Concavity in a vintage capital model with nonlinear utility

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Abstract

An optimization problem in an economic vintage capital model with nonlinear utility is investigated. It is described by non-linear Volterra integral equations with an unknown in the limits of integration. The concavity of the problem is proven, the condition for an extremum is established, and their relevance to applications is demonstrated.

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1. Introduction

Vintage capital models (VCMs) play an important role in simulation of economic–technological development. They are described by non-linear Volterra integral equations. The first and most famous VCM with an endogenous capital lifetime was developed for macroeconomic growth by Solow et al. in [1]. Since 1966, numerous VCMs have been offered in [2–8] to explain existing phenomena related to technological change and creative destruction of capital. However, in contrast to other economic–mathematical models, the optimal dynamics in VCMs is largely unexplored. Certain progress has been reached in the optimization with linear utility [2,4–7] where the short-term (transition) and long-term (balanced growth) optimal dynamics in the Solow VCM and its modifications have been analyzed. The nonlinear utility represents varying consumer preferences. Although great applied interest exists for VCMs with nonlinear utility, their optimal dynamics remains completely unknown. A few existing
results [2] rely on limited numeric simulation. This work investigates a nonlinear-utility optimization problem (OP) in the Solow VCM and represents an essential breakthrough into the VCM theory.

Optimization versions [2,4–7] of the original Solow VCM [1] lead to finding functions \( m(t), a(t), \) and \( y(t), t \in [t_0, T), T \leq \infty, \) which maximize the objective functional:

\[
I = \int_{t_0}^{T} \rho(t)[u(y(t)) - \lambda(t)m(t)]dt \rightarrow \max, \tag{1}
\]

under the state equations:

\[
y(t) = \int_{a(t)}^{t} \beta(\tau, t)m(\tau)d\tau, \tag{2}
\]
\[
P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \tag{3}
\]

the constraint inequalities:

\[
0 \leq m(t) \leq M(t), \quad a'(t) \geq 0, \quad a(t) \leq t, \tag{4}
\]

and the initial conditions:

\[
a(t_0) = a_0 < t_0, \quad m(\tau) = m_0(\tau), \quad \tau \in [a_0, t_0]. \tag{5}
\]

The OP unknown controls are the product output \( y(t), \) capital investment \( m(t) \) and scrapping time \( a(t) \) for obsolete capital, \( t \in [t_0, T). \) The specific capital cost \( \lambda(t) \) and productivity \( \beta(\tau, t), \partial \beta / \partial \tau > 0, \) total labor \( P(t) \) and discount factor, \( 0 < \rho(t) \leq 1, \rho' \leq 0, \) are given. Following [2], the functional \( I \) reflects so-called “social welfare” over the planning horizon \([t_0, T). \) The first term of \( I \) describes the usefulness of output \( y(t) \) expressed via the nonlinear concave utility function \( u(y). \) The concavity of \( u(y) \) reflects the natural economic assumption that an additional unit of \( y(t) \) is more valuable for consumers when the whole output \( y(t) \) is small. The second term \( \lambda m \) in \( I \) describes the current expenses on capital replacement.

Problems of type (1)–(5) have been investigated using discrete analogues or corresponding delay differential systems [2,3,8]. A technique based on integral equations has been developed in [5,7] where the gradient of the OP (1)–(5) has been derived. In the case of linear utility \( u(y), \) the gradient depends only on \( a. \) It leads to new qualitative results such as turnpike properties of the solution \( a \) and exact OP solutions in cases of linear and exponential \( \beta(\tau, t) \) [5–7]. The structure of the exact solutions exposes important features of capital replacement dynamics. However, the case of nonlinear utility \( u(y) \) is extremely interesting for economic and financial applications [2,3]. In this case, the dependence of the gradient on both \( a \) and \( y \) makes the OP investigation essentially more complicated. The present work proves the concavity of the OP (1)–(5) with nonlinear utility \( u(y) \) in Section 2. The concavity leads to the necessary and sufficient condition for an extremum. Discussion of the results obtained is provided in the last section.

2. Main results

The investigation methods for similar OPs [4,6,7,9] are based on variation techniques of optimization theory (e.g., see [10,11]) and express extremum conditions in terms of the OP gradient. We assume that the given functions \( \beta, \lambda, P, \rho, \) and \( M \) are Lipschitz continuous, \( m_0 \) is piecewise continuous, all the functions are positive and satisfy (2)–(5) at \( t = t_0. \)
Let \( m(t), t \in [t_0, T) \), be the independent control variable of the OP \((1)-(5)\). Then the functions \( y(t) \) and \( a(t), t \in [t_0, T) \), are dependent (phase) variables. Following [5,9], we replace the differential constraint \( a'(t) \) with the stricter constraint for the control \( m \) only:

\[
m_{\min}(t) \leq m(t) \leq M(t), \quad \text{where} \quad m_{\min}(t) = \max\{0, P'(t)\}.
\] (6)

**Lemma 1.** For any measurable control \( m \) that satisfies (6) almost everywhere (a.e.) on \([t_0, T)\), a unique a.e. continuous function \( a(t) < t, t \in [t_0, T) \), exists, satisfies (3), (4) and a.e. has \( a'(t) \geq 0 \).

**Proof.** Let us transform Eq. (3) to the following form:

\[
P(t) = \int_{a(t)}^{t_0} m_0(\tau)d\tau + \int_{t_0}^{t} m(\tau)d\tau,
\]

and introduce the notation:

\[
F(a(t)) = \int_{a(t)}^{t_0} m_0(\tau)d\tau; \quad X(t) = P(t) - \int_{t_0}^{t} m(\tau)d\tau.
\]

First, we consider the interval \([t_0, \min(T, t_1))\), where the time \( t_1, t_0 < t_1 \leq T \), is determined from the condition \( X(t_1) = 0 \) (if such a \( t_1 \) exists; otherwise, \([t_0, T)\) is considered). Since \( F'(a(t)) = -m_0(a(t)) \leq 0 \), the function \( F(a) \) does not increase. Hence, the unique a.e. continuous function \( a(t) = F^{-1}(X(t)) \leq t_0 < t \) exists for \( t \in [t_0, \min(T, t_1)] \) and \( a(t_1) = t_0 \). Next, \( X(t) \) does not increase because \( X'(t) = P'(t) - m(t) \leq 0 \) under condition (6). Therefore, \( a(t) \) does not decrease as the composition of two non-increasing functions, so there exists \( a'(t) \) a.e. such that \( a'(t) \geq 0 \).

Now one can prove the same result for the interval \([t_1, \min(T, t_2)]\) such that \( X(t_2) = 0 \), and so on, until the whole interval \([t_0, T)\) is covered. \( \square \)

**Remark 1.** The conversion of Eq. (3) to a nonlinear ODE form is another possible investigation technique but it will raise the smoothness requirements imposed on the given model functions and unknown variables. Such a technique was earlier applied to some VCMs in [2]. The investigation technique [5,7] based on properties of integral equations is more general and leads to deeper mathematical results.

Here and hereafter we consider the OP \((1)-(3), (5)\) and (6). Let us call variations \( \delta m(t), \delta y(t), \delta a(t), t \in [t_0, T) \), admissible if functions \( m(t), y(t), a(t) \) and \( m(t) + \delta m(t), y(t) + \delta y(t), a(t) + \delta a(t) \) do not violate restrictions (2), (3) and (6) at \( t \in [t_0, T) \).

**Theorem 1 (The OP Gradient).** If \( m(t) > 0, t \in [t_0, T) \), then functional (1) is differentiable and for any admissible variations \( \delta m(t), t \in (t_0, T) \), the increment \( \delta I \) of the functional \( I \) is of the form

\[
\delta I = I(m + \delta m) - I(m) = \int_{t_0}^{T} I'(t)\delta m(t)dt + \delta^2 I, \quad \delta^2 I = o(\|\delta m\|).
\] (7)

Here \( I'(t) \) is the gradient of the functional \( I \) in \( m \):

\[
I'(t) = \int_{a(t)}^{t_0} u'(\tau)\rho(\tau)[\beta(t, \tau) - \beta(a(t), \tau)]d\tau - \lambda(t)\rho(t), \quad t \in [t_0, T),
\]

\[
\tilde{a}^{-1}(t) = \begin{cases} a^{-1}(t), & t \in [t_0, a(T)], \\ T, & t \in [a(T), T], \end{cases}
\] (8)
\( a^{-1}(t) \) is the inverse function of \( a(t) \), and the second-order residual is

\[
\delta^2 I = \int_{t_0}^{T} \rho(t) \left\{ u'(y(t)) \int_{a(t)}^{a(t) + \delta a(t)} \left[ \beta(a(t), t) - \beta(\tau, t) \right] m(\tau) d\tau \\
+ \frac{u''(y(t))}{2} [\delta y(t)]^2 \right\} d\tau + o(\|\delta m\|^2) \tag{10}
\]

**Sketch of Proof.** Let us give a small admissible variation \( \delta m(t) \), \( t \in [t_0, T] \), to \( m(t) \), and determine the corresponding variations \( \delta y(t), \delta a(t), t \in [t_0, T] \), \( \delta I \) of the phase variables \( y(t) \) and \( a(t) \) and functional \( I(m) \). Differentiating (2) and using (3), we obtain the following expressions:

\[
\int_{a(t)}^{t} \delta m_{\text{int}}(\tau) d\tau = \int_{a(t)}^{a(t) + \delta a(t)} [m(\tau) + \delta m(\tau)] d\tau, \tag{11}
\]

\[
\delta y(t) = \int_{a(t)}^{t} [\beta(\tau, t) - \beta(a(t), t)] \delta m_{\text{int}}(\tau) d\tau \\
- \int_{a(t)}^{a(t) + \delta a(t)} [\beta(\tau, t) - \beta(a(t), t)] m(\tau) d\tau, \tag{12}
\]

\[
\delta m_{\text{int}}(\tau) = \begin{cases} \delta m(\tau), & \tau \in (t_0, T), \\ 0, & \tau \in [a(t_0), t_0]. \end{cases} \tag{13}
\]

If \( a(t) < t_0 \), then the integration ranges in (11) and (12) involve the prehistory interval where \( m(\tau) = m_0(\tau) \) is given. The introduction of the function \( \delta m_{\text{int}} \) by (13) avoids the variation of the fixed \( m_0 \) (i.e., \( \delta m(\tau) \equiv 0, \tau \leq t_0 \)).

Similarly to the proof of Lemma 1 one can show that Eq. (11) has a unique a.e. continuous function \( \delta a(t), t \in [t_0, T] \), for any given measurable \( m \) and \( \delta m \) and a.e. continuous \( a \). The corresponding \( \delta y \) is explicitly determined from (12). Next, by virtue of (11) and (12), the variations \( |\delta y(t)| \) and \( |\delta a(t)| \) are small and have the order \( \|\delta m\| \) for all \( t \in [t_0, T] \), where \( \| \ldots \| \) is the \( L_\infty \)-norm. Substituting \( m + \delta m, y + \delta y, a + \delta a \) into (7) and using the Taylor expansion for \( u(y) \) up to the third order with respect to \( \delta y \), we obtain that

\[
\delta I = I(m + \delta m) - I(m) = \int_{t_0}^{T} \rho(t) \left\{ u'(y(t)) \delta y(t) + \frac{u''(y(t))}{2} [\delta y(t)]^2 \\
- \lambda(t) \delta m(t) \right\} d\tau + o(\|\delta m\|^2). \tag{14}
\]

Now, we can substitute (12) into (14), exchange the order of integration taking into account that \( \delta m(\tau) \equiv 0, \tau \leq t_0 \), and finally obtain the formulas (7)–(10) for the increment \( \delta I \) of the functional \( I \).

**Remark 2.** In the case of linear \( u(y) \), the result of Theorem 1 has been earlier proven in [5,7]. Then the gradient \( I'(t) \) depends only on the unknown variable \( a \) and does not depend on \( m \). This fact has been intensively exploited in [4,7].

**Theorem 2 (The OP Concavity).** If \( u'(y) > 0, u''(y) < 0, \) and \( \beta(\tau, t)/\partial\tau > 0 \), then \( \delta^2 I < 0 \) for any admissible variations \( \delta m(t), t \in (t_0, T) \), \( \delta m \neq 0 \), i.e., the functional \( I(m) \) is strictly concave downward.
Proof. The proof is based on the special structure of (10). Using the mean value theorem, (10) can be written as

$$\delta^2 I = \int_0^T \rho(t) \left\{ u'(y(t))[\beta(a(t), t) \beta(a(t) + \chi(t), t)] \int_{a(t)}^{a(t) + \delta a(t)} m(\tau) d\tau + \frac{u''(y(t))}{2} [\delta y(t)]^2 \right\} dt,$$

(15)

where $0 < \chi(t) < \delta a(t)$ if $\delta a(t) > 0$ and $\delta a(t) < \chi(t) < 0$ if $\delta a(t) < 0$.

Let us consider an arbitrary variation $\delta m(\tau), \tau \in [t_0, T]$. Since $\delta m \neq 0$, the corresponding $\delta a \neq 0$ and $\delta y \neq 0$. Let us split the interval $[t_0, T]$ into the subintervals $\Delta_i$ where $\delta a(t)$ does not change its sign. Now, let us assume that $\delta a(t) > 0$, $t \in \Delta_1$, where $\Delta_1 \subset [t_0, T]$ is an arbitrary subinterval. Then, $\int_{a(t)}^{a(t) + \delta a(t)} m(\tau) d\tau \geq 0$ at $t \in \Delta_1$. Also, $\beta(a(t), t) - \beta(a(t) + \chi(t), t) < 0$ at $t \in \Delta_1$ because $\partial \beta(\tau, t)/\partial \tau > 0$ and $\chi(t) > 0$. Hence, the coefficient of $u'(y)$ in (15) is non-positive at $t \in \Delta_1$. The case $\delta a(t) < 0$, $t \in \Delta_1$ leads to the same result. Next, the coefficient $[\delta y(t)]^2$ of $u''(y)$ is always non-negative and $u''(y) < 0$. Thus, the whole integrand of (15) is non-positive at $t \in [t_0, T]$ and, hence, $\delta^2 I \leq 0$. Moreover, from (12), $[\delta y(t)]^2$ is positive at least on the subset $\Delta^* \subset [t_0, T]$ where $\delta y(t) \neq 0$. Hence, $\delta^2 I < 0$. Therefore, the functional $I(m)$ is strictly concave downward. □

Corollary 1 (The Necessary and Sufficient Condition for an Extremum). In order for a function $m^*(t), t \in [t_0, T]$, to be a solution of the OP (1)–(6), it is necessary and sufficient that

$$I'(t) \leq 0 \quad \text{at} \quad m^*(t) = m_{\text{min}}(t),$$

$$I'(t) \geq 0 \quad \text{at} \quad m^*(t) = M(t),$$

$$I'(t) \equiv 0 \quad \text{at} \quad m_{\text{min}}(t) < m^*(t) < M(t), \quad t \in [t_0, T).$$

(16)

Proof. The proof of necessity is standard for such OPs and follows from the general necessary extremum condition (e.g., [10,11]) of the form $\delta I = I(m^* + \delta m) - I(m^*) \leq 0$ for any admissible variation $\delta m(\tau), \tau \in [t_0, T]$. Condition (16) is sufficient because of the OP concavity (Theorem 2). □

Remark 3. If the condition $m(t) > 0$ is not valid at some parts of $[t_0, T)$, then in view of (2) the variation $\delta a(t)$ can be finite for an infinitesimal $\delta m(\tau), \tau < t$. In this case, the functional $I(m)$ is not differentiable, and (8) does not represent the gradient of functional (1). However, condition (16) is still valid in this case because of the concavity of the functional $I(m)$ [10,11]. The case $m = 0$ is natural in economics [2–4].

3. Conclusion

The conditions $u'(y) > 0, u''(y) < 0$ of the OP (1)–(6) concavity (Theorem 2) represent the commonly accepted definition of the nonlinear utility in mathematical economics, namely, more product output is better ($u' > 0$) but the benefit of an additional output decreases when the total output is higher ($u'' < 0$). The condition $\beta'_c > 0$ means the presence of the embodied technological change (newer capital is more efficient).

The concavity of the OP has important theoretical implications. First of all, it produces the necessary and sufficient condition for an extremum (16). The OP concavity also means that the OP solution $(m^*, y^*, a^*)$ is unique (if it exists) and therefore delivers the global optimum to the OP.
The concavity provides a theoretical basis for the further qualitative analysis of the OP. In the linear-utility OP, the asymptotic analysis shows the presence of turnpike properties of the optimal lifetime of capital. This means that there is a certain trajectory $\tilde{a}$, turnpike, that attracts the OP solution $a^\ast$. Turnpike theorems indicate basic tendencies in economic dynamics and are good evidence of the quality of economic models. In the nonlinear-utility case, the turnpike trajectory $\tilde{a}$ will depend on the output $y$, which is a new effect in VCM theory.

Qualitative analysis of the Solow VCM with nonlinear utility will provide new insight into general economic mechanisms of capital equipment renovation under varying consumer preferences. Such an analysis is required by economic and financial applications of VCMs and is of interest for rational equipment replacement strategies.

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