable medium, the bifurcation will be coincident with the real dictyostelium discoideum experiment [Lee et al., 1996]. Lee and co-workers showed that using the density as a control parameter, spiral waves dominated the pattern when the density was high whereas circular waves were dominant in the pattern at low densities.

4. Design of sCML Models

The analysis results discussed in the preceding section will now be used to inform the design of simple sCML models that can replicate the important characteristics of the excitable media system. Construction of a new sCML model is essentially an inverse routine of the analysis. The model will be designed to possess the properties of excitable media. In this section, we assume the sCML model to be designed is of the same structure as the first example in Appendix A. A different structure is discussed in Appendix B.

The \(F\)-functions \(f_1(p^m z(t))\) defined in (7) of the shapes in Fig. 5 should be constructed first of all. Obviously, different sCML models can be constructed by selecting different model structures and properly setting the corresponding coefficients. In this section, only the simplest model will be considered. Several basic rules are proposed to construct a simple sCML model.

4.1. Design of F-functions

For simplicity, assume \(f_1(p^m z(t))\) and \(f_2(p^m z(t))\) are of the simplest form \(f_1(p^m z(t)) = f_1(z(t - 1))\) and \(f_2(p^m z(t)) = f_2(z(t - 2))\), that is \(f_1\) is a function of \(z(t - 1)\) and \(f_2\) a function of \(z(t - 2)\). In order to make the \(F\)-functions of the shape in Fig. 5, the simplest structure that can be taken are...
The constant of integration \( C \) can be taken as zero to make \([0,0] \) a fixed point of the system. For excitable media the zero fixed point should be locally stable so that the system will return to the equilibrium state after a small disturbance occurs.

### 4.3. Selection of the diffusion coefficients and the final model

To properly select the diffusion coefficients, two principles should be considered. Firstly, the diffusion should be large enough to excite the resting cells. Secondly, when an inhibitor diffuses faster than an activator, the fixed point may lose the stability because of the Turing bifurcation [Gierer & Meinhard, 1972; Turing, 1952] and then the system cannot be an excitable media system any more. Therefore, the inhibitor is assumed to have a smaller diffusion rate as compared with the activator so that spiral patterns can be generated. That is, the diffusion coefficient \( d_1 \) of \( z(t) \) takes a larger value than the diffusion coefficient \( d_2 \) of \( z(t) \).

All the main parts which a sCML model requires have now been constructed. Collecting all the results together gives the final sCML model which takes the form

\[
z(t) = z(t-1) + \int_{-A_0 z(t-1) - a} (z(t-1) - d) dz(t-1)
+ \int_{A_0 z(t-2) - b} A_0 z(t-2) dz(t-2)
+ d_4 \nabla z(t-1) - d_5 \nabla z(t-2)
\]  

(27)

where \( a < b < c < d \) and \( d_4 > d_5 \).

Equation (27) is the simplest sCML model where the F-functions are a second order polynomial of only one variable. Of course more complex sCML models, for example a model like (A.2), can be constructed to satisfy some specific requirements by appropriately choosing the F-functions. However, even the simplest sCML model can generate complex patterns such as spiral waves and expanding target patterns. This will be illustrated in the next section.

### 5. Illustrative Examples

In this section, illustrative examples will be constructed following the rules developed in the last section. The excitability and bifurcations of this model will then be analyzed. The simulations show...
that this simple model can generate complex spatio-temporal patterns.

5.1. Design of a sCML model
Following the procedure in Sec. 4, a sCML model is constructed as follows.
\[
z(t) = 1.7557z(t - 1) + 0.1979z^2(t - 1) - 0.024z^3(t - 1) - 0.763z(t - 2) - 1.95z^2(t - 2) + 0.0237z^3(t - 2) + 0.2\n\]
where the parameters in (24) as
\[
a = -1.53, \quad b = -1.5, \quad c = 7, \quad d = 7.03, \quad A_0 = 0.072, \quad A_1 = 0.071
\]
and the diffusion coefficients as
\[
d_1 = 0.2, \quad d_2 = -0.1.
\]
Here \(A_0\) and \(A_1\) are chosen to satisfy \(A_0(c-b)/4 = A_1(d-a)/2 = 1.3\).

In a spatially homogeneous condition where the diffusion effect is absent, the behaviors of the system will be dominated by the local dynamics of the system. Dropping the diffusion part of the model initially and simulating the model under perturbations of different strengths, the local dynamics of model (28) are illustrated in Fig. 5 which shows the dynamical system damps off a small-strength perturbation but undergoes a long term response for the stronger perturbation. Figure 5(b) shows the phase portrait of the dynamical system. Observe that in the phase portrait \((z(t-1), z(t))\) starting from point \((0,0,0)\) leaves the symmetry line \(z(t) = z(t - 1)\) (the black line) first and then returns back to the symmetry line. From Fig. 5, \(|f_1(z(t-1) - f_2(z(t-2)))|\) increases with the difference \(|z(t-1) - z(t-2)|\) increasing in region \([b, c]\).

The characteristic equation of \(J\) can be written as
\[
\lambda^2 - \text{tr}(J)\lambda + \text{det}(J) = 0
\]
where \(\text{tr}(J) = 1.7557\) is the trajectory of \(\lambda\) and \(\text{det}(J) = 0.7633\) in the determinant of \(J\). Solving (32) yields \(\lambda_1 = 0.9634\) and \(\lambda_2 = 0.7923\). The resting state \([0, 0]\) is locally stable because \(|\lambda_{1,2}| < 1\).

5.2. The Hopf bifurcation in the designed sCML model
A dynamical system loses stability at a fixed point as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the unit circle in the \(\lambda\)-plane. When this occurs, a Hopf bifurcation happens and a limit cycle may arise from the fixed point. The sCML model may undergo a Hopf bifurcation when the parameters are slightly perturbed. From Fig. 1, maintaining the relationship \(a < b < c < d\), when the distance between \(b\) and \(c\) is shortened, the fixed point at \([0, 0]\) may lose stability and a limit cycle may be born. This is because the value \(\alpha_3\) will increase when the distance between \(b\) and \(c\) is shortened while the relationship \(A_0(c-b)^2/4 = A_1(d-a)^2/4 = 1.3\) remains unchanged. Notice that \(\alpha_3\) is the \(\gamma\)-intercept of the \(F\)-function \(f_1(z(t-1))\) in Fig. 1.

In this example, it is easy to calculate that \(\mu = -5.0833 < 0\) and \(\alpha_3 = 0.7557\). According to the analysis in Sec. 3, \(z = [0, 0]\) is the only fixed point, which corresponds to the resting state of an excitable media. The local stability of the fixed point can be analyzed by considering the characteristic values of the following Jacobian matrix at the fixed point.
\[
J = \begin{bmatrix}
0 & 1 \\
-0.7633 & 1.7557
\end{bmatrix}
\]

In this example, the bifurcation parameter \(\mu = -1.1597\) is again less than zero and \([0, 0]\) is the only fixed point of the system. However, \(\alpha_3\) increases to 1.0314 which is greater than the critical value \(1 + \gamma|\mu - \kappa^2| = 0.9942\). This means the fixed point
at zero loses its stability. Repeating the local stability analysis to give the Jacobian matrix

\[
J = \begin{bmatrix}
0 & 1 \\
-1.0372 & 2.0314
\end{bmatrix}.
\]  

(34)

The characteristic equation is

\[
\lambda^2 - \text{tr}(J)\lambda + \det(J) = 0
\]  

(35)

where \(\text{tr}(J) = 2.0314\), \(\det(J) = 1.0372\). Solving (35) yields \(\lambda_{1,2} = 1.0107 \pm 0.1252i\). But this time the fixed point \([0,0]\) is locally unstable because \(|\lambda_{1,2}| > 1\). This means the excitable media system loses stability point \((0,0)\) when the parameters are adjusted from model (28) to model (33).

Simulating the local dynamics of model (33), the system response to a perturbation and the phase portrait are illustrated in Figs. 6(a) and 6(b). Observe that state \((z(t-1), z(t))\) leaves the initial point \((0,0.4)\) and finally approaches a stable limit cycle which is of a different topology from the fixed point in model (28). This means a Hopf-like bifurcation occurs where a fixed point turns into a limit cycle as the parameters are adjusted.

5.3. Simulation of the designed sCML models (28) and (33)

Combining the local dynamics with the diffusion effects, different patterns can be generated by simulating the sCML models. The sCML model (28) was simulated on a 256 \times 256 square lattice with a periodic boundary condition where a von Neumann neighborhood was selected to approximate the diffusion of components following formula (2). The simulation was started from a purely random initial state. A snapshot of the simulated pattern at \(k = 500\) is illustrated in Fig. 7(a) which shows a typical spiral pattern.

For different values of the parameters, the sCML model exhibits different spatio-temporal patterns. Resetting the parameters as in the model (33) and simulating the model on the same lattice, this time with a zero initial condition and a spot disturbance in the center, produces typical expanding target-like patterns. A snapshot of the simulated patterns at \(t = 500\) is shown in Fig. 7(b).

6. Conclusions

Scalar coupled map lattice models are a new class of model for the description, identification and analysis of excitable media. Guo et al. [2010] have shown...
the advantages of the new sCML model in identification. In practical cases, the analysis of excitable media systems based on identified models is often of a more significant meaning than the identification itself. The current paper has presented a detailed analysis of the sCML model from different aspects to reveal the relationships between the new sCML model and the observed physical phenomena including the interpretation of the complex behavior of excitable media.

The new sCML model possesses some significant advantages compared with the existing models. In this model, only one continuous measurement variable is involved to reconstruct the dynamics of excitable media. The sCML model is computationally efficient and descriptively powerful. The new model is of a simple structure and is closely related to the phenomena in real excitable media. This study intensively analyzed the behaviors of excitable media based on the new sCML model and revealed the connection between the physical phenomena and the model parameters. A complete procedure for the design of a sCML model has also been proposed. It has been shown how new pattern formations can arise so that different spatio-temporal patterns can be generated by appropriately adjusting the model parameters.

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References

Appendix A
In this section, two sCML models which have been separately identified from a simulated spatio-temporal pattern and from a real Belousov–Zhabotinsky pattern [Guo et al., 2010] are analyzed...
to extract some of the common properties of the sCML models for excitable media. These two models, the simulated pattern model and the real pattern model, are given as (A.1) and (A.2) below.

\[
\begin{align*}
 z(t) &= 1.7981z(t - 1) + 1.88849z^2(t - 1) - 1.70921z^3(t - 1) - 0.820409z(t - 2) \\
 &\quad - 1.83672z^2(t - 2) + 1.65721z^3(t - 2) + 0.459467\nabla^2 z(t - 1) - 0.42561\nabla^2 z(t - 2) \quad (A.1) \\
 z(t) &= 0.95013z(t - 1) - 0.0424791z(t - 2) + 1.86899 \times 10^{-5}z^4(t - 1) \\
 &\quad - 5.73529 \times 10^{-5}z^5(t - 1)z(t - 2) + 6.28918 \times 10^{-5}z(t - 1)z^2(t - 2) - 0.00310803z^3(t - 1) \\
 &\quad + 0.118357z^2(t - 1) - 0.185982z(t - 1)z(t - 2) + 0.00779509z^2(t - 1)z(t - 2) \\
 &\quad + 3.0064 \times 10^{-5}z^2(t - 2) - 2.71051 \times 10^{-5}z(t - 1)z^2(t - 2) - 0.0068082z(t - 1)z^2(t - 2) \\
 &\quad + 0.0020784z^3(t - 2) + 0.071285z^2(t - 2) + 0.743265\nabla^2 z(t - 1) - 0.354318\nabla^2 z(t - 2). \quad (A.2)
\end{align*}
\]

The local dynamics of models (A.1) and (A.2) are illustrated in Figs. 8(a) and 8(b). Figure 8 shows the excitability of both models, that is, a cell in a resting state is stable for a small perturbation (shown by the blue curves) while a perturbation with strength greater than a certain threshold can cause this cell to undergo a large excursion (shown by the red curves).

Rewriting model (A.1) in the form of (6), the \( G \)-function is given as

\[
\begin{align*}
 g(p^{\mu_2}z) &= 0.7981z(t - 1) - 0.820409z(t - 2) \\
 &\quad + 1.65721z^3(t - 2) - 1.83672z^2(t - 2) \\
 &\quad + 1.88849z^2(t - 1) - 1.70921z^3(t - 1). \quad (A.3)
\end{align*}
\]
Calculating the $F$-functions of model (A.1) with respect to $z(t-1)$ and $z(t-2)$ according to formula (7) yields

$$f_1(g(p^m z)) = 0.7981 + 2 \times 1.88849z(t-1) - 3 \times 1.70921z^2(t-1)$$

$$f_2(g(p^m z)) = -0.8204092 - 2 \times 1.83072z(t-2) + 3 \times 1.65721z^2(t-2)$$

(4.4)

$g(p^m z) = -0.0499z(t-1) - 0.0424791z(t-2) + 1.686899 \times 10^{-5}z^2(t-1)$

$$- 5.73529 \times 10^{-5}z^3(t-1)z(t-2) + 6.28916 \times 10^{-5}z^2(t-1)z^2(t-2)$$

$$- 0.0310803z^3(t-1) + 0.1183573z^2(t-1) - 0.185982z(t-1)z(t-2)$$

$$+ 0.00779090z(t-1)z(t-2) + 3.0084 \times 10^{-6}z(t-2)$$

$$- 2.71051 \times 10^{-5}z(t-1)z^3(t-2) - 0.0068082z(t-1)z^2(t-2)$$

$$+ 0.00207884z^3(t-2) + 0.0712835z^2(t-2)$$

(4.5)

$$f_1(g(p^m z)) = -0.0499 + 4 \times 1.868999 \times 10^{-5}z^2(t-1) - 3 \times 5.73529 \times 10^{-5}z^2(t-1)z(t-2)$$

$$+ 2 \times 6.28918 \times 10^{-5}z(t-1)z^2(t-2) - 3 \times 0.0310803z^2(t-1)$$

$$+ 2 \times 0.1183573z(t-1) - 0.185982z(t-2) + 2 \times 0.00779090z(t-1)z(t-2)$$

$$- 2.71051 \times 10^{-5}z(t-1) - 0.0068082z^2(t-2)$$

(4.6)

$$f_2(g(p^m z)) = -0.0424791 - 5.73529 \times 10^{-5}z^3(t-1) + 2 \times 6.28916 \times 10^{-5}z^2(t-1)z(t-2)$$

$$- 0.185982z(t-1) + 0.00779090z(t-1) - 4 \times 3.0084 \times 10^{-6}z(t-2)$$

$$- 3 \times 2.71051 \times 10^{-5}z(t-1)z^3(t-2) - 2 \times 0.0068082z(t-1)z(t-2)$$

$$+ 3 \times 0.00207884z^2(t-2) + 2 \times 0.0712835z(t-2).$$

$F$-functions of model (A.1) are illustrated in Fig. 9. In this example, $f_1$ and $f_2$ collapse to functions of only one variable in model (A.1) that is $f_1(p^m z) = f_1(z(t-1))$ and $f_2(p^m z) = f_2(z(t-2)).$ Observe that $F$-functions $f_1$ and $f_2$ are similar in shape but take opposite signs.

Consider sCML model (A.2) which was identified directly from real Belousov-Zhabotinsky reaction data. Repeating the same calculation, the $G$-function and $F$-functions of model (A.2) are given as (A.5) and (A.6) separately.

$F$-functions of model (A.2) are illustrated in Fig. 9. (a) $f_1(z(t-1), z(t-2))$, (b) $f_2(z(t-1), z(t-2))$. 

(a)

(b)
will only take values near the cross-sections along then $f$ for two totally different systems.

If $z(t - 1)$ and $z(t - 2)$ always take close values, then $f_1(z(t - 1), z(t - 2))$ and $f_2(z(t - 1), z(t - 2))$ will only take values near the cross-sections along $z(t - 1) = z(t - 2)$, namely, $f_1(p^m z(t))|_{z(t - 1) = z(t - 2)}$ and $f_2(p^m z(t))|_{z(t - 1) = z(t - 2)}$.

The cross-sections of $f_1$ and $f_2$ in (A.6) along $z(t - 1) = z(t - 2)$ are shown in Fig. 11. Comparing the curves in Figs. 9 and 11, it is easy to observe that the F-functions (or cross-sections of F-functions) of both model (A.1) and model (A.2) are of similar shapes although these two models were identified from two totally different systems.

Appendix B

In Appendix B, the F-functions in Figs. 9 and 11 will be analyzed in more detail. The F-functions in Fig. 9 are symmetrical along $z(t) = 0.47$ while the F-functions of the sCML model identified from a real chemical experiment are asymmetrical. In this section the advantages of asymmetrical F-functions in the design of sCML model are analyzed and a slightly more complex structure will be introduced.

The model (28) was simulated where a large initial perturbation with strength of 2 was applied. The response of the local dynamics model (28) is shown in Fig. 12. A small-amplitude oscillation can be observed at the beginning of the refractory phase. Obviously, this is not a characteristic of excitable media and should be avoided in the design of sCML models.

The oscillation happens because of the big difference between the amplitudes of the two F-functions at the far right of Fig. 1 which shows that $|f_1(z(t - 1))| < |f_2(z(t - 2))|$ when $z(t - 1) > z(t - 2)$. This makes the state of the excitable media refract from the excited state. However, a large step may make the value of $z(t - 1)$ significantly less than $z(t - 2)$ so that $|f_1(z(t - 1))| > |f_2(z(t - 2))|$ and an oscillation occurs. A intuitive idea to avoid the oscillation is to reduce the difference between $f_1$ and $f_2$ by decreasing the values of $A_e$ and $A_c$. However, this may lead to another problem that the threshold for excitation may increase. That is, in order to excite the excitable media from a resting state a perturbation with greater strength will be needed. This makes the system hard to be excited.

The oscillation in the response to a large perturbation.

The asymmetrical F-functions.
In this condition, asymmetrical $F$-functions provide flexibility for the design of $s$CML models. Define the asymmetrical $F$-functions as

\[
\begin{align*}
    f_1(z(t-1)) &= -A_a(z(t-1) - b) \\
                 &\times (z(t-1) - c)(z(t-1) - w) \\
    f_2(z(t-2)) &= A_i(z(t-2) - a) \\
                 &\times (z(t-2) - d)(z(t-2) - w)
\end{align*}
\]

(B.1)

by introducing an auxiliary point "$w$" into Fig. 1 where $a < b < 0 < c < d < w$ and $c - b > w - c$. This makes the $F$-functions between $e$ and $c$ flat and retains the shape of the $F$-functions between $b$ and $e$. The asymmetrical $F$-functions are shown in Fig. 13. When $w - c \gg c - b$, the existence of "$w$" will have little influence on the shape of the $F$-functions in interval $[b, c]$ and the asymmetrical $F$-functions work like symmetrical $F$-functions in this interval. Simulations showed that the oscillation phenomena existing in the asymmetrical $F$-function $s$CML models can be successfully avoided by adopting the asymmetrical $F$-functions.