FIR Smoothing of Discrete-Time Polynomial Signals in State Space

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Abstract—We address a smoothing finite impulse response (FIR) filtering solution for deterministic discrete-time signals represented in state space with finite-degree polynomials. The optimal smoothing FIR filter is derived in an exact matrix form requiring the initial state and the measurement noise covariance function. The relevant unbiased solution is represented both in the matrix and polynomial forms that do not involve any knowledge about measurement noise and initial state. The unique i-degree unbiased gain and the noise power gain are derived for a general case. The widely used low-degree gains are investigated in detail. As an example, the best linear fit is provided for a two-state clock error model.

I. INTRODUCTION

POLYNOMIAL models often well formalize a priori knowledge about processes [1] and systems whose states change slowly with time. Relevant signals are typically represented with finite-degree polynomials to fit a number of practical needs. Examples can be found in signal processing [2], timescales and clock synchronization of digital communication networks [3], image processing [4], speech processing [5], etc. Polynomial models are usually processed on finite horizons of \(N\) points that typically yield a nice restoration.

To improve the performance of filters and determine initial conditions, smoothing is commonly used. Soon after a solution to the linear filtering was obtained by Kalman and Bucy [6], the smoothing problem for recursive linear structures was formulated by Rauch [7]. Thereafter, a number of solutions to the problem have been proposed to have a fixed lag, fixed interval, or fixed point. Summarizing, Moore stated in [8] that there are an infinity of smoothing algorithms.

Researchers mostly developed the recursive infinite impulse response (IIR) smoothing structures. Owing to the analytical complexity and large computation time and in spite of the inherent advantages in stability and robustness, the transversal finite impulse response (FIR) smoothers have not been used for decades. We find only a few substantial results in recent years. For polynomial models, FIR smoothers were used by Wang in [9] to design a nonlinear filter and by Zhou and Wang in the FIR-median hybrid filters [10]. In state space, order-recursive FIR smoothers were proposed by Yuan and Stuller in [11]. Most recently, the general receding horizon FIR smoother theory has been developed by W. H. Kwon et. al. and others in [12]–[16].

In this paper, we develop the approach used to derive the unbiased FIR filter [18] and propose a new smoothing FIR filter (optimal and unbiased) for discrete time-invariant polynomial models represented in state space with the finite-degree Taylor series. The rest of the paper is organized as follows. In Section II, we describe the model and formulate the problem. In Section III, the gains for the optimal and unbiased smoothing FIR filters are derived. The unbiased smoothing FIR filter is considered in detail in Section IV. Section V pays special attention to the exact and unique low-degree unbiased polynomial gains to cover an overwhelming majority of applications. An example of the two-state clock model is given in Section VI and concluding remarks are drawn in Section VII.

II. POLYNOMIAL SIGNAL MODEL AND PROBLEM FORMULATION

Consider a signal \(x_{kn}\) representing the \(k\)th state, \(k \in [1, K]\), of a \(K\)-state system in discrete-time \(n\). If a signal projects ahead on a horizon of \(N\) points from \(n - N + 1 - p\) to \(n - p\), where \(p\) is a discrete time shift, then the smoothing FIR filtering can be provided at a current point \(n\) with a lag \(^1\) \(p\), \(p < 0\), as shown in Fig. 1a. If to project a signal ahead from \(n - N + 1\) to \(n\), then the FIR smoothing can be organized at a past point \(n + p\) following Fig. 1b. Because structures of these two estimators are convertible by changing a variable, below we focus our attention only on the smoothing FIR filtering (Fig. 1a).

Suppose that a polynomial signal \(x_{1n}\), \(k = 1\), representing the first state is projected from \(n - N + 1 - p\) to \(n\) (Fig. 1a).

\(^1\)To unify the model as in [17], we let the smoother lag to be negative, \(p < 0\), in contrast to the prediction step [19] that is positive, \(p > 0\). The case of \(p = 0\) considered in [18] corresponds to filtering.
with the finite Taylor series expansion of order \( K - 1 \) (all higher order terms are supposed to be zero) as follows [20]:

\[
x_{1n} = \sum_{q=0}^{K-1} \tau^q (N - 1 + p)^q q! = \frac{x_1(n-N+1-p) + x_2(n-N+1-p)\tau(N - 1 + p)}{2} \frac{\tau^2(N - 1 + p)^2}{(K - 1)!} + \cdots + x_K(n-N+1-p) \frac{\tau^{K-1}(N - 1 + p)^{K-1}}{(K - 1)!},
\]

(1)

where \( x_{(q+1)}(n-N+1-p) \), \( q \in [0, K - 1] \), can be called the \((q+1)\)-state at \( n - N + 1 - p \) and the signal thus characterized with \( K \) states, from 1 to \( K \). Here, \( \tau \) is the sampling time.

Also suppose that a signal \( x_{kn} \) is coupled with \( x_{(k-1)n} \), starting with \( k = 2 \), by the time derivative in continuous time.

Then, most generally, we have an expansion [19]

\[
x_{kn} = \sum_{q=0}^{K-k} \tau^q (N - 1 + p)^q q! = \frac{x_k(n-N+1-p) + x_{k+1}(n-N+1-p)\tau(N - 1 + p)}{2} \frac{\tau^2(N - 1 + p)^2}{(K - k)!} + \cdots + x_{K-n+1}(n-N+1-p) \frac{\tau^{K-k}(N - 1 + p)^{K-k}}{(K - k)!},
\]

(2)

such that \( x_{1n} \) is provided by (1) and \( x_{Kn} = x_{K(n-N+1-p)} \) holds true for \( k = K \).

If we now assume that \( x_{1n} \) (1) is measured as \( s_n \) in the presence of noise \( v_n \) having zero mean, \( E\{v_n\} = 0 \), and arbitrary distribution and covariance \( Q = E\{v_nv_n^T\} \), then the signal and the measurement can be represented in state space, using (2), with the state and measurement equations as, respectively,

\[
x_n = A^{N-1+p}x_{n-N+1-p},
\]

\[
s_n = Cx_n + v_n,
\]

(3)

(4)

where the \( K \times 1 \) state vector is given by

\[
x_n = [x_{1n} x_{2n} \ldots x_{Kn}]^T.
\]

(5)

The \( K \times K \) triangular matrix \( A \) is specified as [20]

\[
A = \begin{bmatrix}
1 & \tau & \frac{\tau^2}{2} & \ldots & \frac{\tau^{K-1}}{(K-1)!} \\
0 & 1 & \tau & \ldots & \frac{\tau^{K-2}}{(K-2)!} \\
0 & 0 & 1 & \ldots & \frac{\tau^{K-3}}{(K-3)!} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix},
\]

\[
A^i = \begin{bmatrix}
1 & \tau & \frac{\tau^2}{2} & \ldots & \frac{\tau^{K-1}}{(K-1)!} \\
0 & 1 & \tau & \ldots & \frac{\tau^{K-2}}{(K-2)!} \\
0 & 0 & 1 & \ldots & \frac{\tau^{K-3}}{(K-3)!} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix},
\]

(6)

and the \( 1 \times K \) measurement matrix is

\[
C = \begin{bmatrix}
1 & 0 & \ldots & 0
\end{bmatrix}.
\]

(7)

The problem now formulates as follows. Given the state space model (3) and (4), we would like to find optimal and unbiased gains for the smoothing FIR filter (Fig. 1a), to be formally defined below, in order to produce the relevant estimate\(^2\) \( \hat{x}_{n|n-p} \), \( p < 0 \), at \( n \). It is implied that measurement is available from \( n - N + 1 - p \) to \( n - p \).

### III. Smoothing FIR Filter

In FIR filtering, an estimate can be obtained via the discrete convolution applied to measurement [20]–[23]. That may be done by matching the model with the averaging horizon of \( N \) points as shown in [18], [24]–[27]. Referring to Fig. 1a, we thus need to represent the model (3) and (4) on a horizon from \( n - N + 1 - p \) to \( n - p \). Similarly to [25], the recursively computed forward-in-time solutions give us

\[
X_N(p) = A_Nx_{n-N+1-p},
\]

\[
S_N(p) = C_Nx_{n-N+1-p} + U_N(p),
\]

(8)

(9)

where

\[
X_N(p) = [x^T_{n-p} \ldots x^T_{n-N+1-p}]^T,
\]

\[
S_N(p) = [s_{n-p} \ldots s_{n-N+1-p}]^T,
\]

\[
U_N(p) = [v_{n-p} \ldots v_{n-N+1-p}]^T.
\]

(10)

(11)

(12)

\[
A_N = \begin{bmatrix}
A^{N-1+p} \\
A^{N-2+p} \\
\vdots \\
A^p
\end{bmatrix},
\]

\[
C_N = \begin{bmatrix}
CA^{N-1+p} \\
CA^{N-2+p} \\
\vdots \\
CA^p
\end{bmatrix} = \begin{bmatrix}
(A^{N-1+p})_1 \\
(A^{N-2+p})_1 \\
\vdots \\
(A^p)_1
\end{bmatrix}.
\]

(13)

(14)

Here \( (Z)_1 \) means the first row of a matrix \( Z \). Note that the matrix \( C \) given with (7) sifts out the first row of each power of \( A \) in (14).

Given (8) and (9), the smoothing FIR filtering estimate of the first state \( x_{1n} \) can be obtained as follows. Utilize \( N \) measurements from \( n - N + 1 - p \) to \( n - p \), as in (11), use

\(^2\)Here and in the following, \( \hat{x}_{n|n} \) means the estimate of \( x_n \) at \( n \) via measurement from the past to \( m \).
the discrete convolution, and find the estimate \( \hat{x}_{1n|n-p} \) of \( x_{1n} \) at \( n \) as

\[
\hat{x}_{1n|n-p} = \sum_{i=p}^{N-1+p} h_{li}(p)s_{n-i} \quad (15a)
\]

\[
= W^T_l(p)S_N \quad (15b)
\]

\[
= W^T_l(p)[C_Nx_{n-N+1-p} + U_N(p)], \quad (15c)
\]

where \( h_{li}(p) \equiv h_{li}(N,p) \) is the FIR filter gain [20] represented with the \( l \)-degree polynomial (to be formally defined below) dependent on \( N \) and \( p \) [19], [27] and the \( l \)-degree and \( 1 \times N \) filter gain matrix is given by

\[
W^T_l(p) = [h_{l0}(p) h_{l(1+p)}(p) \ldots h_{l(N-1+p)}(p)]. \quad (16)
\]

Note that \( h_{li}(p) \) in (15a) and (16) can be specified in different senses, e.g., 1) minimum mean square error (MSE), 2) unbiased, and 3) minimum variance, depending on applications. Below, we investigate this gain in the sense of minimum MSE and minimum bias.

A. Optimal Gain

To specify \( h_{li}(p) \) optimally in the sense of minimum MSE, the cost function can be written as

\[
J = E \{ (x_{1n} - \hat{x}_{1n|n-p})^2 \}
\]

\[
= E \{ (x_{1n} - W^T_l(p)[C_Nx_{n-N+1-p} + U_N(p)])^2 \}, \quad (17)
\]

where \( E \) means an average of the succeeding expression. It has been shown in [25] that the optimal gain matrix \( W_{0l} \) that minimizes \( J \) can be computed by using the orthogonality condition [25, eq. 29]

\[
E \{ x_{1n} - W^T_{0l}(p)[C_Nx_{n-N+1-p} + U_N(p)] \} \times [C_Nx_{n-N+1-p} + U_N(p)]^T = 0, \quad (18)
\]

in which \( x_{1n} \) must be substituted with the deterministic model

\[
x_{1n} = (A^{N-1+p})_1x_{n-N+1-p}, \quad (19)
\]

taken from (3). Supposing that the initial state and the measurement noise are mutually uncorrelated and independent for all \( p \), one can find an average in (18) and arrive at

\[
W^T_{0l}(p) = (A^{N-1+p})_1R_0(p)C^T_N[Z_0(p) + \Phi_U(p)]^{-1}, \quad (20)
\]

where the initial state covariance matrix \( R_0(p) \) and an auxiliary one \( Z_0(p) \) are specified with, respectively,

\[
R_0(p) = E \{ x_{n-N+1-p}x_{n-N+1-p}^T \}, \quad (21)
\]

\[
Z_0(p) = C_NR_0(p)C^T_N, \quad (22)
\]

the measurement noise covariance function matrix is

\[
\Phi_U(p) = E \{ U_N(p)U_N^T(p) \}, \quad (23)
\]

and \( C_N \) is given by (14).

B. Unbiased Estimate

The unbiased smoothing FIR filtering estimate can be found if we start with the unbiasedness condition

\[
E \{ \hat{x}_{1n|n-p} \} = E \{ x_{1n} \}. \quad (24)
\]

Combining \( x_{1n} \), modeling (19), with the estimate \( \hat{x}_{1n|n-p} \) in (15c) leads to the unbiasedness (or deadbeat [23]) constraint

\[
(A^{N-1+p})_1 = W^T_l(p)C_N, \quad (25)
\]

where \( W_l(p) \) means the \( l \)-degree unbiased gain matrix [20]. We notice that the constraint (25) has been discussed in different forms by many authors [13], [18]–[20], [22]–[29], representing the fundamental property of unbiased filters.

It follows from an analysis of (20) that the optimal gain reduces to the unbiased gain \( W_l(p) \) specified with the constraint (25) if the initial state error dominates the measurement noise covariance or the state space model is deterministic (deadbeat property [23]). In fact, by supposing that the components of \( Z_0(p) \) dominate in orders of magnitude the components of \( \Phi_U(p) \) and accounting for (22), we go from (20) to

\[
\bar{W}^T_l(p) = (A^{N-1+p})_1R_0(p)C^T_N[R_0(p)C^T_N]^{-1}. \quad (26)
\]

Further, multiplying both sides of (26) with \( C_NR_0(p)C^T_N \), referring to the fact that neither \( (A^{N-1+p})_1 \) nor \( W^T_l(p)C_N \) has zero components and \( R_0(p)C^T_N \) has full row rank, and then removing \( R_0(p)C^T_N \) from the both sides produces (25). That means that the unbiased gain \( W_l(p) \) does not depend on the initial state matrix \( R_0(p) \). This fundamental property has been postulated in many paper, [18]–[20], [23], [24], meaning that any \( R_0(p) \) can be supposed in (26), provided that the inverse in (26) exists.

It has also been shown in [28] that the optimal and unbiased FIR estimates of polynomial models converge and become indistinguishable with large \( N \). Therefore, the unbiased FIR filters still produce accurate estimates for many applications.

IV. UNBIASED SMOOTHING FIR FILTER

Although (26) is exact for unbiased smoothing FIR filtering, the formula is redundant, since the initial state matrix \( R_0(p) \) does not affect the unbiased gain due to (25). Moreover, (26) requires large computation burden when the averaging horizon is large. Below, we shall show that, alternatively, \( \bar{W}_l(p) \) can be described via \( h_{li}(p) \) represented in a short polynomial form suitable for engineering applications.

A. Unbiased Gain

To find the unbiased gain \( \bar{W}_l(p) \), let us equate the components of the row matrices in (25) and, similarly to [18], arrive at the equations (27) (see next page). Further inserting the first identity in the remaining ones of (27) leads to the fundamental properties of the \( p \)-lag unbiased smoothing FIR filter gain:
\[ 1 = \sum_{i=p}^{N-1+p} h_{li}(p), \]
\[ \tau(N-1+p) = \sum_{i=p}^{N-1+p} h_{li}(p)\tau(N-1-i+p) + \sum_{i=p}^{N-1+p} h_{li}(p), \]
\[ \frac{\tau(N-1+p)}{(K-1)!} = \sum_{i=p}^{N-1+p} h_{li}(p)\frac{\tau(N-1-i+p)}{(K-1)!} + \ldots + \sum_{i=p}^{N-1+p} h_{li}(p)\tau(N-1-i+p) + \sum_{i=p}^{N-1+p} h_{li}(p). \]

A compact matrix form of (28) and (29) is thus
\[ \bar{W}_I^T(p) V(p) = J^T, \]
where
\[ J = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, \]
and the \( p \)-dependent and \( N \times (l+1) \) Vandermonde matrix [30] is specified by
\[ V(p) = \begin{bmatrix} 1 & p & p^2 & \cdots & p^{(l+1)} \\ 1 & 1+p & (1+p)^2 & \cdots & (1+p)^{(l+1)} \\ 1 & 2+p & (2+p)^2 & \cdots & (2+p)^{(l+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & N-1+p & (N-1+p)^2 & \cdots & (N-1+p)^{(l+1)} \end{bmatrix}. \]

The product \( V^T(p)V(p) \) with \( V(p) \) specified by (32) is a \((l+1) \times (l+1)\) matrix, whose elements increase with growing the row and column indexes. Therefore, the product \( V^T(p)V(p) \) has a nonzero determinant and its inverse thus always exists. Then multiply the right-side of (30) with the identity matrix \( [V^T(p)V(p)]^{-1} V^T(p)V(p) \). Because \( \bar{W}_I^T(p) \) is not zero-valued, remove \( V(p) \) from both sides, and finally arrive at the fundamental solution for the gain,
\[ \bar{W}_I^T(p) = J^T[V^T(p)V(p)]^{-1} V^T(p), \]
that can be used for unbiased smoothing FIR filtering of the polynomial models.

\[ h_{li}(p) = \sum_{i=p}^{l} a_{ij}(p)j^i, \]
where \( l \in [1, K], i \in [p, N-1+p], \) and \( a_{ij}(p) \neq a_{ij}(N,p) \) are still unknown coefficients. Substituting (34) to (33) and rearranging the terms lead to a set of linear equations, having a compact matrix form of
\[ J = D(p) Y(p), \]
where
\[ Y = \begin{bmatrix} a_0(0) & a_1(0) & \cdots & a_{K-1}(0) \\ a_0(1) & a_1(1) & \cdots & a_{K-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ a_0(K-1) & a_1(K-1) & \cdots & a_{K-1}(K-1) \end{bmatrix}, \]
and a low dimensional, \( l \times l \), symmetric matrix \( D(p) \) is specified via the Vandermonde matrix (32) as
\[ D(p) = V^T(p)V(p) \]
\[ = \begin{bmatrix} d_0(p) & d_1(p) & \cdots & d_l(p) \\ d_1(p) & d_2(p) & \cdots & d_{l+1}(p) \\ \vdots & \vdots & \ddots & \vdots \\ d_l(p) & d_{l+1}(p) & \cdots & d_{2l}(p) \end{bmatrix}. \]

The components in (38) may be developed as [27]
\[ d_m(p) = \sum_{i=p}^{N-1+p} i^m, \quad m = 0, 1, \ldots, 2l, \]
\[ = \frac{1}{m+1} [B_{m+1}(N+p) - B_{m+1}(p)], \]
\[ \text{Unbiasedness is also achieved with the redundant filter degree, } l > K-k, \text{ although with larger noise [26].} \]
where $B_n(x)$ is the Bernoulli polynomial [20].

An analytic solution to (35), with respect to the coefficients $a_{ji}(p)$ of a polynomial (34), gives us

$$a_{ji}(p) = (-1)^{j} \frac{M_{j+1,1}(p)}{|D(p)|},$$  \hspace{1cm} (41)

where $|D(p)|$ is the determinant of $D(p)$ is the minor of $D(p)$.

In order to determine $a_{ji}(p)$ and $h_{li}(p)$, the unbiased smoothing FIR filter of the polynomial signal $x_{1n}$ is provided by the following lemma.

**Lemma 1:** Given a discrete time-invariant polynomial state space model, (3) and (4). Then the $p$-lag unbiased smoothing FIR filtering estimate of the model $x_{1n}$ having $K$ states is provided at $n$ on a horizon of $N$ points using the data taken from $n - N + 1 - p$ to $n - p$, $p < 0$, by

$$\hat{x}_{1n|n-p} = \sum_{i=p}^{N-1+p} h_{l(K-i)}(p) s_{n-i},$$  \hspace{1cm} (42)

$$= \bar{W}_l^T(p) S_N(p),$$  \hspace{1cm} (43)

where $\bar{W}_l(p)$ is specified with (33), $h_{li}(p)$ with (34) and (41), and $S_N(p)$ is the data vector (11).

**Proof:** The derivation of the unbiased gain $\bar{W}_l^T(p)$ is provided by the collection of equations (8)–(33) and $h_{l(K-i)}(p)$ has been justified with (34)–(41). The proof is complete.

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**D. Estimate Variance**

For the zero-mean measurement noise $v_n$, having arbitrary distribution and covariance, the variance of the smoothing FIR filtering estimate can be found using the MSE (17)

$$J = E \left \{ [x_{1n} - \bar{W}_l^T(p) C_N x_{N-n+1-p} - \bar{W}_l^T(p) U_N(p)]^2 \right \}$$

$$= E \left \{ [(A_{N-1+p}) x_{N-n+1-p} - \bar{W}_l^T(p) C_N x_{N-n+1-p} - \bar{W}_l^T(p) U_N(p)]^2 \right \}.$$  \hspace{1cm} (48)

Employing the unbiasedness (25) and incorporating the commutativity $\bar{W}_l^T U_N = U_N^T \bar{W}_l$, the MSE (48) indicates the variance

$$\sigma^2(p) = E \left \{ |\bar{W}_l^T(p) U_N(p)|^2 \right \}$$

$$= E \left \{ \bar{W}_l^T(p) U_N(p) \bar{W}_l^T(p) U_N(p) \right \}$$

$$= \bar{W}_l^T(p) E \left \{ U_N(p) U_N^T(p) \right \} \bar{W}_l(p)$$

$$= \bar{W}_l^T(p) \Phi_U(p) \bar{W}_l(p).$$  \hspace{1cm} (49)

In an important special case when the measurement noise $v_n$, whose components are collected in $U_N(p)$ (12), is a white sequence, having a constant variance $\sigma^2(v_i)$. (49) becomes

$$\sigma^2(p) = \bar{W}_l^T(p) \text{diag}(\sigma^2_1 \sigma^2_2 \ldots \sigma^2_N) \bar{W}_l(p)$$

$$= \sigma^2_{\bar{W}} g_i(p),$$  \hspace{1cm} (50)

where the noise power gain (NPG) $g_i(p) \triangleq g_1(N,p)$ is specified by

$$g_i(p) = \bar{W}_l^T(p) \bar{W}_l(p)$$  \hspace{1cm} (51a)

$$g_i(p) = \sum_{i=p}^{N-1+p} h_{li}(p)$$  \hspace{1cm} (51b)

$$a_{0l}(p),$$  \hspace{1cm} (51c)

which implies that diminishing $g_i(p)$ leads to reducing $a_{0l}(p)$ in (34).

**V. LOW-DEGREE POLYNOMIAL GAINS FOR UNBIASED SMOOTHING FIR FILTERS**

Typically, smoothing of signals is provided on a horizon of some points with low-degree polynomials. Below, we derive and investigate the relevant unique gains for the uniform, linear, quadratic, and cubic signal models covering an overwhelming majority of practical needs.

**A. Uniform Model**

A signal that is constant over an averaging horizon of $N$ points is the simplest one. The relevant system is characterized with one state and the filter gain is represented, by (34), with the 0-degree polynomial as

$$h_{0i}(p) = h_{0i} = \begin{cases} \frac{1}{N}, & p \leq i \leq N - 1 + p \next \bar{W}_l(p) = \bar{W}_l$$

$$= \sum_{i=p}^{N-1+p} h_{li}(p)$$

$$= \sum_{i=p}^{N-1+p} h_{li}(N,p)$$

$$= \sum_{i=p}^{N-1+p} h_{li}(N,p)$$

$$= a_{0l}(N,p),$$  \hspace{1cm} (47)

where $a_{0l}(N,p)$ is the zero-order coefficient in (34).

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4Numerical investigations show that the determinant of $D(p)$ is $p$-invariant.
By (51c), the NPG of this filter is $p$-invariant, namely $g_0(p) = g_0 = \frac{1}{N}$. Because (52) is associated with simple averaging, it is also optimal for a common task [31]: reducing random noise while retaining a sharp step response. No other filter is better than the simple moving average in this sense. However, this gain is not good in terms of the estimate bias that reaches 50% when a signal changes linearly. Therefore, best smoothing is obtained by (52) at a center of the averaging horizon, namely when $p = -(N - 1)/2$.

B. Linear Model

For the linearly changing signal, the $p$-dependent gain, existing from $p$ to $N - 1 + p$, becomes a ramp

$$h_{1i}(p) = a_{01}(p) + a_{11}(p)i,$$  \hspace{1cm} (53)

having the coefficients

$$a_{01}(p) = \frac{2(2N - 1)(N - 1) + 12p(N - 1 + p)}{N(N^2 - 1)},$$  \hspace{1cm} (54)

$$a_{11}(p) = -\frac{6(N - 1 + 2p)}{N(N^2 - 1)}.$$  \hspace{1cm} (55)

At a center of the averaging horizon provided with $p = -\frac{N - 1}{2}$, the ramp gain transforms to the uniform one (52),

$$h_{1i} \left(N, -\frac{N - 1}{2}\right) = g_1 \left(N, -\frac{N - 1}{2}\right) = h_{0i} = g_0 = \frac{1}{N},$$  \hspace{1cm} (56)

and also becomes optimal at this point having zero bias and minimum possible noise produced by simple averaging.

Another important special application of (53) is smoothing the initial signal value with $p = -N + 1$. That is provided with

$$h_{1i}(N, -N + 1) = \frac{2N - 1 + 3i}{N(N + 1)}.$$  \hspace{1cm} (57)

Figure 2 exhibits an evolution of the ramp gain (53), by increasing $|p|$. As can be seen, changing $p$ from 0 to $-(N - 1)/2$ results in the negative slope reduction such that, with $p = -(N - 1)/2$, the ramp gain becomes uniform with zero slope (Fig. 2a). Further changing $p$ from $-(N - 1)/2$ to $-N + 1$ leads to the opposite effect. The function slope becomes positive and such that, with $p = -N + 1$, the plot shown in Fig. 2b looks like symmetrically reflected from that sketched in Fig. 2a.

Definitely, an ability of the ramp gain of becoming uniform with $p = -(N - 1)/2$ must affect the noise amount in the smoothing estimate. Investigation of noise reduction can be provided using the NPG

$$g_1(p) = a_{10}(p) = \frac{2(2N - 1)(N - 1) + 12p(N - 1 + p)}{N(N^2 - 1)}.$$  \hspace{1cm} (58)
Figure 3 illustrates (58) for different lags \( p \). Here, the case of \( p = 0 \) corresponds to filtering and a dashed line is the lower bound featured to simple averaging. Instantly one realizes that noise in the smoother has lower intensity than in the filtering estimate \( (p = 0) \). Indeed, when \( p \) ranges as \(-N + 1 < p < 0\), the NPG traces below the bound sketched by \( p = 0 \). The NPG rises dramatically, when \( p < -N + 1 \). One is surprised by this fact, because smoothing with lags exceeding an averaging horizon is nothing more than the backward prediction inherently producing noise larger than in filtering [19].

C. Quadratic Model

For a signal changing quadratically on an averaging horizon, the polynomial gain (34) can be written as

\[
h_{2i}(p) = a_{02}(p) + a_{12}(p)i + a_{22}(p)i^2,
\]

with the coefficients defined by (60)–(62) (see next page).

An evolution of \( h_{2i}(p) \), by increasing \(|p|\), is shown in Fig. 4. As well as the ramp gain, the quadratic one has several special points. Namely, by the lags

\[
p_{21} = -\frac{N - 1}{2} + \sqrt{\frac{N^2 - 1}{12}},
\]

and, at the initial signal point \( p = -N + 1 \), we have

\[
h_{2i}(N, -N + 1) = 3 \frac{3N^2 - 3N + 2 + 6(2N - 1)i + 10i^2}{N(N + 1)(N + 2)}.
\]

The NPG associated with the quadratic gain (59) is specified with

\[
g_2(p) = a_{02}(p),
\]

where \( a_{02}(p) \) is given by (60). For different lags, a set of functions (67) is sketched in Fig. 5.

Unlike the ramp gain (53) having the NPG lower bound \( 1/N \), the relevant bound for the quadratic gain (59) traces upper (Fig. 5) as

\[
g_{2\text{min}} = \frac{3(3N^2 - 2)}{5N(N^2 - 1)}.
\]

This value can be found by putting to zero the derivative of \( g_2(N, p) \) with respect to \( p \) and then finding the roots of the polynomial. Two lags correspond to (68), namely

\[
p_{23} = -\frac{N - 1}{2} + \frac{1}{2} \sqrt{\frac{N^2 + 1}{5}},
\]

and

\[
p_{24} = -\frac{N - 1}{2} - \frac{1}{2} \sqrt{\frac{N^2 + 1}{5}}.
\]

Like the ramp gain case, here noise in the smoothing estimate is lower than in the filtering one, if \( p \) does not exceed an averaging horizon. Otherwise, we watch in Fig. 5 for the dramatic increase in the error.
values of \( p \) of this gain, by changing \( p \) of converting to the quadratic gain. Fig. 6 shows an evolution demonstrating several important features, including an ability with the coefficients given by (72)–(75).

The cubic model

The \( p \)-dependent cubic gain can now be derived in a similar manner to have a polynomial form of

\[
h_{3i}(p) = a_{03}(p) + a_{13}(p)i + a_{23}(p)i^2 + a_{33}(p)i^3,
\]

with the coefficients given by (72)–(75).

As well as the ramp and quadratic gains, the cubic one demonstrates several important features, including an ability of converting to the quadratic gain. Fig. 6 shows an evolution of this gain, by changing \( p \) from zero to \(-N+1\). Special values of \( p \) depicted in this figure are given below:

\[
p_{31} = -\frac{N-1}{2} + \frac{1}{10} \sqrt{5(3N^2-2)},
\]

\[
p_{32} = -\frac{N-1}{2} + \frac{\sqrt{105}}{210} \times \sqrt{33N^2 - 17 + 2 \sqrt{36N^4 + 507N^2 - 2579}},
\]

\[
p_{33} = -\frac{N-1}{2} + \frac{\sqrt{105}}{210} \times \sqrt{33N^2 - 17 - 2 \sqrt{36N^4 + 507N^2 - 2579}},
\]

The lags \( p_{31}, p = -\frac{N-1}{2} \) and \( p_{36} \) convert the cubic gain to the quadratic one. These lags are therefore preferable from the standpoint of estimation accuracy, because the quadratic gain produces lower noise. The lags \( p_{32}, p = -\frac{N-1}{2}, \) and \( p_{35} \) correspond to minima on the smoother NPG characteristic. The remaining lags, \( p_{33} \) and \( p_{34} \), cause two maxima in the range of \(-N+1 < p < 0\).

The NPG corresponding to the cubic gain (71) is given by

\[
g_3(p) = a_{03}(p),
\]

where \( a_{03}(p) \) is specified with (72). Function (82) is sketched in Fig. 7 for small and large values of \( p \). As can be seen, (82) ranges above the lower bound

\[
g_{3\text{min}} = \frac{3(3N^2-7)}{4N(2N^2-4)}
\]

and, by \( p = \text{const} \), it asymptotically approaches \( g_3(N, 0) \), with increasing \( N \). As well as in the quadratic gain case, noise in the cubic smoother can be much lower than in the relevant filter \( (p = 0) \). On the other hand, the range of uncertainties is broadened here to \( N = 3 \) and the smoother becomes thus low inefficient on short horizons. The latter is neatly seen in Fig. 7. In fact, to the left of the minimum placed on the lower bound (83), the NPG increases rapidly. When it exceeds unity, the smoothing filter loses an ability of denoising and its use becomes hence meaningless.

E. Generalizations

Several important common properties of the unbiased smoothing FIR filters can now be outlined.

Effect of the lag \( p \) on the NPG of these filters is reflected in Fig. 8. One infers that the NPG of the ramp gain is exactly that of the uniform gain, when \( p = -(N - 1)/2 \). By \( p = p_{21} \) and \( p = p_{22}, \) where \( p_{21} \) and \( p_{22} \) are specified by (63) and (64),
respectively, the NPG of the quadratic gain becomes equal to that of the ramp gain. Also, by $p = p_{31}$ (76), $p = -\frac{N-1}{2}$, and $p = p_{36}$ (81), the NPG of the cubic gain is reduced to that of the quadratic gain. Similar deductions can be made for higher degree gains.

It can also be noticed that all of the functions shown in Fig. 8 are symmetric about $p = -(N-1)/2$. Therefore, errors in FIR smoothers with $p < -N + 1$ and in FIR predictors with $p > 0$ grow equally.

The following generalizations can also be provided for a two-parameter family of the $l$-degree and $p$-lag, $p < 0$, unbiased smoothing FIR filters specialized with the gain $h_{l1}(N, p)$ and NPG $g_l(N, p)$:

- Any smoothing FIR filter with the lag $-(N-1) < p < 0$ produces smaller random errors than the relevant FIR filter with $p = 0$.
- Without loss in accuracy, the $l$-degree filter can be substituted, for some special values of $p$, with a reduced $(l-1)$-degree one. Namely, the $l$-degree gain can be substituted with the $0$-degree gain for $p = -(N-1)/2$, the $2$-degree gain with the $1$-degree gain for $p = p_{21}$ and $p = p_{22}$, and the $3$-degree gain with the $2$-degree gain, if $p = p_{31}$, $p = -\frac{N-1}{2}$, or $p = p_{36}$.
- Beyond the averaging horizon, the error in the smoothing FIR filter with $p < -N + 1$ is equal to that in the predictive FIR filter [19] with $p > 0$.
- The NPG lower bounds for such filters with the ramp gain, $g_{l1\min}$, quadratic gain $g_{l2\min}$, and cubic gain, $g_{l3\min}$, are given with, respectively,

$$g_{l1\min} = \frac{1}{N},$$

$$g_{l2\min} = \frac{3(3N^2 - 2)}{5N(N^2 - 1)} \bigg|_{N\geq 1} \approx \frac{9}{5N},$$

$$g_{l3\min} = \frac{3(3N^2 - 7)}{4N(N^2 - 4)} \bigg|_{N\geq 1} \approx \frac{9}{4N}.$$  

- With large $N$, noise in the $l$-degree filter is defined for $p = -N + 1$ by the NPG

$$g_l(N, -N + 1) |_{N\gg 1} \approx \frac{(l+1)^2}{N}.$$

The initial conditions can hence be ascertained using the ramp and quadratic gains with the NPGs $\approx 4/N$ and $\approx 9/N$, respectively.

- By increasing $N$ for a constant $p$ such that $|p| \ll N$, the noise variance in the unbiased smoothing FIR filter asymptotically approaches that in the relevant FIR filter with $p = 0$.

**VI. EXAMPLE: THE BEST LINEAR FIT FOR A TWO-STATE CLOCK MODEL**

To demonstrate efficiency of the proposed solution, we find the best fit for the time interval error (TIE) of a crystal clock measured each second during 357332 s using the Stanford Frequency Counter SR620 for the reference cesium clock (Symmetricom CsSI). The TIE function is shown in Fig. 9 as “$x_n + noise$”. On the measured time interval of $N$ points, the clock was identified to have two states. For the two-state model, the ramp FIR smoother can be organized by changing a variable in the ramp gain (53). Accordingly, we obtain the smoothing estimate at $n + p$, $p < 0$, with

$$\tilde{x}_{n+p} = \sum_{i=0}^{N-1} \tilde{h}_{l11}(N, p) s_{n-i},$$

where

$$\tilde{h}_{l11}(N, p) = \frac{2(2N-1) - 6i + 6\rho(N-1-2i)}{N(N+1)} + \frac{6\rho(N-1-2i)}{N(N^2-1)}.$$  

To find the best fit, all the data must be involved. We thus substitute $N$ with $n + 1$ and modify (88) to

$$\tilde{x}_{n+p} = \sum_{i=0}^{n} \tilde{h}_{l11}(n+1, p) s_{n-i},$$

where

$$\tilde{h}_{l11}(n+1, p) = \frac{2n(2n+1) - 6in + 6\rho(n-2i)}{n(n+1)(n+2)}.$$
Fig. 6. Evolution of the cubic gain by increasing $|p|$: (a) $p_{31} < p < 0$, (b) $p_{33} < p < p_{31}$, (c) $-\frac{N-1}{2} < p < p_{33}$, (d) $p_{34} < p < -\frac{N-1}{2}$, (e) $p_{36} < p < p_{34}$, and (f) $N - 1 < p < p_{36}$.

Fig. 7. NPG of the unbiased smoothing FIR filter with a cubic gain for a set of negative lags $p$. The case of $p = 0$ corresponds to filtering.

Fig. 8. Effect of $p$ on the NPG of the low-degree unbiased smoothing FIR filters.

Fig. 9. FIR filtering of the crystal clock first state.
The best linear fit can be found if to smooth the data at the initial point with $p = -n$ as $\hat{x}_0$ and filter at the current point with $p = 0$ as $\hat{x}_n$. The relevant straight line $\hat{x}$, passing through these two points is provided with

$$
\hat{x}_{n+p} = \frac{n}{n+p} \hat{x}_0 + \frac{n+p}{n} \hat{x}_n
$$

(92a)

$$
= \sum_{i=0}^{n} \frac{2(2n+1) - 6i + 6p(n-2i)}{n(n+1)(n+2)} s_{n-i}
$$

(92b)

$$
= \sum_{i=0}^{n} \hat{h}_{1i}(n,p) s_{n-i},
$$

(92c)

if we fix $n$ and change the lag $p$ from $-n$ to 0, as a variable. As can be seen, the gain in (92c) is exactly that (91) of the $p$-dependent ramp unbiased FIR filter; that is $\hat{h}_{1i}(n,p) = \hat{h}_{1i}(n+1,p)$. Note that the best fit holds true only for the observed database. It would be corrected for every new measurement point added to the data [32].

VII. CONCLUSION

In this paper, we proposed a new smoothing FIR filter for discrete-time polynomial state space models. We found both the optimal and unbiased solutions. The gain for the optimal smoothing filter was found in the matrix form, requiring the initial state and the covariance function of the measurement noise. The gain for the unbiased smoothing filter had been developed in the unique polynomial form (Lemma 1) that does not involve any knowledge about noise and initial state, thus having strong engineering features. Most widely used the low-degree gains were represented in simple engineering forms and investigated in detail. The results are supported with an application to the nonstationary time error of a crystal clock. Another important application for hybrid median FIR filtering of images is currently under investigation.

REFERENCES


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