TECHNICAL NOTE

Optimal Dynamic Joint Inventory-Pricing Control for Multiplicative Demand with Fixed Order Costs and Lost Sales

Yuyue Song
Faculty of Business Administration, Memorial University of Newfoundland,
St. John’s, Newfoundland, Canada A1B 3X5, ysong@mun.ca

Saibal Ray, Tamer Boyaci
Desautels Faculty of Management, McGill University, Montreal, Quebec, Canada H3A 1G5
{saibal.ray@mcgill.ca, tamer.boyaci@mcgill.ca}

This note studies the optimal dynamic decision-making problem for a retailer in a price-sensitive, multiplicative demand framework. Our model incorporates lost sales, holding cost, fixed and variable procurement costs, as well as salvage value. We characterize the structure of the retailer’s (discounted) expected profit-maximizing dynamic inventory policy for both finite and infinite selling horizon problems.

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1. Introduction and Literature Overview

We consider a retailer facing multiplicative end customer demand in each period. Unsatisfied demands result in lost revenue; leftover inventory at the end of a period is charged a holding cost, whereas leftovers at the end of the selling horizon can be salvaged. The retailer’s replenishment cost includes a variable cost per unit and a fixed order cost for any positive purchase quantity. Our objective is to determine the optimal dynamic inventory control policy (from which the optimal price to charge in each period can be deduced) that maximizes the retailer’s total discounted expected profit over the selling horizon (either finite or infinite). For expositional convenience, we present our analysis assuming stationary demand and cost parameters, although all the results can be generalized to nonstationary settings under some mild conditions (see §3).

Dynamic joint inventory-pricing control in the existing operations management literature can be categorized based on whether the excess demand in each period is backordered or lost, and whether the demand form is multiplicative or additive. For additive demand, the optimality of the \((s, S, p)\) policy has been established for backordering and lost-sales models by Chen and Simchi-Levi (2004a, b) and Chen et al. (2006), respectively. Using an alternative approach, Huh and Janakiraman (2008; henceforth referred to as H&J) also prove the optimality of the \((s, S, p)\) policy for both cases under quite general conditions.

For multiplicative demand and backordering, Chen and Simchi-Levi (2004a, b) have shown that the optimal policy is of \((s, S, p)\) form in the infinite-horizon case, but has an \((s, S, A, p)\) structure for the finite-horizon scenario. To the best of our knowledge, the structure of the optimal policy for a lost-sales, multiplicative setting remains an open question. According to Polatoglu and Sahin (2000), multiple optimal order-up-to levels might exist in that case (also refer to Chen and Simchi-Levi 2004a, p. 892).\(^1\)

Our note complements existing literature by establishing the optimality of the \((s, S, A, p)\) policy for a modelling paradigm with multiplicative demand, lost sales, and a fixed cost associated with any replenishment for finite selling horizon problems. Moreover, we show that for the special case with zero fixed order costs, the optimal policy reduces to a base-stock policy, whereas for the stationary infinite-horizon case, an \((s, S, p)\) policy is optimal. As discussed at the end of §3, our technique readily applies to the backordering model, reestablishing Chen and Simchi-Levi’s (2004a) optimal policy result, although our approach requires certain conditions that Chen and Simchi-Levi do not impose.

Because some of our proofs borrow concepts and results from H&J, we briefly discuss that paper before presenting...
our results. H&J identify two conditions: Condition 1 and a more restrictive Condition 2, which guarantee the optimality of the \((s, S, p)\) policy for (stationary) infinite- and finite-horizon problems, respectively. However, verification of these conditions for specific scenarios is nontrivial. H&J show that Condition 1 is valid for: (i) backordering models for both additive and multiplicative demand forms, and (ii) additive, lost-sales models. They also verify that Condition 2 holds true for additive demand models (both lost sales and backordering). However, they do not provide any result for the lost-sales model under multiplicative demand, which is the focus of this paper.

The remainder of this note is organized as follows. We develop our stationary model framework in §2, whereas in §3 we present the optimal policy results. At the end of §3, we point out the conditions for validity of our results under backordering and nonstationary settings. In the interest of space, we provide proofs of only the main results. Some of these proofs are lengthy (Lemmas 2 and 4, Proposition 1), in which case we only provide a high-level sketch. Complete details of the abbreviated and omitted proofs are provided in the online appendix. An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

2. Model Framework

Consider a periodic-review, finite selling horizon with \(T\) periods, indexed forward by period index \(t, 1 \leq t \leq T\). If the retail price charged in period \(t\) is \(p\), the demand in that period is \(D(p)\varepsilon_t\), where \(D(p)\) is a strictly decreasing, deterministic function defined on \((0, P^*)\). Note that \(P^*\) is the lowest positive retail price such that \(D(P^*) = 0\), i.e., the “null price.” If \(D(p) > 0\) for all \(p > 0\), let \(P^* = +\infty\). Without loss of generality, we also assume that \(P^* > w\), where \(w > 0\) is the per-unit purchasing cost. On the other hand, \(\varepsilon_t\), \(1 \leq t \leq T\), are independent and identically distributed random variables that are defined and positive on \((L, U)\), \(0 < L < U < \infty\). Without loss of generality, we assume that \(E[\varepsilon_t] = 1\). Let \(f(u)\) and \(F(u)\) be the density and distribution functions of \(\varepsilon_t\), respectively. Also, \(\lim_{u \to +\infty} f(u) = 0\) and \(f(u) = 0\) for any \(u \in [0, L] \cup [U, +\infty]\).

Given an initial stock level \(x (\geq 0)\) before ordering at the beginning of period \(t\), the retailer needs to decide on the order-up-to inventory level \(y(x)\) and the retail price \(p\), before any demand is realized. The objective is to maximize the total discounted expected profit from period \(t\) until the end of the planning horizon \(T\). For an order of \((y - x)\) from the manufacturer, the retailer’s replenishment cost is given by \(K(y - x) + w(y - x)\), where \(K (\geq 0)\) is the fixed order cost, and \(\delta(y - x) = 1\) if \(y > x\) and zero otherwise. Once the order is placed, it is received immediately by the retailer. This is a standard assumption in the related literature (see also H&J and Chen and Simchi-Levi 2004a, b).

Subsequently, demand in period \(t\) is realized. Any demand not directly satisfied from stock results in lost revenue of \(p\), whereas any leftover inventories are charged a holding cost at the rate of \(h\) per unit.\(^3\)

Let \(\Pi_t(x)\) be the optimal discounted expected total profit from period \(t\) until the end of the planning horizon \(T\) less the purchase cost of \(x\) units, when the starting inventory level in period \(t\) is \(x\). We define \(LO(y, p, u) = \max\{y - D(P(y)|u, 0)\}\) as the leftover inventory level at the end of period \(t\) given \(y, p\) and the realization of \(\varepsilon_t\). If \(0 < \alpha < 1\) denotes the (given) discount factor, then the retailer’s maximization problem can be formally stated as

\[
\Pi_t(x) = \max_{(p, y; x)} \left\{ -K\delta(y - x) + \Gamma(y, p) \right\}
+ \alpha \int_0^{+\infty} \Pi_{t+1}(LO(y, p, u)) f(u) du \right\}
\]

(1)

Here, \(\Gamma(y, p)\) is the expected total profit function for a price-setting newsvendor with a salvage value \(v = aw - h\) and zero initial stock. \(\Gamma(y, p)\) can be expressed as

\[
\Gamma(y, p) = pD(p) \left[ 1 - \Theta \left( \frac{y}{D(p)} \right) \right]
+ (aw - h)\Lambda \left( \frac{y}{D(p)} \right) D(p) - wy,
\]

(2)

where \(\Lambda(z) = \int_z^{+\infty} (z - u)f(u) du\) and \(\Theta(z) = \int_z^{+\infty} (u - z)f(u) du\) for any \(z \in (0, \infty)\) represent the relative overage and underage functions, respectively, and \(y/D(p)\) denotes the stocking factor (as in Petruzzi and Dada 1999). We assume that \(p > v \geq 0\), where \(p > v\) guarantees positive profit. On the other hand, \(w > v = aw - h\), i.e., \(w + h > aw\), implies that it is cheaper to procure a unit than to carry it over from the previous period, eliminating the “speculative” motive for holding inventory. At the end of the period \(T\), the remaining stock can be returned to the manufacturer for full credit, i.e., the salvage value is \(w (\geq 0)\). Then, \(\Pi_{T+1}(x) = 0\) for any \(x \geq 0\). Because any excess demand is lost, we define \(\Pi_t(x) = \Pi_t(0)\) if \(x < 0\).

Our analysis requires the deterministic part \((D(p))\) and the random part \((\varepsilon_t)\) of the demand to have the following properties.

**ASSUMPTION 1.** \(D(p)\) is positive and strictly decreasing for \(p \in (0, P^*)\) and \(\lim_{p \to P^*} pD(p) = 0\). Moreover, \(D(p)\) is continuously differentiable and the elasticity \(\eta(p) = -p(D'(p)/D(p))\) \((> 0)\) is increasing for \(p \in (0, P^*)\). Also, \(D(p)/D'(p)\) is monotone and concave, whereas \(p + D(p)/D'(p)\) is strictly increasing for \(p \in (0, P^*)\).

Assumption 1 implies that the curvature of \(D(p)\), defined as \(E(p) = D(p)(D'(p)/D'(p))\), should not be highly positive and it should increase in \(p\). Assumption 1 is satisfied by most of the common demand functions. Examples include concave functions \((D(p) = a - p^k\) \((a > 0, k > 1)\), \(D(p) = (a - kp)^\gamma\) \((a > 0, k > 0, 0 < \gamma < 1)\), as well as convex ones \((D(p) = ap^{-k}\) \((a > 0, k < 1)\), \(D(p) = (a - kp)^\gamma\) \((a > 0, k > 0, \gamma > 1\) or \(a > 0, k < 0, \gamma < -1)\) (refer to Cowan 2004, Ziya et al. 2004, and Song et al. 2008 for more details).
ASSUMPTION 2. \( r(u) = uf(u)/(1 - F(u)) \) is increasing in \( u \) on \( u \in (L, U) \).

Assumption 2 implies an increasing generalized failure rate (IGFR) for \( \epsilon_u \), and is a mild requirement satisfied by distributions such as Uniform, Gamma with shape parameter \( \geq 1 \), Beta with both parameters \( \geq 1 \), Normal, and Exponential (refer to Lariviere 2006 for more details).

We use the expression \( V(z) = (1 - z[1 - F(z)]/ (1 - \Theta(z)))^{-1} \) \( \forall z \in (L, U) \) throughout this study. If \( z \) represents the stocking factor, then \( V(z) \) is a one-to-one function of the elasticity of expected sales with respect to \( z \) represented by \( z[1 - F(z)]/1 - \Theta(z) \) (Petruzzi 2004). It turns out that \( V(z) \) exhibits the following properties (refer also to Song et al. 2008):

**Lemma 1.** \( V(z) \) is strictly decreasing on \( z \in (L, U) \), \( \lim_{z \to L} V(z) = +\infty \), and \( \lim_{z \to U} V(z) = 1 \). Furthermore, \( \Lambda(z) \) and \( \Theta(z) \) can be rewritten as

\[
\Lambda(z) = zF(z) - \int_0^z uf(u) \, du \quad \text{and} \quad \Theta(z) = 1 - z[1 - F(z)] - \int_0^z uf(u) \, du.
\]

(3)

### 3. Model Analysis

We first analyze a price-setting newsvendor problem, and then, utilizing the results of this single-period model, we characterize the optimal policy for the multiperiod model setting.

#### 3.1. Analysis of the Single-Period Model

For an initial inventory level \( x \), a price-setting newsvendor retailer's profit is given by \(-K\delta(y - x) + \Gamma(y, p) + wx\), where \( \Gamma(y, p) \) is given by (2). If \( y/D(p) \leq L \) for some pair \( (y, p) \), then \( \Gamma(y, p) \) can be simplified as \( (p - w)y \), and it is always increasing in terms of \( p \). Therefore, for any \( (y, p) \) satisfying \( y/D(p) \leq L \), there exists an \( \hat{y} \) such that \( \hat{y}/D(\hat{p}) > L \) and \( \Gamma(\hat{y}, \hat{p}) > \Gamma(y, p) \). Hence, in the remainder of the note we assume that \( y/D(p) > L \). For any \( y \) \( \in (x, +\infty) \) at the beginning of the period, the corresponding feasible range of \( p \) is \( \{\max\{D^{-1}(y/L), v\}, P^0\} \) (recall we assume that \( p > v \geq 0 \)).

The next proposition summarizes the properties of \( \Gamma(y, p) \) and the optimal policy characteristics for the single-period model.

**Proposition 1.** For a single-period model, given any order-up-to inventory level \( y \geq x \), there exists a unique \( P(y) \), solution of \( \partial \Gamma(y, p)/\partial p = 0 \), such that \( \Gamma(y, p) \) is maximized. \( P(y) \) is strictly decreasing and \( \Gamma(y, P(y)) \) is concave in \( y \). There also exists a unique maximizer of \( \Gamma(y, P(y)) \). Moreover, the following are true:

1. Let \( S \) be the unique maximizer of \( \Gamma(y, P(y)) \) and \( s \) \( \leq S \) be the maximal inventory level such that \( \Gamma(s, P(s)) < \Gamma(S, P(S)) - K \). If there is no such \( s \), define \( s = 0 \). Then, an \((s, S, P)\) policy is optimal for the retailer.

2. Let \( Z(y) = y/D(P(y)) \) for any \( y > 0 \). \( Z(y) \) is increasing in \( y \), and so is the leftover inventory at the end of the period, i.e., \( LO(y, P(y)) \) is a maximizer of \( \Gamma(y, P) \) for any realization \( u \) of \( \epsilon, \frac{5}{5} \).

**Proof.** For any given order-up-to inventory level \( y \in [x, +\infty) \) at the beginning of the period, let \( P(y) \in \{\max\{D^{-1}(y/L), v\}, P^0\} \) be the maximizer of \( \Gamma(y, p) \). We first show that there is a unique \( P(y) > v \geq 0 \), which satisfies the first-order condition

\[
V\left(\frac{y}{D(p)}\right) + p\frac{D'(p)}{D(p)} - v\frac{D'(p)}{D(p)} = 0.
\]

(4)

We establish this irrespective of whether \( D'(p)/D(p) \) is increasing or decreasing. Taking the derivative of (4) on both sides with respect to \( y \), we can show that \( P(y) \) is strictly decreasing, while \( y/D(P(y)) \) is strictly increasing. It then follows that \( LO(y, P(y), u) = \max\{D(P(y))[Z(y) - u], 0\} \) is also increasing in \( y \) for any realization \( u \) of \( \epsilon \). This proves part 2 of the proposition.

Then, based on the first and second derivatives of \( \Gamma(y, P(y)) \) with respect to \( y \), we show that \( \Gamma(y, P(y)) \) is concave for \( y \in [x, +\infty) \). Consequently, there is a unique maximizer of \( \Gamma(y, P(y)) \) on \( y \in [x, +\infty) \), which we define as \( S \). Defining \( s \leq S \) as the maximal inventory level such that \( \Gamma(s, P(s)) < \Gamma(S, P(S)) - K \) (if there is no such \( s \), let \( s = 0 \)), the optimality of the \((s, S, P)\) policy in part 1 of Proposition 1 then follows. \( \square \)

#### 3.2. Analysis of the Multiperiod Model

Consider a given period \( t \) and an initial inventory level \( x \). The retailer maximizes the expected total profit given by

\[-K\delta(y - x) + \Gamma(y, p) + w\int_0^{+\infty} \Pi_{t+1}(LO(y, p, u))f(u) \, du\]

by selecting the order-up-to inventory level \( y \) and the retail price \( p \).

We approach this optimization problem sequentially. We first determine the optimal price for a given \( y \), and then analyze the resulting one-variable problem in terms of \( y \).

Define

\[H_t(y) = \max_p \left\{ \Gamma(y, p) + \alpha \int_0^{+\infty} \Pi_{t+1}(LO(y, p, u))f(u) \, du \right\}.
\]

(5)

Let us denote \( p_t(y) \) as the optimal price for a given \( y \) (if there are multiple optimal prices, we define \( p_t(y) \) as the smallest maximizer), and \( y^*_t \) as the maximizer of \( H_t(y) \). We now present a crucial property of \( \Gamma(y, p, u) \), which will help us to analyze \( H_t(y) \).

**Lemma 2.** For any two given order-up-to levels \( y^1 \) and \( y^2 \) such that \( S \leq y^1 < y^2 \), and a given retail price \( p^2 \), there exists a retail price \( p^1 \) such that \( \Gamma(y^1, p^1) \geq \Gamma(y^2, p^2) \) and \( LO(y^1, p^1, u) \leq LO(y^2, p^2, u) \).
The proof of this lemma follows a similar logic to that of the proof of Proposition 2 in H&J. For any given $S < y' < y^2$ and $p^2$, there are two cases to consider to prove the lemma:

Case 1: $p^2 \geq P(y^2)$. Choose $p^1 = P(y^1)$. The lemma then follows directly from Proposition 1.

Case 2: $p^2 < P(y^2)$. Let $I_2$ be the constant such that $(y^i, p^i)$ is on $C^{i_2}$. Based on the relation between $C$ and $C^{i_2}$ (i.e., $P(y)$ and $P(y^i)$), we need to analyze three subcases (see Figure 1). However, before analyzing the subcases, we need to define the following. Let $P(y)$ be the unique positive solution of $V(y)D(p) = l$ for a given positive constant $l \geq 1$ and $Z(l) = y/D(P(y))$. We also define $C = \{(y, P(y)) \mid y > 0\}$ and $C^i = \{(y, P(i)) \mid y > 0\}$. The common point on both $C$ and $C^i$, if any, is denoted by $(y', p^i)$. Note that it can be shown that both $P(y)$ and $P(y)$ are decreasing, but $P(y)$ is decreasing faster than $P(y)$ at the common point $(y', p^i)$ (if any).

Subcase 2(a): There is a common point $(y^i, p^i)$ on both $C^{i_2}$ and $C$, and $y_1 < y^i$. Let $p^1 = P(y^1)$. Then, from Proposition 1, we have $\Gamma(y^1, p^1) > \Gamma(y^i, p^i) \geq \Gamma(y^2, p(y^2))$. Then only need to show that $LO(y^2, p^2, u) \geq LO(y^1, p^1, u)$. We then only need to show that $LO(y^2, p^2, u) \geq LO(y^1, p^2, u)$. This follows because on $C^i$, the leftover $D(p)[y/D(p) - u]$ is decreasing in $p$ (note that $y/D(p)$ is constant on $C^i$).

Subcase 2(b): There is a common point $(y^i, p^i)$ on both $C^{i_2}$ and $C$, and $y_1 > y^i$. Let $(y', p^i)$ be the point on both $C^{i_2}$ and $\{(y, p) \mid y = y^1\}$, and $l_1 < l_2$ be the constant such that $C^{i_2}$ is passing through the point $(y', P(y^i))$. For a fixed $y'$ ($i = 1$ or 2), the retail price $p'(l)$ of the point $(y', p(l))$ on $C^i(l \leq l \leq l_2)$ is a function of $l$ and satisfies $V(y'/D(p'(l))) = l$. After deriving $d\Gamma(y^i, p'(l))/dl$ ($i = 1, 2$), based on the expression for $dp'(l)/dl$, we first establish that

$$\Gamma(y^i, p^1) - \Gamma(y^i, P(y^i)) = \int_{l_1}^{l_2} d\Gamma(y^i, p^1(l)) \geq \int_{l_1}^{l_2} d\Gamma(y^2, p^2(l)) = \Gamma(y^2, p^2) - \Gamma(y^2, P(y^2))$$

On the other hand, $\Gamma(y^1, P(y^1)) \geq \Gamma(y^2, P(y^2)) \geq \Gamma(y^2, P(y^2))$ from the unimodality of $\Gamma(y, P(y))$ and the optimality of $P(y)$. Combination of the above two results yields $\Gamma(y^1, p^1) \geq \Gamma(y^2, p^2)$. $LO(y^2, p^2, u) \geq LO(y^1, p^1, u)$ then follows based on the same argument as in Subcase 2(a).

Subcase 2(c): No common point on $C^{i_2}$ and $C$. Note that for any point $(y, p)$ below $C$, we have $\partial \Gamma(y, P(y))/\partial p > 0$ (from the definition of $P(y)$ and the unimodality of $\Gamma(y, P(y))$ in terms of $p$). Because $(y^2, p^2)$ on $C^{i_2}$ is below $C$, we always have $I_2 + (p - u)(D(p)/D(p)) > 0$ for any $p > v$. To prove the result for this subcase, we investigate the location of the maximizer of $\Gamma(y, p)$ on $C^{i_2}$ by varying the curve parameter $I_2$. For notational simplicity, we denote this curve parameter by $I$. Note that $I > 1$ and $(p - u)/(D(p)/D(p)) > 0$ for any $p > v$, and the profit on $C^i$ is given as

$$\Gamma(p) = pD(p[1 - \Theta(Z(l)) + v\Lambda(Z(l))D(p) - wZ(l)D(p)].$$

We show that $\Gamma(p)$ is unimodal with a unique maximizer $p(Z(l))$ and a corresponding $y(Z(l)) = D(p(Z(l)))Z(l)$, which is no more than $S$. Now, taking $l = I_2$ and $p^1 = P^{i_2}(y^1)$, from the unimodality of $\Gamma(p)$ and because $y(Z(l)) \leq S$, we can establish that $\Gamma(y^1, p^1) \geq \Gamma(y^2, p^2)$. The proof of $LO(y^1, p^1, u) \geq LO(y^2, p^2, u)$ is then similar to Subcases 2(a) and 2(b).

The result in Lemma 2 is slightly stronger, and hence implies Condition 1 of H&J for multiplicative demand with lost sales. In nontechnical terms, it means that: (i) the nearer the inventory level at the beginning of period $t$ to $S$, the better it is for the retailer from the viewpoint of expected profit; and (ii) if the retailer starts with a higher inventory level at the beginning of the period, this will result in higher leftovers at the end of the period.

Lemma 2 enables us to characterize the optimal replenishment strategy when the initial stock level is “high” (i.e., $x \geq 0$) using H&J’s results.

Lemma 3. It is optimal for the retailer to order nothing at the beginning of period $t$ if the initial stock level $x \geq 0$. 

![Figure 1. Illustration of subcases for Case 2.](image-url)
Theorem 1 clearly establishes the optimal replenishment policy when the starting inventory level is either low \((x \leq s)\) or high \((x \geq S)\). The complication arises in the intermediate range \((s < x < S)\), where it is not possible to ascertain the exact behavior of the profit function \(H_t(y)\). Polatoglou and Sahin (2000) indicate that multiple order-up-to levels might exist (for general \(D(p)\) functions). Nevertheless, we show that for a sufficiently large group of demand functions, there is a unique order-up-to level, whenever it is optimal to order. The optimal price to charge in each period is based on the postreplenishment inventory level. Theorem 1 reveals that the structure of the optimal policy is of the form \((s, S, A, p)\), where \(A\) denotes the set of inventory levels \((s, S)\) for which it is optimal to order, as is shown to be optimal for the backordering case by Chen and Simchi-Levi (2004a).

A simple upper bound on the optimal order-up-to level \(y^*_t\) can be derived from the characterization of \(H_t(y)\) in Lemma 4, which is useful for computational purposes.

**Theorem 2.** Let \(m (\geq S)\) be the maximal \(y\) such that \(\Gamma(y, P(y)) \geq \Gamma(S, P(S)) - K\). Then, \(y^*_t \in [s, m]\).

Furthermore, if \(K = 0\), then from the proof of Lemma 4 it is clear that \(s = S\) and the set \(A\) disappears, implying that a base-stock policy is optimal at the beginning of period \(t\). Likewise, we can characterize the optimal policy for a stationary, infinite planning horizon scenario.

**Theorem 2.** For the stationary infinite-horizon problem, an \((s, S, p)\) policy is optimal.

**Proof.** Follows from Lemma 2 and Theorem 1 in H&J. □

We note that the optimal policy results for the lost-sales case continue to hold even if excess demands in each period are backordered at a cost of \(b (\geq 0)\) per unit, where \(0 \leq h < b\). We then require \(D(p)\) to satisfy the following assumption (we still require Assumption 2):

**Assumption 3.** The demand function \(D(p)\) is positive, strictly decreasing, and continuously differentiable in terms of \(p\) on \((0, P^*)\) and \(\lim_{p \to P^*} pD(p) = 0\). Furthermore, \(p + D(p)/D'(p)\) is strictly increasing for \(p \in (0, P^*)\). □

The strictly increasing property of \(p + D(p)/D'(p)\) for \(p \in (0, P^*)\) is exactly equivalent to Assumption 2 in Chen and Simchi-Levi (2004a, p. 888). Expectedly, the backordering scenario will result in some changes in the analytical expressions. We do not repeat the detailed derivations and proofs here. However, we can reestablish Chen and Simchi-Levi’s (2004a) result that the optimal policy for the finite-horizon, backordering model is also of the \((s, S, A, p)\) form if the inventory holding and backordering costs are linear. This immediately leads to results analogous to zero fixed cost and stationary infinite-horizon cases like before. However, note that Chen and Simchi-Levi’s results for the backordering scenario is valid under more general conditions than ours (e.g., they allow nonlinear holding and
backordering costs, and more general demand structures than those satisfying Assumption 3).

As a final remark, we would like to point out that although we have presented the note for stationary demand and cost parameters, all our policy results (for both lost-sales and backordering scenarios) are valid under the following assumption about the nature of nonstationarity:

**Assumption 4.** (a) \( K_t \geq \alpha K_{t+1} \), and (b) \( S_t \leq S_{t+1} \) for any \( t (\leq l < T) \).

The above assumption is prevalent in the related literature (e.g., see H&J for (a), and see H&J and Chen and Simchi-Levi 2004a for (b)). Obviously, we would also require \( p_i > v_i \geq 0 \), where \( v_i = \alpha w_{i+1} - h_i \) and \( w_i < P_i^u \).

### 4. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

### Endnotes

1. For a more detailed literature review, refer to H&J, Chen et al. (2006), and Chen and Simchi-Levi (2004a).
2. Throughout this note, we use “increasing” and “decreasing” in the weak sense, unless otherwise stated.
3. Note that we are not able to definitely prove whether the results of this note continue to hold if there is an explicit penalty cost for lost sales, in addition to \( p \).
4. If \( L = 0 \), we define \( D^{-1}(y/L) = 0 \).
5. \( Z(y) \) is the stocking factor corresponding to the optimal price \( P(y) \) for any given order-up-to level \( y \).
6. Condition 1 of H&J can be stated in our context as: \( \Gamma(y) = \max_y \Gamma(y, p) \) is quasiconcave, and for any \( y_1 \) and \( y_2 \) satisfying \( S \leq y_1 < y_2 \) and \( P_1^u \), there exists a \( p_1 \) such that \( \Gamma(y_1, p_1) \geq \Gamma(y_2, P^u) \) and \( L_1(y_1, p_1, u) \leq \max[S, L_1(y_2, P^u, u)] \).
7. Note that \( y \) is a local variable, which is an arbitrary point as defined in the lemma.
8. Chen and Simchi-Levi (2004a) assume that \( p(D)D \) is concave in \( D \), where \( p(D) \) is the inverse of \( D(p) \).
9. Details are available from the authors on request.

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### References


