Anticipation echoes in vintage capital models

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Abstract

The paper analyzes the structure of optimal trajectories in the one-sector vintage capital model that optimizes the endogenous lifetime of age-structured capital under technological change. A new repetition pattern (anticipation echoes) is demonstrated in both optimal investment and optimal capital lifetime in the case of finite-horizon optimization. The anticipation echoes are caused by expectation of the future “no-investment” policy at the end of the planning horizon. It is shown that the anticipation echoes represent a general structural property of dynamic systems with endogenous delay. Mathematically, the problem under study is the optimal control of non-linear integral equations with unknowns in the integration limits.

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1. Introduction

Vintage capital models (VCMs) describe the replacement of productive capital in the presence of technological change (TC) [2,4,37,38]. They consider the age structure of capital and are described by the integral equations of a special type. Such models are a promising direction in economics and Operations Research [3,5,8–13,16–27]. The economic theory of VCMs has been developed during the last fifty years. R.Solow first proposed macroeconomic VCMs in [37] and introduced the endogenous lifetime of capital into these models in [38]. A VCM with the endogenous lifetime of capital was introduced in [38]. A VCM for a separate firm was suggested and analyzed in [30]. Two-sector macroeconomic VCMs were investigated in [11,26]. Studies on VCMs were intensified in the nineties [4,5,7–10,16–18,20]. Van Hilten [16] investigated the optimal capital lifetime in a finite-horizon VCM and emphasized the zero-investment policy at the end of the planning horizon. Boucekkine et al. [4] developed an optimization version of the Solow VCM [38] and revealed replacement echoes in the infinite-horizon case. Yatsenko and Hritonenko [17–19] discovered turnpike properties of the optimal capital lifetime in one- and two-sector VCMs. The VCMs have also been applied to problems such as conversion of defense industry [31], modernization of agricultural manufacturing systems [12], technological renovation in hierarchical economic–ecological systems [18]. In Operations Research, VCMs with endogenous equipment lifetime and their discrete-time analogues have been explored in [1,3,6,14,15,28,29,32–36].

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Mathematically, the VCMs lead to the optimal control of non-linear integral equations with unknowns in the integration limits. Such models do not have a systematic mathematical theory. This paper provides a rigorous optimization analysis of the structure of optimal trajectories in a one-sector VCM for both finite and infinite horizons. The considered optimization problem is essential for the optimal asset replacement of a separate firm. The structural analysis of the optimal trajectories reveals the long-run and transition dynamics of optimal capital replacement under the embodied TC. In particular, new types of echo effects are revealed (the anticipation echoes) in the finite-horizon case. Such echoes appear in both optimal investment and capital lifetime because of changes in the optimal dynamics at the end of the planning horizon.

The paper is organized as follows. Section 2 describes the one-sector VCM with endogenous capital lifetime and formulates the optimization problem (OP) under study. Section 3 explains the investigation technique and presents the necessary theoretic background. In Section 4, the structure of OP solutions is investigated in the case of the infinite-horizon optimization. Section 5 analyzes the optimal trajectories in the finite-horizon case. Section 6 discusses the obtained results. The proofs are given in Appendix.

2. Optimization in one-sector vintage capital model

The first and most known VCM with endogenous capital lifetime was developed for macroeconomic growth in [38]. It can be reduced [18,20,22] to the following one-sector VCM:

\[
\begin{align*}
    y(t) &= \int_{a(t)}^{t} \beta(\tau, t) m(\tau) d\tau, \\
    P(t) &= \int_{a(t)}^{t} m(\tau) d\tau,
\end{align*}
\]

where \(y(t)\) is the total output of an aggregate product at time \(t\), \(P(t)\) is the total labour, \(\beta(\tau, t)\) is the specific productivity of the capital put into service at time \(\tau\), \(m(\tau)\) is the investment into new capital, and \(a(t)\) is the scrapping time of obsolete capital. Then \(L(t) = t - a(t)\) is the lifetime of capital replaced at time \(t\).

We suggest that the TC is embodied in new capital (new vintages of machines), that is, newer capital is more efficient. It means that \(\beta(\tau, t)\) increases in \(\tau\). Possible dependence of \(\beta(\tau, t)\) on \(t-\tau\) reflects the capital deterioration and its dependence on \(t\) reflects the autonomous TC and price fluctuations. The VCMs provide a convenient tool to consider the capital lifetime as an endogenous, that is, unknown variable. To determine this endogenous variable, we formulate an OP for the capital replacement policy.

In the Solow macroeconomic VCM [38], the output \(y(t)\) is distributed between the total consumption \(c(t)\) and the new investment \(i(t)\) as \(y(t) = c(t) + i(t)\). The investment \(i\) is defined in output units as \(i(t) = \lambda(t)m(t)\), where \(\lambda(t)\) is the specific cost of new capital (per one worker). Boucekkine et al. has developed in [4] an optimal growth version of the Solow VCM, which maximizes the discounted consumption flow

\[
I = \int_{t_0}^{T} \rho(t)[y(t) - \lambda(t)m(t)]dt
\]

on the time horizon \([t_0, T]\) under linear consumer preferences, given labour resource \(P\), and the equality constraint (1). In (2), \(\rho(t)\) is the discount factor, \(0 < \rho(t) < 1, \rho'(t) < 0\).

The OP (1) and (2) also plays a significant role as the replacement problem of a firm under improving technology [14,30,36]. It maximizes the discounted profit of the firm and describes the optimal replacement policy for the firm’s fleet of productive equipment (capital) under the given total amount \(P\) of capital. Starting from [14,30,36], many similar models have been constructed in discrete and continuous time [1,3,6,15,16,18,24,28,29,32–35]. Their study often uses sophisticated mathematical and numeric techniques but qualitative properties of the optimal replacement are still not clear [24].

In the OP (1) and (2), the functions \(m(t)\) and \(a(t)\) are the unknown decision variables and the functions \(\beta(\tau, t), P(t), \lambda(t)\), and \(\rho(t)\) are given on \([t_0, T]\). The unknown functions should satisfy certain restrictions. First of all, \(m(t) \geq 0\) and \(m(t) \leq M(t)\), where the maximal possible investment \(M(t)\) is determined by financial budgeting constraints of a firm. In macroeconomics, the natural boundary for \(M(t)\) is the total output \(y(t) / \lambda(t)\). In the case of a separate firm, we consider \(M\) to be a given function. We also assume that the scrapped capacity cannot be used again,
i.e. \( a'(t) \geq 0 \). Finally, as model (1) and (2) is defined on \([t_0, T]\), a specific vintage structure of the capital is known and defined by the investment \( m_0(\tau) \) already made on the pre-history interval \([a(0), t_0]\).

Summarizing the above, the OP consists of finding the functions \( m(t) \) and \( a(t) \), \( t \in [t_0, T] \), \( T \leq \infty \), which maximize the objective functional:

\[
I = \int_{t_0}^{T} \rho(t) \left[ \int_{a(t)}^{t} \beta(\tau, t)m(\tau)d\tau - \lambda(t)m(t) \right] dt \to \max_{a,m},
\]

under the constraint equality:

\[
P(t) = \int_{a(t)}^{t} m(\tau)d\tau,
\]

the constraint inequalities:

\[
0 \leq m(t) \leq M(t),
\]

\[
a'(t) \geq 0, \quad a(t) < t, \quad t \in [t_0, T),
\]

and the initial conditions:

\[
a(t_0) = a_0 < t_0, \quad m(\tau) = m_0(\tau), \quad \tau \in [a_0, t_0].
\]

The presence of unknown functions in the integration limits creates serious difficulties in the OP study. An investigation technique for such OPs was suggested in [17–25,39,40].

3. Investigation technique

This section presents some known theoretical results about the OP (3)–(7), which are necessary for further analysis. Let the given functions \( \beta, \lambda, \rho, M, \) and \( P \) be continuously differentiable, \( m_0 \) be piecewise continuous, and all these functions be positive on \([t_0, T]\).

We choose \( m(t), \ t \in [t_0, T), \) as the independent control variable of the OP, then \( a(t), \ t \in [t_0, T), \) is the dependent phase unknown variable. As in \([17,19,22]\), \( m \) is assumed to be Lebesgue measurable on \([t_0, T)\) and \( U \) is the set of measurable controls \( m \) that satisfy (5) almost everywhere (a.e.) on \([t_0, T]\).

Difficulties caused by the restrictions on phase variables are well known in optimization theory. To avoid them, let us analyze constraints (6). Because of (4), the restriction \( a(t) < t \) in (6) is never active at \( P > 0, \ m \geq 0, \) and can be removed from the OP statement. Next, following [18,22], the differential constraint \( a'(t) \geq 0 \) in (6) is replaced with a stricter constraint for the control \( m \):

\[
m_{\min}(t) = \max\{0, P'(t)\} \leq m(t) \leq M(t), \quad \text{a.e. } t \in [t_0, T).
\]

**Lemma 1** ([22,23]). For any measurable control \( m \) that satisfies (8), a unique a.e. continuous function \( a(t) < t, \ t \in [t_0, T), \) exists, satisfies (3), and a.e. has \( a'(t) \geq 0 \).

**Remark 1.** Lemma 1 defines the smoothness requirements for the endogenous phase variable \( a \). By (4), \( m(a(t))da/dt = m(t) - P'(t) \). Hence, the following cases are possible at any \( t \in [t_0, T) \):

- \( a(t) \) is continuous and \( a'(t) > 0 \) exists if \( m(a(t)) > 0 \);
- \( a(t) \) is continuous and \( a'(t) = \infty \) if \( m(a(t)) = 0 \) but \( m(\tau) > 0 \) in a small neighborhood of \( a(t) \);
- \( a(t) \) is discontinuous and has a finite increasing jump if \( m(\tau) = 0 \) at \( \tau \in [a(t), a(t+)] \), \( t_1 > t \).

The minimum possible investment \( m_{\min} \) in (8) plays a special role in the analysis below. It means that one of two situations can take place:

- if \( P'(t) \geq 0, \ t \in [t_1, t_2] \subset [t_0, T) \), then \( m_{\min}(t) = P'(t) \geq 0 \) and from (4) \( a'(t) = 0, \ a(t) \equiv a(t_1) \) for \( t \in [t_1, t_2] \);
- if \( P'(t) \leq 0, \ t \in [t_1, t_2] \subset [t_0, T) \), then \( m_{\min}(t) = 0 \) and \( a'(t) = -P'(t)/m(a'(t)) \geq 0 \) for \( t \in [t_1, t_2] \).
From the economic point of view, \( m \equiv m_{\text{min}} \) is a trivial regime. As shown in [18,19], the minimum possible capital renovation \( m_{\text{min}} \) is not optimal in the presence of the embodied TC if the horizon \([t_0, T]\) is large.

Let \( U_1 \) be the set of measurable variables \( m \) that satisfy (8). Under (8), the differential constraint (6) is satisfied automatically and can be removed from the OP statement. Thus, the OP (3)–(7) is reduced to an OP without phase constraints by narrowing of the control \( m \) domain \( U \) to \( U_1 \subset U \). Here and thereafter, we consider the OP (3)–(8) with the narrower domain \( U_1 \) of the admissible controls instead of the OP (3)–(7).

The OP is investigated at \( T \leq \infty \). If \( T = \infty \), then the conditions

\[
\int_{t_0}^{\infty} \rho(t) \beta(t, t) P(t) \, dt < \infty, \quad \int_{t_0}^{\infty} \rho(t) \lambda(t) \, dt < \infty, \tag{9}
\]

should be held to ensure the convergence of the improper integral in (3).

The solvability of the state equation (4) with respect to \( a \) (Lemma 1) makes it possible to derive the extremum condition using standard variation techniques of the optimization theory.

**Lemma 2** ([22]). The functional \( I(m) \) in OP (3)–(8) is differentiable and the increment \( \delta I \) of functional (3) is of the form:

\[
\delta I = I(m + \delta m) - I(m) = \int_{t_0}^{T} I'(t) \delta m(t) \, dt + \delta^2 I,
\]

where

\[
I'(t) = \int_{t}^{\tilde{a}^{-1}(t)} \rho(\tau)[\beta(t, \tau) - \beta(a(\tau), \tau)] \, d\tau - \lambda(t) \rho(t), \quad t \in [t_0, T), \tag{10}
\]

is the Fréchet derivative \( I'(t) \) of the functional \( I \) with respect to \( m \),

\[
\tilde{a}^{-1}(t) = \begin{cases} a^{-1}(t), & t \in [t_0, a(T)], \\ T, & t \in [a(T), T), \tag{11} \end{cases}
\]

\( a^{-1}(t) \) is the inverse function of \( a(t) \), and

\[
\delta^2 I = \int_{t_0}^{T} \rho(t) \left\{ \int_{a(t)}^{a(t) + \delta a(t)} [\beta(a(t), \tau) - \beta(t, \tau)] [m(\tau) + \delta m(\tau)] \, d\tau \right\} \, dt
\]

is the second variation in (10).

**Remark 2.** The inverse function \( a^{-1}(t) \) always exists because \( a'(t) \geq 0 \) under the Lemma 1 conditions. The function \( a^{-1}(t) \) is continuous if \( a'(a^{-1}(t)) \) \( > 0 \) or discontinuous (with a finite increasing jump) at a point \( t = t_1 \) if \( a'(a^{-1}(t)) = 0 \) and \( a(t) \equiv t_1 \) at \( t \in [a^{-1}(t_1), a^{-1}(t_2)] \), \( t_2 > t_1 \).

**Lemma 3** ([23]). If \( \beta(t, \tau)/\delta \tau > 0 \), then \( \delta^2 I < 0 \) for any admissible variations \( \delta m(t), t \in (t_0, T), \delta m \neq 0 \), i.e., the functional \( I(m) \) is strictly concave downward.

The strict concavity of the OP allows us to derive a necessary and sufficient condition for an extremum. Let \((m^*, a^*)\) denote the solution of the OP (if it exists).

**Lemma 4** ([23]). In order for a function \( m^*(t), t \in [t_0, T), T \leq \infty, \) to be a solution of OP (3)–(8), it is necessary and sufficient that:

\[
\begin{align*}
I'(a^*, t) & \leq 0 \quad \text{at} \quad m^*(t) = m_{\text{min}}(t), \\
I'(a^*, t) & \geq 0 \quad \text{at} \quad m^*(t) = M(t), \\
I'(a^*, t) & \equiv 0 \quad \text{at} \quad m_{\text{min}}(t) < m^*(t) < M(t), t \in [t_0, T).
\end{align*}
\]
The strict concavity also means that the OP solution \((m^*, a^*)\) is unique (if it exists) and delivers the global optimum to the OP. So, the qualitative properties of optimal trajectories established below are global.

The derivative \(I'(t)\) in \(m\) (11) depends only on the unknown \(a\) and does not depend on \(m\). This fact has essential economic implications. To reflect it, we denote \(I'(t)\) as \(I'(a, t)\) here and thereafter. The integral-functional equation \(I'(a, t) = 0\) or

\[
\int_{\tau}^{\tau^{-1}(t)} \rho(t)[\beta(t, \tau) - \beta(a(t), \tau)]d\tau = \lambda(t)\rho(t),
\]

then Eq. (15) has the unique solution \(a^- (t) < t, da^- /dt > 0, t \in [t_0, \infty), \) such that:

- if \(c_1 > c_2,\) then \(t - a^- (t) \rightarrow 0\) at \(t \rightarrow \infty;\)
- if \(c_1 = c_2,\) then \(a^- (t) \equiv t - L, t \in [t_0, \infty),\) where the constant \(L\) is found from

\[
c_3 \exp(-c_1 L) - c_1 \exp(-c_3 L) = (c_3 - c_1)(1 - c_3 \lambda_0 / \beta_0).
\]

If \(0 \leq c_1 < c_3 < 1,\) then \(L \approx \sqrt{2L_0 / (\beta_0 c_1)}.\)

Next, the structure of the solution \((m^*, a^*)\) to the OP (3)–(8) is studied. The cases \(T = \infty\) and \(T < \infty\) of the OP appear to be quite different and are analyzed separately.

4. Infinite-horizon optimization

The structure of the OP solution appears to be pretty simple in the case of infinite horizon.

**Theorem 1** (The Solution Structure at \(T = \infty\)). If a function \(a^- (t), t \in [t_0, \infty),\) exists such that \(I'(a^-, t) \equiv 0\) for \(t \in [t_0, \infty)\) and the corresponding \(m^-\) satisfies (8), then OP (3)–(8) has the unique solution \((m^*, a^*)\) of the following form:

\[
m^*(t) = \begin{cases} 
m_{\min}(t) & \text{or} \ M(t), \quad t \in [t_0, \mu), \\
m^-(t), & \quad t \in [\mu, \infty), 
\end{cases}
\]

where

\[
a^*(t) = \begin{cases} 
\alpha_{\min}(t), & \text{if} \ a_0 > a^- (t_0), \\
\alpha^- (t), & \text{if} \ a_0 < a^- (t_0), 
\end{cases}
\]

\[
a_{\min} \text{ and } a_{\max} \text{ correspond to } m^* \equiv m_{\min} \text{ and } m^* \equiv M \text{ in virtue of (4), the function } m^- \text{ is found from (4) at } a \equiv a^*, \text{ and the instant } \mu \text{ is determined from the condition } a_{\lim}(\mu) = a^- (\mu). \text{ The value } \mu - t_0 \text{ is defined only by the value of the deviation } |a^- (t_0) - a_0| \text{ if } a^- (t_0) = a_0.
\]

**Proof.** See the Appendix.

Theorem 1 can be interpreted as the turnpike theorem in the strongest form for the OP in the case \(T = \infty\) [18, 22]. It states that, starting from some instant \(\mu \geq t_0,\) the OP solution \(a^*(t)\) coincides with the turnpike \(a^- (t)\) that does not necessarily satisfy the initial condition \(a(t_0) = a_0.\) As it follows from Lemma 5, the unique turnpike \(a^- (t), t \in [t_0, \infty),\) always exists in the case (16) of exponential \(\beta, \lambda,\) and \(\rho\) at \(0 < c_1 < c_2 < c_3.\)
Fig. 1. The optimal capital scrapping time $a^*(t)$, investment $m^*(t)$, and the derivative $l'(t)$ on the infinite horizon ($T = \infty$). The optimal $m^*$ possesses the replacement echoes. The OP solution $(m^*, a^*)$ is illustrated in Fig. 1 for the case $a_0 > a \sim(t_0)$. For simplicity, here $P = \text{const}$ and $m_0 = \text{const}$. The turnpike trajectory $a \sim$ is indicated with the dashed line. For comparison, the dotted line shows the constant lifetime trajectory $t - L$ in the case $c_2 = c_1$.

Let us discuss the structure of the optimal investment trajectory $m^*(t)$, $t \in [t_0, \infty)$. Fig. 1 demonstrates a clear repetition pattern in the optimal trajectory $m^*$. This effect is known as the replacement echoes in the VCM theory [4]. This was demonstrated for one- and two-sector VCMs in [18] (Figs. 4.1 and 8.1).

The replacement echoes appear because of the impact of the initial condition (7) imposed on the unknown $a$. In the general case $a(t_0) = a_0 \neq a \sim(t_0)$, the solution $m^*(t)$ is boundary by (18) (minimum $m_{\text{min}}(t)$ or maximum $M(t)$) at the beginning part $[t_0, \mu]$ of the planning horizon (the transition dynamics). After that, $m^*(t) \equiv m \sim (t)$ is found from (4) as $m \sim (t) = P'(t) + m(a \sim (t))d a \sim /dt$. The last formula demonstrates that the initial boundary-valued section of $m^*$ is repeated throughout the whole horizon $[t_0, T]$. Namely, when we reach the part $[a \sim^{-1}(t_0), a \sim^{-1}(\mu)]$ of $[t_0, T)$, then $m \sim (t)$ is similar to $m(a \sim (t))$ at $[t_0, \mu]$, at least, at small $|P'|$ and $da \sim /dt \approx 1$. The same situation appears on the interval $[a \sim^{-1}(a \sim^{-1}(t_0)), a \sim^{-1}(a \sim^{-1}(\mu))]$, and so on.

So, the replacement echoes disseminate the transition dynamics of the optimal investment $m^*(t)$ to the future infinite period. These echoes do not decline. On the other side, the optimal capital lifetime $a^*(t)$ does not have any irregularities after the instant $\mu$. If the initial condition is “perfect”: $a_0 = a \sim(t_0)$, then the replacement echoes are absent and the optimal $a^*(t)$ coincides with the turnpike $a \sim(t)$ indicated with the dashed line in Fig. 1 from the very beginning, $\mu = t_0$. As we shall see below, in the finite-horizon case $T < \infty$, there exists even a more powerful echo pattern caused by the “zero-investment period” $(\Theta, T)$.

5. Finite-horizon optimization

At $T < \infty$ the structure of the OP solutions is more complicated as compared to the $T = \infty$ case. First of all, the optimal policy is no investment at the end of planning horizon. It follows directly from the extremum condition
(14) and expression (11) for $I'(a^*, t)$. Indeed, $I'(a^*, t) < 0$ by (11) and $I'(a^*, t)$ is continuous in $t$, hence, a “zero-investment period” $(\Theta, T)$ exists such that $t_0 \leq \Theta < T$, $I'(a^*, t) < 0$ and $m^*(t) \equiv m_{\min}(t)$ for $t \in (\Theta, T)$ (see also [16,18,19]).

The further analysis is based on the special structure (11) of the derivative $I'(a, t)$ that does not depend on the control $m$. It allows us to introduce an adjusted turnpike trajectory ($T$-trajectory) $a_q$ that will play the role of the turnpike $a^\sim$ in the finite-horizon OP.

**Definition 2.** The $T$-trajectory of the OP (3)–(8) is a continuous function $a_q(t), t \in [t_0, T]$, that satisfies the extremum condition (14) and does not necessarily satisfy the initial condition $a(t_0) = a_0$ in (7).

By Definitions 1 and 2, the $T$-trajectory $a_q$ in the infinite-horizon case $T = \infty$ simply coincides with the turnpike $a^\sim$. At $T < \infty$, the $T$-trajectory is different from the turnpike $a^\sim$ because of the zero-investment period. As it is shown below, $a_q$ can be determined explicitly in special cases.

If a $T$-trajectory $a_q$ exists, then one can expect that the optimal trajectory $a^*$ coincides with $a_q$ everywhere, except for an initial finite interval $[t_0, \mu)$. On the interval $[t_0, \mu)\), the structure of the OP solution will be similar to the infinite-horizon case, namely, the solution will be boundary: $m^* \equiv m_{\min}$ or $m^* \equiv M$. The corresponding $m^*(t), t \in [t_0, T)$, is determined from (4) and always depends on the initial function $m_0$.

To obtain economic meaningful results, we restrict ourselves to the case (16) with the unique turnpike trajectory $a^\sim$. Moreover, we assume for simplicity that $c_1 = c_2 = c$, i.e.,

$$\beta(\tau, t) = \exp(\kappa \tau), \quad \lambda(t) = \lambda_0 \exp(\kappa t), \quad \rho(t) = \exp(-c_3 t), \quad 0 < c < c_3, \lambda_0 c_3 < 1. \quad (21)$$

Then, by Lemma 5, the unique turnpike is $a^\sim(t) = t - L, t \in [t_0, \infty)$, where the constant capital lifetime $L$ is defined by the non-linear equation (17). Also, let us assume that

$$P'(t) \geq 0, \quad t \in [t_0, T]. \quad (22)$$

Under condition (22), the $T$-trajectory $a_q$ does not depend on $m$. Indeed, in view of (4)

$$m_0(a(t))a'(t) = m(t) - P'(t) \geq 0 \quad (23)$$

for any admissible $m$. Then the boundary-valued regime $a(t) = a_{\min}(t)$ at $[t_1, t_2] \subset [t_0, T)$ means $a'(t) \equiv 0$, $a(t) \equiv a(t_1)$, and $m_{\min}(t) = P'(t) \geq 0$ for $t \in [t_1, t_2]$. Hence, the $a_{\min}(t), t \in [t_1, t_2]$, depends on the value $a_{\min}(t_1)$ only and does not depend on $m_{\min}$.

Under conditions (21), the Freshet derivative (11) becomes

$$I'(a; t) = \int_t^{\tilde{a}^{-1}(t)} e^{-c_3 \tau} \left[ e^{\kappa \tau} - e^{\kappa a(\tau)} \right] d\tau - \lambda_0 e^{(-c_3 + \kappa) t}, \quad t \in [t_0, T]. \quad (24)$$

**5.1. Construction of the adjusted turnpike trajectory**

The technique is the following:

1. We start constructing the $T$-trajectory $a_q$ from the right end of $[t_0, T)$ where $a_q(t) = a_{\min}(t) = a_q(\Theta)$ on the zero-investment interval $(\Theta, T], \Theta \geq t_0$.

2. If $\Theta > t_0$, then we try to build $a_q$ recurrently from right to left by adjusting $I'(a_q, t)$ to zero and keeping its value zero as long as possible.

The differentiation of the equality $I'(a; t) = 0$ in $t$ on an interval $\Delta \subset [t_0, T)$ leads to the following relation between the functions $a$ and $a^{-1}$:

$$t - a(t) = -\frac{1}{e} \ln \left\{ 1 - \lambda_0 (c_3 - c) + \frac{e}{c_3} \left[ e^{(c_3 - \kappa a^{-1}(t) - c)} - 1 \right] \right\}, \quad t \in \Delta. \quad (25)$$

Since $a(t) < t$, then $a^{-1}(t) > t$ and relation (25) is a recurrent formula from right to left.
It appears that there exists a set of the special trajectories $a_i, i = 1, 2, \ldots$, such that $a_q(t)$ coincides with one of $a_i(t)$ on $\Delta \subset [t_0, T)$ in order to be interior to the domain (6) at $t \in \Delta$. Knowing $a_q(t) = a_{\min}(t) \equiv a_q(\Theta), t \in (\Theta, T]$, we can determine $a_1(t)$ from (25) as

$$a_1(t) = t + \frac{1}{c} \ln \left\{ 1 - \lambda_0(c_3 - c) + \frac{c}{c_3} e^{c_3(T-t)} - 1 \right\} \tag{26}$$

on the interval $[a_q(\Theta), \Theta]$. Next, we determine $a_2(t)$ on the interval $[a_1(a_q(\Theta)), a_q(\Theta)]$, and so on. This process leads to the following set of trajectories:

$$a_{i+1}(t) = t + \frac{1}{c} \ln \left\{ 1 - \lambda_0(c_3 - c) + \frac{c}{c_3} e^{c_3(a_i^{-1}(t))} - 1 \right\}, \quad i = 2, 3, \ldots \tag{27}$$

The trajectories $a_i(t)$ depend on the constant $T$ and functions $\beta, \lambda, \rho$ and do not depend on the function $P$ and the initial conditions $t_0, a_0, m_0$. The analysis of the recurrent relation (27) shows that $a_i(t) < a_{i+1}(t), da_i(t)/dt > 1$ at $t \in (-\infty, T)$; and $a_{i+1}(t) \to a^\sim(t) = t - L$ as $i \to \infty$ and $T - t \to \infty$, where $a^\sim$ is the turnpike trajectory from Lemma 5.

So, we can separate the interval $[t_0, T)$ of any finite length into the parts $[a_q(\Theta), \Theta], [a_q(a_q(\Theta)), a_q(\Theta)], [a_q(a_q(a_q(\Theta))), a_q(a_q(\Theta))], \ldots$, and assign the $T$-trajectory $a_q$ to the trajectories $a_i(t), i = 1, 2, 3, \ldots$. To obtain a continuous trajectory, we connect the separate pieces of $a_i$ with the boundary-valued trajectories $a_{\min}$ (which are the horizontal lines $a_{\min}'(t) \equiv 0$ by (22)). The obtained continuous function $a_q$ is unique and represents the continuous $T$-trajectory in the sense of Definition 1.

5.2. The structure of solutions in the finite-horizon OP

Similarly to the infinite-horizon OP, the OP solution $a^*$ coincides with the $T$-trajectory $a_q$ everywhere except for the initial interval $[t_0, \mu)$. This result is formalized as the following theorem:

**Theorem 2 (The OP Solution Structure at $T < \infty$).** Under conditions (21) and (22), there exists a $T$-trajectory $a_q(t)$ to the finite-horizon OP (3)–(8) of the form:

$$a_q(t) = \begin{cases} a_i(\alpha_i), & \text{if } I'(t) < 0, t \in (\alpha_i, \beta_i) \\ a_i(t), & \text{if } I'(t) = 0, t \in (\beta_i+1, \alpha_i), \quad i = 1, 2, \ldots, t \in [t_0, T), \end{cases} \tag{28}$$

where the trajectories $a_i, i = 1, 2, 3, \ldots$, are determined by (26) and (27) and the parameters $\alpha_i, \beta_i, i = 1, 2, 3, \ldots$, are uniquely determined, $\beta_i+1 < \alpha_i, \alpha_i < \beta_i, \beta_1 = T$. The OP (3)–(8) has the unique solution $(m^*, a^*)$ of the following form:

$$m^*(t) = \begin{cases} m_{\min}(t) & \text{or } M(t), \quad t \in [t_0, \mu), \\ m_q(t), & \text{if } t \in [\mu, T), \end{cases} \tag{29}$$

$$a^*(t) = \begin{cases} a_\mu(t), & \text{if } t \in [t_0, \mu), \\ a_q(t), & \text{if } t \in [\mu, T), \end{cases} \tag{30}$$

where $a_\mu(t) = \begin{cases} a_{\min}(t) & \text{if } a_0 > a^* - (t_0), \\ a_{\max}(t) & \text{if } a_0 < a^* - (t_0), \end{cases}$ the function $m_q$ is found from (4) at $a \equiv a_q$, the functions $a_{\min}$ and $a_{\max}$ are defined from (4) and correspond to the minimum $m = m_{\min}$ or maximum $m = M$, and the instant $\mu$ is determined from the condition $a_\mu(\mu) = a_q(\mu)$.

**Proof.** See the Appendix. ■

**Remark 3.** Our technique is not restricted to the case (21) with the constant turnpike lifetime. This case has been chosen for the sake of simplicity of analytic formulas only.

5.3. Replacement and anticipation echoes

**Theorem 2** allows us to analyze the nature of irregularities in the optimal control functions $m^*$ and $a^*$, which are caused by the OP initial and final conditions. In particular, the formulas (28) and (30) demonstrate a new repetition
effect that appears in the optimal $a^*$. We refer to it as *anticipation echoes* because their source is the anticipation of future changes at the end of the planning horizon. Let us analyze the structure of the optimal investment control $m^*$. For the sake of clarity, we consider the case $P(t) \equiv \text{const}$, then two boundary-valued regimes $a^\text{min}_\text{m} \equiv 0$ and $m^\text{min}_\text{m} \equiv 0$ coincide. If the initial condition is $a_0 = a^\sim(t_0)$ and the optimal $a^*(t)$ coincides with the turnpike $a^\sim(t)$ from starting $t_0$.

**Corollary (The Existence of Anticipation Echoes).** Let (21) and (22) be true, $a_0 = a^\sim(t_0)$, and $P'(t) \equiv 0$. Then, starting from the zero-investment period $(\Theta, T)$, the optimal $m^*$ has zero-investment parts $m^*(t) = 0$ on a finite set of the repetitive intervals $(\alpha_i, \beta_i)$, $i = 1, 2, 3, \ldots$, until it becomes $t = t_0$. The instants $\alpha_i, \beta_i, i = 1, 2, 3, \ldots, \beta_{i+1} < \alpha_i$, $\alpha_i < \beta_i$, $\alpha_i < \beta_1 = T$, are uniquely determined by Theorem 2.

**Proof.** By (28) and (30), $a^\prime(t) \equiv 0$ and $m^*(t) \equiv 0$ at $t \in (\alpha_1, T]$, $\alpha_1 = \Theta$. For $t_0 < t < \alpha_1$, the optimal $m^*(t)$ is found from (4) as

$$m^*(t) = P'(t) + m(a^*(t)) \frac{da^*}{dt}$$

(31)

from the left to the right, starting with the initial condition $m(\tau) = m_0(\tau), \tau \in [a_0, t_0]$. By Theorem 2, the optimal trajectory $a^*$ has the irregular parts $[\alpha_i, \beta_i]$ where $a^\prime(t) \equiv 0$. When we reach such an irregular part, then $m^*(t) = m^\text{min}_\text{m} = P' \equiv 0$, $t \in [\alpha_i, \beta_i]$ in view of (31) regardless of its behavior before and after $[\alpha_i, \beta_i]$.

Fig. 2 illustrates the optimal $a^*$ and $m^*$ in the Corollary case. Then $a^*$ coincides with the $T$-trajectory $a_q$ on the whole interval $[t_0, T]$ and the infinite-horizon turnpike $a^\sim(t)$ is indicated with the dotted line (the lifetime $t - a^\sim(t)$ is constant by Lemma 5).

Summarizing, there exist two echo effects in the finite-horizon OP:

5.3.1. The replacement echoes

As in the infinite-horizon case $T = \infty$, the “imperfect” initial condition $a(t_0) = a_0 \neq a^\sim(t_0)$ on the left end $t = t_0$ of the planning horizon $[t_0, T]$ causes the appearance of the initial boundary-valued regime $m(t) \equiv 0$ or $M$ on $[t_0, \mu]$ in the optimal new investment trajectory $m^*$. This boundary section disseminates the corresponding *replacement*
Fig. 3. An example of the interaction of the anticipation (a) and replacement (r) echoes in the finite-horizon case \((T < \infty)\).
The structural analysis of OP solutions provides a new insight into the optimal dynamics of capital replacement in the one-sector VCM. In particular, it demonstrates the existence of irregularities in the optimal replacement process caused by the conditions on both ends of the planning horizon. First, the given initial condition at the beginning of the planning horizon causes the appearance of an initial boundary-valued section in the optimal investment trajectory \( m^* \) and the dissemination of the corresponding replacement echoes throughout the whole horizon \([t_0, T]\). In the finite-horizon case, a “zero-investment period” \((\theta, T]\) appears at the right end of the horizon \([t_0, T]\). The anticipation of the future policy change on \((\theta, T]\) causes a repetition pattern (anticipation echoes) before \(\theta\) in both optimal capital lifetime \(a^*\) and investment \(m^*\). The anticipation echoes propagate backward throughout the whole horizon and their intensity decreases as \(T - t\) increases.

The anticipation echoes appear to be a general structural property of the optimal investment policy. In [25], similar echoes are caused in an infinite-horizon problem by the anticipation of a future jump in the productivity function \(\beta(t)\) at some instant \(\theta\) (a future technological breakthrough). Both optimal controls \(a^*\) and \(m^*\) in [25] possess irregular controls (echoes) on \([t_0, \theta]\) where the controls are boundary: \(a^*(t) \equiv \text{const}\) and \(m^*(t) \equiv 0\). The behavior of these anticipation echoes is similar to the ones demonstrated above. They propagate backward throughout the interval \([t_0, \theta]\) and become smaller as \(k\) increases. So, we can consider the anticipation echoes as a general response of dynamic systems with endogenous delay to possible perturbations in given parameters.

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Appendix

A.1. Proof of Theorem 1

Proof. First, a direct check shows that the constructed pair \((m^*, a^*)\) satisfies all restrictions (4)–(8), hence, \((m^*, a^*)\) is admissible in the domain \(U_1\). Next, using the necessary and sufficient condition for an extremum, we can prove that the functions \((m^*, a^*)\) represent an OP solution. Let us show that \((m^*, a^*)\) satisfies Lemma 4.

Let us consider the case \(a_0 < a^-(t_0)\). Then \(m^*(t) = M(t)\), \(a^*(t) = a_{\lim}(t) < a^-(t)\), \(a_{\lim}^{-1}(t) > a^-1(t)\) for \(t \in [t_0, \mu]\); where \(\mu\) is the intersection point of the lines \(a_{\lim} = a_{\max}\) and \(a^-\). The corresponding derivative (11) takes the following form:

\[
I'(a^*, t) = \int_1^\mu \rho(\tau)[\beta(t, \tau) - \beta(a_{\lim}(\tau), \tau)]d\tau + \int_{a_{\lim}^{-1}(t)}^{a_{\lim}^{-1}(t)} \rho(\tau)[\beta(t, \tau) - \beta(a^-(\tau), \tau)]d\tau + \lambda(t)\rho(t)
\]

\[
> \int_{a_{\lim}^{-1}(t)}^{a_{\lim}^{-1}(t)} \rho(\tau)[\beta(t, \tau) - \beta(a^-(\tau), \tau)]d\tau + \lambda(t)\rho(t)
\]

\[
> \int_{a_{\lim}^{-1}(t)}^{a_{\lim}^{-1}(t)} \rho(\tau)[\beta(t, \tau) - \beta(a^-(\tau), \tau)]d\tau + \lambda(t)\rho(t) = 0
\]

for \(t \in [t_0, \mu]\). Hence, \(I'(a^*, t) > 0\) for \(t \in [t_0, \mu]\) where \(m^*(t) = M(t)\) and \(I'(a^*, t) = I'(a^-, t) \equiv 0\) for \(t \in [\mu, \infty)\). Therefore, the necessary and sufficient condition for an extremum holds true, and \((m^*, a^*)\) is a solution to the OP.

To prove the uniqueness of \((m^*, a^*)\), we assume that another solution \((m^+, a^+)\) exists. Let us analyze the increment (10) of the functional \(I\) caused by the variations \(\delta m = m^+ - \hat{m}, \delta a = a^+ - \hat{a}\). Since \(I'(t) \equiv 0\) at \(t \in [\mu, \infty)\), then from Lemma 2 we have \(\delta I = \int_{t_0}^{t_*} I'(a^*, t)\delta m(t)dt + \delta^2 I\). By the construction of \((m^*, a^*)\), \(I'(a^*, t) > 0\) and \(\delta m(t) \leq 0\) for \(t \in [t_0, \mu]\), hence, the first integral is non-positive. Next, Lemma 3 states that \(\delta^2 I < 0\) for any variation \(\delta m(t), t \in [t_0, \infty)\). \(\delta m \neq 0\). Noticing that \(\delta m \neq 0\) because of \(m^+ \neq \hat{m}\), we conclude that \(\delta I < 0\) and \((m^+, a^+)\) is not a solution. Hence, \((m^*, a^*)\) is the unique OP solution.

The case \(a_0 > a^-(t_0)\) is investigated similarly. The theorem is proved. ■
A.2. Proof of Theorem 2

**Proof.** The proof consists of two parts: (a) the construction of the $T$-trajectory $a_q$; (a) the construction of the solution $(m^*, a^*)$ and the verification of its optimality.

Part 1. The construction of $a_q$ starts from the right end $T$ of the horizon $[t_0, T]$. Since $I'(a_q, T) = -\lambda(T)\rho(T) < 0$ by (11), the derivative $I'(a_q, t)$ will stay negative on some interval $(\bar{\Theta}, T]$. Hence, $a_q(t) = a_{\text{min}}(t) \equiv a_q(T)$ at $t \in (\bar{\Theta}, T]$ (see Fig. 4). It means that $a_q^{-1}(t) \equiv T$ at $t \in [\Theta, T]$. After $I'(a_q, t)$ increases up to zero at $t = \Theta$, we will keep the identity $I'(a_q, t) = 0$ to the left of $\Theta$. In virtue of expressions (25) and (26), the only way to do this is to keep $a_q(t)$ on the curve $a_1(t)$ at $t < \Theta$. So, we need to find the first point $(\Theta, a_1(\Theta))$ on the curve $a_1(t)$ that satisfies $I'(a_q, \Theta) = 0$. To show that such $\Theta$ exists, let us investigate the asymptotic dynamics of the function $I'(a_q, \Theta)$. The substitution of $a_q(t)$ and $a_q^{-1}(t)$, $t \in [\Theta, T]$, into (24) leads to

$$I'(a_q; \Theta) = \int_{\Theta}^T e^{-c_3\tau}[e^{c\tau} - e^{c\Theta}]d\tau - \lambda_0 e^{(-c_3+c)\Theta}$$

$$= e^{(-c_3+c)\Theta}[1 - e^{c(\Theta-\Theta)}][1 - e^{-c_3(T-\Theta)}]/c_3 - \lambda_0].$$

Analyzing the asymptotics of formula (26) for $a_1(t)$, we obtain that $a_1(t) \rightarrow t + \ln[1 - \lambda_0(c_3 - c) - c/c_3]$ as $T - t \rightarrow \infty$. Substituting this asymptotic expression for $a(\Theta)$ in (32), we obtain that

$$I'(a_q; \Theta) = e^{(-c_3+c)\Theta}[1 - 1 + \lambda_0 c_3 - \lambda_0 c + c/c_3][1 - e^{-c_3(T-\Theta)}]/c_3 - \lambda_0]$$

$$= e^{(-c_3+c)\Theta}/c_3[[\lambda_0 c_3 + c/c_3(1 - \lambda_0 c_3)][1 - e^{-c_3(T-\Theta)}] - \lambda_0 c_3].$$

(33)

Taking into account that $\lambda_0 c_3 < 1$ by (21), we get $I'(a_q, \Theta) > 0$ for $T - \Theta \gg 1$. Since $I'(a_q, T) < 0$ and $I'(a_q, \Theta)$ is continuous in $\Theta$, at $T - t_0 \gg 1$ a unique moment $\Theta = \alpha_1$ exists that satisfies the equality $I'(a_q, \Theta) = 0$.

Let us construct the next piece of $a_q$ on the interval $[\bar{\alpha}_2, \alpha_1]$ (see Fig. 4). We put $a_q(t) = a_1(t)$ to keep $I'(a_q, t) \equiv 0$ to the left of $\alpha_1$ on some interval $[\beta_2, \alpha_1]$. Because of the symmetry of the inverse functions, the inverse $a_q^{-1}(t)$, $t \in (a_1(\alpha_1(\Theta)), T]$, is already defined by $a_q(t)$, $t \in (a_1(\Theta), T]$. So, the trajectory $a_q(t)$ has to leave $a_1(t)$ at some point $\beta_2$, before $a_q^{-1}(t)$ jumps from $T$ to $a_1^{-1}(t)$ at $t = a_1(\Theta)$. To the left of $\beta_2$, $a_q(t)$ follows the boundary minimum trajectory $a_q(t) = a_q(\beta_2) = a_1(\beta_2)$ until it reaches the second line $a_2(t)$ at some point $\alpha_2 < \beta_2$. The points $\alpha_2$ and $\beta_2$ are found from the condition $I'(a_q, \alpha_2) = 0$ on the new curve $a_q(t)$. To show that the point $\alpha_2$ exists, we estimate
the derivative \( I'(a_q, t) \) at \( \alpha'_2 = a_1(\Theta) \) and at the point \( \alpha'_2 < \alpha'_2 \) defined by \( \beta_2a_1(\Theta) \) and \( a_2(\alpha'_2) = a_1(\beta'_2) \):

\[
I'(a_q, \alpha'_2) = \int_{\alpha'_2}^{a_1^{-1}(\alpha'_2)} e^{-ct} [e^{ca_2(t)} - e^{ca_2(\tau)}] d\tau - \lambda_0 e^{(-c+\epsilon)c}\alpha'_2
\]

\[
= \int_{\alpha'_2}^{a_1^{-1}(\alpha'_2)} e^{-ct} [e^{ca_2(t)} - e^{ca_2(\tau)}] d\tau - \lambda_0 e^{(-c+\epsilon)c}\alpha'_2 + \int_{\alpha'_2}^{a_1^{-1}(\alpha'_2)} e^{-ct} [e^{ca_2(t)} - e^{ca_2(\tau)}] d\tau
\]

\[
= \int_{\alpha'_2}^{a_1^{-1}(\alpha'_2)} e^{-ct} [e^{ca_2(t)} - e^{ca_2(\tau)}] d\tau > 0,
\]

\[
I'(a_q, \alpha'_2) = \int_{\alpha'_2}^{\alpha'_2} e^{-ct} [e^{ca_2(t)} - e^{ca_2(\tau)}] d\tau - \lambda_0 e^{(-c+\epsilon)c}\alpha'_2
\]

\[
= \int_{\alpha'_2}^{\alpha'_2} e^{-ct} [e^{ca_2(t)} - e^{ca_2(\tau)}] d\tau - \lambda_0 e^{(-c+\epsilon)c}\alpha'_2 - \int_{\alpha'_2}^{\beta_2} e^{-ct} [e^{ca_1(\beta_2)} - e^{ca_1(\tau)}] d\tau
\]

\[
= - \int_{\alpha'_2}^{\beta_2} e^{-ct} [e^{ca_1(\beta_2)} - e^{ca_1(\tau)}] d\tau < 0.
\]

Because of the continuity of \( I'(a_q, t) \) in \( t \), a unique moment \( \alpha'_2 \), \( \alpha'_2 < \alpha'_2 \), \( \alpha'_2 \), exists such that \( I'(a_q, \alpha'_2) = 0 \).

Let us show that \( I'(a_q, t) \) is less than 0 at \( t \in (\alpha_2, \beta_2) \), i.e., when the \( T \)-trajectory \( a_q(t) \) leaves the curve \( a_1(t) \) at \( t = \beta_2 \) and until it reaches \( a_2(t) \) at \( t = \alpha_2 \) (see Fig. 4). By construction, \( I'(a_q, t) \equiv 0 \) on \([\beta_2, \alpha_1]\) and \( d[I'(a_q, t)]/dt \equiv 0 \) on \((\beta_2, \alpha_1)\). We investigate its derivative in \( t \):

\[
d[I'(a, t)]/dt = -\frac{c}{c_3} \left[ e^{-c_3a_1(\tau)} - e^{-c_3t} \right] e^{ca_1(t)} + \left[ e^{ca_1(t)} - e^{ca_1(\tau)} \right] e^{-c_3t} + \lambda_0 (c_3 - c)e^{-(c_3-c)t}, \quad t \in [t_0, T]. \tag{34}
\]

Let us first consider a small neighborhood of the instant \( t = \beta_2 \). Here \( a_q(t) > a_1(t) \) at \( t < \beta_2 \), hence \( I'(a_q, t) \)/\( dt > d[I'(a_q, t)]/dt = 0 \) because of (34). Similarly, \( d[I'(a_q, t)]/dt < d[I'(a_2, t)]/dt = 0 \) at some neighborhood \( t > \alpha_2 \). Therefore, \( I'(a_q, t) < 0 \) on \((\alpha_2, \beta_2)\).

The previous step contains a complete iteration in constructing the \( T \)-trajectory \( a_q(t) \) on the interval \([\alpha_2, \alpha_1]\) (see Fig. 4). At the beginning, \( a_q(t) = a_1(t) \), \( t \in (\beta_2, \alpha_1) \), hence \( a_q^{-1}(t) = a_1^{-1}(t) \) at \( t < a_1(\beta_2) \). According to (27), the new curve is \( a_2(t) \) on some interval to the left of \( \beta_2 \). The trajectory \( a_q(t) \) is the minimum possible \( a_q(t) = a_q(\beta_2) = a_1(\beta_2) \) until it intersects \( a_2(t) \) at some point \( \alpha_2 < \beta_2 \). Then the corresponding \( a_q^{-1}(t) \) is found, the iteration is repeated, and so on. Providing these steps, one can build the \( T \)-trajectory \( a_q(t) \) on the whole \([t_0, T] \) as represented by (28).

The “switch” points \( \alpha_i, \beta_i, i = 1, 2, \ldots \), where the \( T \)-trajectory \( a_q(t) \) leaves one curve \( a_i(t) \) for another, are found from the condition \( I'(a_q, a_i) = 0 \) on the new line \( a_{i+1} \). Taking an arbitrary sought-after point \( \beta_i \) on the old line and the corresponding \( a_i \) on the new one, then solving \( I'(a_q, a_i) = 0 \), we verify that the \( T \)-trajectory \( a_q(t) \) satisfies the extremum conditions (14). Namely, \( a_q(t), t \in [t_0, T] \), is constructed in such a way that \( I'(a_q, t) < 0 \) on \((\alpha_i, \beta_i)\) where the \( T \)-trajectory \( a_q = a_{min} \equiv const \) is boundary-valued, and \( I'(a_q, t) = 0 \) on \((\beta_{i+1}, \alpha_i)\) where \( a_q = a_i \) is interior to (6).

Part 2. The optimal solution \( a^* \) is obtained by the adjustment of the \( T \)-trajectory \( a_q \) to the initial conditions (7).

Two cases are possible:

Case 1: \( a_0 > a_q(0) \). Then we choose \( m^*(t) = m_{min}(t) \) at \( t > t_0 \) and move on the corresponding \( a_{min}(t) \equiv a_0 \) until it crosses the trajectory \( a_q(t) \). Then the point of intersection is \( \mu \) and the line of the movement is \( a_\mu(t), t \in [0, \mu] \). At \( t > \mu \), we set the solution \( a^*(t) \) to coincide with the \( T \)-trajectory \( a_q(t) \). Under condition (22), \( a_\mu(t) \equiv a_0, t \in [0, \mu] \), and the corresponding derivative

\[
I'(a^*, t) = \int_{\mu}^{a_\mu^{-1}(t)} e^{-ct} [e^{ca_\mu(t)} - e^{ca^*(\tau)}] d\tau - \lambda_0 e^{(-c+\epsilon)c}\mu
\]

\[
= \int_{\mu}^{a_\mu^{-1}(t)} e^{-ct} [e^{ca^*_t - e^{ca_\mu(t)}]} d\tau - \lambda_0 e^{(-c+\epsilon)c}\mu + \int_{\mu}^{a_\mu(t)} e^{-ct} [e^{ca_\mu(t)} - e^{ca^*_t}] d\tau
\]

\[
= \int_{\mu}^{a_\mu(t)} e^{-ct} [e^{ca_\mu(t)} - e^{ca^*_t}] d\tau < 0, \quad t \in [t_0, \mu],
\]
i.e., the pair \((a^*, m^*)\) satisfies the extremum condition (14) at \(t \in [t_0, \mu]\). The corresponding \(m^*(t), t \in [\mu, T]\), is determined from (23), hence, it will be 0 on \((a_i, \beta_i)\) where \(I'(a^*, t) < 0\) and an internal value (8) between \(m_{\text{min}}\) and \(M\) on \((\beta_{i+1}, a_i)\) where \(I'(a^*, t) = 0, i = 1, 2, 3, \ldots\) (see Fig. 4). Therefore, according to Lemma 4 the pair \((a^*, m^*)\) constructed by (29) and (30) brings the optimal solution to the OP (3)–(8).

Case 2: \(a_0 < a_\mu(t_0)\) is investigated similarly. In this case \(m(t) = M\) brings \(a_\mu(t)\) up to the point of its intersection with the trajectory \(a_i\) and \(I'(a^*, t) > 0\) on \(t \in [t_0, \mu]\).

The proof of the \((m^*, a^*)\) uniqueness is similar to the proof of Theorem 1. We assume that another solution \((m^+, a^+)\) exists and analyze the increment \(\delta I\) (10) of the functional \(I\) caused by the variation \(\delta m = m^+ - m\). Next, Lemma 3 on the OP strict concavity leads to \(\delta I < 0\). The theorem is proved.

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