Splitting-based schemes for numerical solution of nonlinear diffusion equations on a sphere

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Abstract
We provide an advanced study of our recently developed method for the numerical solution of nonlinear diffusion equations on a sphere. In particular, we analyse the method in detail when applied to solving diverse diffusion phenomena, with specific conditions on the smoothness of the solution, the degree of nonlinearity, and the initial data and sources. The main idea of the method consists in splitting the original differential operator by coordinates and subsequent constructing finite difference schemes for the split one-dimensional problems using different coordinate maps for the sphere at the two split time intervals. The essential advantage of this technique is that each split 1D equation can be equipped with a periodic boundary condition, despite the sphere being not a doubly periodic domain. Therefore, unlike the existing methods, this one does not require applying special numerical procedures for careful computing the solution near the poles, which is always a challenge. Each split 1D equation is approximated by a second- or a fourth-order finite difference scheme that keeps all the substantial properties of the differential problem: it is balanced and dissipative, since the spatial finite difference operator is negative definite. The developed algorithm is cheap-to-implement from the computational point of view. The theoretical results are confirmed numerically by simulating various nonlinear diffusion processes. A numerical example, in particular, shows that the competition of the three basic mechanisms—the nonlinear interaction, forcing and dissipation—can generate wave solutions, whose spatial structures on the sphere are subjected to the alternating influence of the processes of self-organisation and self-destruction. The accuracy of the method is evaluated by comparing the numerical solutions versus the analytical ones obtained with specially chosen forcings. The convergence of the numerical solution to the analytics is verified by refining spatial grids.

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1. Introduction

A large number of important natural phenomena, e.g., heat transfer in ionised gases, gas percolation through porous media, concentration waves in distributed chemical reactors, combustion, viscous processes in diverse media, as well as many others are described by (nonlinear) diffusion equations [1–10].

Possibly, the most obvious and well-studied applications of the diffusion equation lie in atmospheric and hydrodynamic sciences. Hundreds of papers providing researches on some or other issues of the environment are published every year.
Among the applications there are air and water pollution, viscous flows in river channels—just to mention a couple of them [9,10]. The governing equations are specific for each application, but in general they contain a hyperbolic part responsible for the advection process (velocity field) and a parabolic term aimed to simulate fluctuation from the mainstream, or diffusion.

Another important application of the diffusion equation, especially involving nonlinearity, is blow-up. In [8], among other issues, a nonlinear diffusion Cauchy problem in $\mathbb{R}^n$ is studied and numerical experiments are performed, simulating blow-up in a bounded domain.

A not less interesting, though a little less obvious problem described by the diffusion equation is the simulation of time evolution of the orientational distribution function of non-spherical particles due to interactions with each other and external fields [11].

In many papers on diffusion equations boundary value problems are studied [12–15], while much less papers are dedicated to the numerical simulation of nonlinear diffusion processes on closed manifolds. The latter differ from bounded domains as having no boundaries, which makes it nontrivial to develop efficient numerical methods for solving diffusion equations in such domains. Although for some particular cases these problems can be reduced to boundary value problems with boundary conditions of special types (e.g., a torus can be represented as a doubly periodic domain, so that periodic boundary conditions in both directions can be employed), this cannot be done for a more or less complicated manifold.

A simple, but important example of such a manifold is a two-dimensional sphere. The case of the sphere can be seen as a step before more general manifolds, which can be interesting, e.g., in nanoelectronics [16]. The point is that the sphere is not a doubly periodic domain, since it is periodic in the longitude, but is not in the latitude due to the presence of two poles. Therefore, a numerical procedure designed to solve the original 2D problem will be computationally cumbersome, because the matrix of the resulting linear system will be of a general type, hence not permitting to apply some or other fast linear solvers. For instance, in [11] a Voronoi tessellation of the sphere is used and then a finite volume method is applied. As the matrix of the resulting linear system will be of a general type, hence not permitting to apply some or other fast linear solvers. For instance, in [11] a Voronoi tessellation of the sphere is used and then a finite volume method is applied. As for making a preliminary splitting of the 2D equation [17], this will bring to the necessity of constructing mathematically and physically correct boundary conditions or involving special numerical procedures near the poles (e.g., matrix bordering) when computing in the latitudinal direction. These are always a problem, as the poles represent an artificial boundary appearing exclusively due to the latitudinal-longitudinal coordinate system, and the construction of proper artificial boundary conditions is a serious independent question [18].

The nonlinear diffusion equation on a two-dimensional sphere $S = \{(\lambda, \varphi) : \lambda \in [0, 2\pi), \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$ has the form

$$\frac{\partial T}{\partial t} = AT + f \equiv \frac{1}{R \cos \varphi} \left[ \frac{\partial}{\partial \lambda} \left( \frac{\mu T^2}{R \cos \varphi} \frac{\partial T}{\partial \lambda} \right) + \frac{\partial}{\partial \varphi} \left( \frac{\mu T^2 \cos \varphi}{R} \frac{\partial T}{\partial \varphi} \right) \right] + f. \tag{1}$$

It is equipped with a suitable initial condition. Here $A$ is the diffusion operator, $T = T(\lambda, \varphi, t) \geq 0$ is the unknown function, $\mu T^2$ is the diffusion coefficient, $\mu = \mu(\lambda, \varphi, t) \geq 0$ is the amplification factor, $f = f(\lambda, \varphi, t)$ is the source function, $R$ is the radius of the sphere with $\lambda$ as the longitude (positive eastward) and $\varphi$ as the latitude (positive northward). The parameter $\alpha$ is usually a positive integer number that determines the degree of nonlinearity of the diffusion process; if $\alpha = 0$ then we are dealing with linear diffusion.

Introducing on $S$ the scalar product as

$$\langle \zeta, \eta \rangle = \int_S \zeta \eta dS, \tag{2}$$

multiplying (1) by $\eta$ and integrating over the domain $S$, we find

$$\left\langle \frac{\partial T}{\partial t}, \eta \right\rangle = \langle AT, \eta \rangle + \langle f, \eta \rangle. \tag{3}$$

In the case of $\eta = 1$ the first summand on the right-hand side of (3) vanishes [19] and the equality turns into the balance equation

$$\frac{d}{dt} \int_S TdS = \int_S f dS, \tag{4}$$

which, if $f = 0$, provides the mass conservation law

$$\frac{d}{dt} \int_S TdS = 0. \tag{5}$$

In the case of $\eta = T$ due to the negative definiteness of $A$ it holds

$$\langle AT, \eta \rangle \leq 0, \tag{6}$$

and so

$$\frac{1}{2} \frac{d}{dt} \int_S T^2 dS \leq \int_S f dS, \tag{7}$$

from where under $f = 0$ we obtain the solution’s dissipation in the $L_2(S)$-norm
\[ \frac{1}{2} \frac{d}{dt} \int_S T^2 dS \leq 0. \] (8)

Our aim is to develop an accurate and efficient numerical method for solving Eq. (1). The accuracy implies that the method should provide physically correct numerical solutions which would possess properties (4) and (8). The efficiency means that the method should be computationally inexpensive. First results on this issue were presented by the authors in [20], so in this paper we study the developed method in detail, analysing how it works when applied to solving diverse diffusion phenomena, with specific conditions on the smoothness of the solution, the degree of nonlinearity, and the initial data and sources. Besides, we explicitly demonstrate the constructed finite difference operators inherit important properties of the original diffusion operator.

The paper is organised as follows. In Section 2 we first linearise the original nonlinear differential equation and then apply the method of splitting [21] to the resulting equation on the sphere. Employing the procedure of alternation of two coordinate maps for representing the same sphere at each split time step, we become able to develop a numerical algorithm as if the method of splitting [21] to the resulting equation on the sphere. Employing the procedure of alternation of two coordinate maps, we construct two second-order finite difference schemes for the split 1D diffusion equations. The schemes are shown to be dissipative and balanced, according to the properties of the original differential problem, as well as easy-to-implement from the computational standpoint. Then, because of the periodicity of the boundary conditions we increase the approximation’s accuracy to the fourth order in space. In Section 3 we provide results of numerical experiments performed to test the developed method. In several examples, from the simplest linear case to a complex nonlinear one, we demonstrate the schemes are accurately simulating diffusion phenomena. We prove the schemes inherit the essential properties of the differential operators and hence yield physically adequate numerical solutions. We comprehensively study the developed method, providing a comparison of the numerical solutions with the analytics. We also give a numerical example (see Experiment 4) demonstrating the synergistic action of the external forcing, nonlinearity and dissipation. These three mechanisms produce solutions, whose spatial structures assert the presence of the processes of self-organisation and self-destruction, which phenomenon is of interest in nonlinear science. In Section 4 we give a conclusion.

2. Mathematical foundation of the method

2.1. Operator splitting on the sphere

In the standard manner we define the grid spacing in time as \( \tau = t_{n+1} - t_n \) and in every time interval \( (t_n, t_{n+1}) \) we linearise (1) and then split the linearised equation by coordinates [22,23]

\[
\frac{\partial T}{\partial t} = A_\lambda T + \frac{f}{2} = \frac{1}{R \cos \phi} \left( \frac{\partial}{\partial \lambda} \left( \frac{D}{R \cos \phi} \frac{\partial T}{\partial \lambda} \right) + \frac{f}{2} \right),
\]

\[ \frac{\partial T}{\partial t} = A_\phi T + \frac{f}{2} = \frac{1}{R \cos \phi} \left( \frac{D}{R} \frac{\partial T}{\partial \phi} \right) + \frac{f}{2}, \] (9)

where \( D = \mu (T^n)^2 \). \( T^n = T(\lambda, \phi, t_n) \). Hereafter the split equations will be considered in time successively: the solution to Eq. (9), solved in \( \lambda \), will be used as the initial condition for Eq. (10); the latter, solved in \( \phi \), will supply the initial condition for (9) in the next time interval \( (t_{n+1}, t_{n+2}) \), and so on.

Because the metric term \( R \cos \phi \) vanishes at \( \phi = \pm \pi/2 \), both equations are meaningless at the poles. Therefore, for covering the entire sphere we assume \( \Delta \lambda = \lambda_{k+1} - \lambda_k \) and \( \Delta \phi = \phi_{l+1} - \phi_l \) and define on \( S \) a grid, making a half step shift in \( \phi \) [24], i.e.

\[
S^{(1)}_{\lambda,\lambda,\phi} = \left\{ (\lambda_k, \phi_l) : \lambda_k, \phi_l \in \left[ \frac{\Delta \lambda}{2}, 2\pi + \frac{\Delta \lambda}{2} \right], \phi_l \in \left[ -\frac{\pi}{2}, \frac{\Delta \phi}{2}, \pi - \frac{\Delta \phi}{2} \right] \right\}. \] (11)

Thereby we exclude the pole singularities (Fig. 1), so that the subsequent finite difference equations will have sense everywhere on \( S^{(1)}_{\lambda,\lambda,\phi} \). For solving (9) one has, evidently, to use the periodic boundary condition in \( \lambda \), as the sphere \( S \) is a periodic domain in the longitude. (Note that a half step grid shifting was also applied in [14,25], and, compared to the classical approach [12], it provided benefits in the solution.)

As for solving Eq. (10), one faces the difficulty since the sphere is not a periodic domain in \( \phi \). Several approaches can be used to resolve it, e.g., one could involve a matrix bordering procedure or try to paste the solution from the opposite meridians at the poles. A disadvantage of these techniques is that in the case of an implicit temporal approximation on the right-hand side of (10) the resulting matrix will be of a general type, and so no fast algorithms of linear algebra can be used for computing the solution. Alternatively, an explicit temporal discretisation can be used, but this will impose serious restrictions on the timestep, especially when \( \varepsilon \) is large. However, this is exactly the method of splitting that allows to avoid these undesired procedures which, besides, if performed inaccurately, may easily result in introducing nonphysical modes into the solution. Specifically, due to the splitting, for computing the solution in \( \phi \) we change the coordinate map from (11) to
\[
S^{(2)}_{\Delta \lambda \Delta \varphi} = \left\{ (\lambda_k, \varphi_l) : \lambda_k \in \left[ \frac{\Delta \lambda}{2}, \pi - \frac{\Delta \lambda}{2} \right], \varphi_l \in \left[ -\frac{\pi}{2} + \frac{\Delta \varphi}{2}, \frac{3\pi}{2} + \frac{\Delta \varphi}{2} \right] \right\}.
\]

Obviously, grid (12) contains the same nodes as (11) (Fig. 2). The use of the two coordinate maps allows employing the same numerical algorithm with the periodic boundary conditions in both directions, \( \lambda \) and \( \varphi \) [26,27,20]. The only change we have to make in (10) if using (12) is to replace \( \cos \varphi \) with \( |\cos \varphi| \), as well.

### 2.2. Second-order finite difference schemes

According to the splitting, in each time interval \( (t_n, t_{n+1}) \) we discretise the temporal derivatives in (9) and (10) as

\[
\frac{\partial T}{\partial t} \bigg|_{t=t_n} \approx \frac{T_{n+1}^{m} - T_n^{m}}{\tau},
\]

and

\[
\frac{\partial T}{\partial t} \bigg|_{t=t_n+\frac{1}{2}} \approx \frac{T_{n+1}^{m+\frac{1}{2}} - T_n^{m+\frac{1}{2}}}{\tau},
\]

respectively. In its turn, for the spatial derivatives we take the second-order central finite difference stencils—

\[
A_{\lambda} T \big|_{\lambda=\lambda_k} \approx A_{\lambda \lambda} T \big|_{\lambda=\lambda_k} \approx \frac{1}{R^2 \cos^2 \varphi} \frac{D_{k+1/2} T_{k+1} - D_{k-1/2} T_{k-1}}{\Delta \lambda},
\]

and

\[
A_{\varphi} T \big|_{\varphi=\varphi_l} \approx A_{\varphi \varphi} T \big|_{\varphi=\varphi_l} \approx \frac{1}{R^2 |\cos \varphi|} \frac{(D |\cos \varphi|)_{l+1/2} T_{l+1} - (D |\cos \varphi|)_{l-1/2} T_{l-1}}{\Delta \varphi},
\]

Fig. 1. Shift of the original grid (dotted lines) a half step in \( \varphi \) (solid lines) allows excluding the pole singularities, thereby keeping the equations to have sense on the entire sphere. The selected black parallel on the sphere corresponds to that on the plane.
where

\[
D_{k+p/2} := \frac{D_{k+(p+1)/2} + D_{k+(p-1)/2}}{2} \quad \text{for } p = \pm 1,
\]

\[
(D \cos \varphi)_{k+p/2} := \frac{(D \cos \varphi)_{k+(p+1)/2} + (D \cos \varphi)_{k+(p-1)/2}}{2} \quad \text{for } p = \pm 1.
\]

(In order not to overload the formulas, the nonvarying index \(l\) in (15) and the nonvarying index \(k\) in (16) are omitted.) The external forcing \(f\) is computed at the intermediate time moment \(t_{n+1/2}\), i.e. \(f_{n+1/2} = f(\lambda', \varphi, \frac{k-1}{C_0})\).

The finite difference operators in (15) and (16) are correct in the sense of keeping the property of negative definiteness (cf. (6)). Indeed, for (15), multiplying the right-hand side by \(T_k\) (recall, \(l\) is fixed while computing in \(\lambda\)) and summing all over the \(k\)'s, we obtain

\[
\sum_k \frac{1}{R^2 \cos^2 \varphi_j} \frac{D_{k+1/2} T_{k+1/2} - D_{k-1/2} T_{k-1/2}}{\Delta \lambda} T_k = \frac{1}{R^2 \cos^2 \varphi_j \Delta \lambda^2} \left( \sum_k D_{k+1/2} (T_{k+1} - T_k) T_k - \sum_k D_{k-1/2} (T_k - T_{k-1}) T_k \right)
\]

\[
= \frac{1}{R^2 \cos^2 \varphi_j \Delta \lambda^2} \left( \sum_k D_{k+1/2} (T_{k+1} - T_k) T_k - \sum_{k'} D_{k'+1/2} (T_{k'+1} - T_{k'}) T_{k'+1} \right) = \frac{1}{R^2 \cos^2 \varphi_j \Delta \lambda^2} \sum_k D_{k+1/2} (T_{k+1} - T_k) (T_k - T_{k+1})
\]

\[
= - \frac{1}{R^2 \cos^2 \varphi_j \Delta \lambda^2} \sum_k D_{k+1/2} (T_{k+1} - T_k)^2 \leq 0.
\]

Here we denoted \(k' = k - 1\) and used the periodicity of the solution in \(\lambda\) on \(S_{\Lambda, \Delta \lambda, \lambda}^{(1)}\) to return to the original index \(k\). Analogously, for (16), due to the periodicity of the solution in \(\varphi\) on \(S_{\Lambda, \Delta \lambda, \varphi}^{(2)}\) we have

Fig. 2. The map swap, possible due to the splitting, allows using the periodic boundary condition in \(\varphi\), avoiding the question of constructing boundary conditions at the poles. The selected black meridian on the sphere corresponds to that on the plane.
Since both the approximation and stability are fulfilled, the discrete solutions to (28)–(31) converge to the solutions to the corresponding 1D differential problems. Hence, due to the periodic boundary conditions in $\lambda$ and in $\varphi$, provided by the map swap, the eventual 2D solution to (28)–(31) converges to the solution to the unsplit linearised differential problem. The issue
of convergence of the numerical solution to the original nonlinear problem (1) will be addressed in the next section (see Experiment 3).

Aside from being dissipative, the constructed schemes are also balanced, according to (4). Specifically, multiplying the left-hand side of (24) by $\tau R^2 \Delta \lambda \Delta \phi \cos \phi_I$, we have

$$
\begin{align*}
& \left( -T_{k+1}^{n+1} m_{k+1} + T_k^{n+1} \left( \frac{1}{\tau} + m_k \right) - T_{k-1}^{n+1} m_{k-1} \right) \tau R^2 \Delta \lambda \Delta \phi \cos \phi_I \\
& = - \left( T_{k+1/2}^{n+1} D_{k+1/2} + T_{k-1/2}^{n+1} D_{k-1/2} \right) \tau R^2 \Delta \lambda \Delta \phi \cos \phi_I - \frac{1}{2R^2 \cos^2 \phi_I \Delta \lambda^2} + T_k^{n+1} (D_{k+1/2} + D_{k-1/2}) \tau R^2 \Delta \lambda \Delta \phi \cos \phi_I \\
& \times \frac{1}{2R^2 \cos^2 \phi_I \Delta \lambda^2} + T_k^{n+1} R^2 \Delta \lambda \Delta \phi \cos \phi_I.
\end{align*}
$$

(33)

Summing all over the $k, l$'s and using the periodicity of the solution in $\lambda$, we find the terms with $D_{k+1/2}$ and $D_{k-1/2}$ vanish, and there only remains $R^2 \Delta \lambda \Delta \phi \sum_i \cos \phi_I \sum_{l} T_{kl}^{n+1/2}$. Doing the same with the right-hand side of (24), we shall eventually obtain

$$
R^2 \Delta \lambda \Delta \phi \sum_i \cos \phi_I \sum_k T_{kl}^{n+1/2} = \tau \sum_i \cos \phi_I \sum_k T_{kl}^{n+1/2},
$$

(34)

which is the discrete analogue of the balance Eq. (4). Calculations for (26) and (27) are similar. Under $f = 0$ we get the mass conservation law (cf. (5))

$$
R^2 \Delta \lambda \Delta \phi \sum_i \cos \phi_I \sum_k T_{kl}^{n+1/2} = R^2 \Delta \lambda \Delta \phi \sum_i \cos \phi_I \sum_k T_{kl}^{n+1/2}.
$$

(35)

It is to emphasise that all the constructed finite difference schemes are systems of linear algebraic equations with tridiagonal positive definite matrices. Hence, the solution can easily be computed by the Sherman–Morrison formula [29].

2.3. Fourth-order finite difference schemes

Due to the map swap (11) and (12) and periodic boundary conditions in $\lambda$ and in $\phi$ we can involve, generally speaking, arbitrary order finite difference approximations when discretising the differential operators $A_{\lambda}, A_{\phi}$ in (9) and (10). In particular, applying the standard fourth-order stencils [30] to

$$
\frac{\partial}{\partial \lambda} \left( D \frac{\partial T}{\partial \lambda} \right) = \frac{\partial D}{\partial \lambda} \frac{\partial T}{\partial \lambda} + D \frac{\partial^2 T}{\partial \lambda^2},
$$

(36)

$$
\frac{1}{\cos \phi \frac{\partial}{\partial \phi}} \left( D \cos \phi \frac{\partial T}{\partial \phi} \right) = \left( \frac{\partial D}{\partial \phi} - D \frac{\partial \tan \phi}{\partial \phi} \right) \frac{\partial T}{\partial \phi} + D \frac{\partial^2 T}{\partial \phi^2},
$$

(37)

we shall eventually obtain the fourth-order finite difference scheme in $\lambda$

$$
T_{k+1}^{n+1} m_{k+1} + T_k^{n+1} \left( \frac{1}{\tau} + m_k \right) - T_{k-1}^{n+1} m_{k-1} \times \frac{1}{2R^2 \cos^2 \phi_I \Delta \lambda^2} + T_k^{n+1} R^2 \Delta \lambda \Delta \phi \cos \phi_I.
$$

(38)

where

![Fig. 3. Experiment 1: graph of the $L_2$-norm of the second-order solution.](image-url)
Fig. 4. Experiment 1: second-order numerical solution at several time moments.
\[ m_k = \frac{30D_k}{24R^2 \cos^2 \varphi \Delta \lambda^2}, \]
\[ m_{k+p} = \frac{1}{R^2 \cos^2 \varphi_1} \left( \frac{16D_k}{24\Delta \lambda^2} + \text{sgn}(p) \frac{8M_k}{24\Delta \lambda} \right) \text{ for } p = \pm 1, \]
\[ m_{k+p} = \frac{1}{R^2 \cos^2 \varphi_1} \left( \frac{D_k}{24\Delta \lambda^2} + \text{sgn}(p) \frac{M_k}{24\Delta \lambda} \right) \text{ for } p = \pm 2, \]
\[ M_k = \frac{-D_{k+2} + 8D_{k+1} - 8D_{k-1} + D_{k-2}}{12\Delta \lambda}, \]

and in \( \varphi \)

\[ \tau_{i+2}^{n+1} m_{i+2} - \tau_{i+1}^{n+1} m_{i+1} + T_i^{n+1} \left( \frac{1}{\tau} + m_t \right) - \tau_{i-1}^{n+1} m_{i-1} + T_i^{n+1} m_{i-2} \]
\[ = -\tau_{i+1}^{n+1} m_{i+1} + T_i^{n+1} m_{i+1} + T_i^{n+1} \left( \frac{1}{\tau} - m_t \right) + \tau_{i-1}^{n+1} m_{i-1} - \tau_{i+1}^{n+1} m_{i-2} + \frac{\rho_{n+1}^{i+1}}{2}, \]  \( (40) \)

where

\[ m_t = \frac{30D_t}{24R^2 \Delta \varphi^2}, \]
\[ m_{t+p} = \frac{1}{R^2} \left( \frac{16D_t}{24\Delta \varphi^2} + \text{sgn}(p) \frac{8M_t}{24\Delta \varphi} \right) \text{ for } p = \pm 1, \]
\[ m_{t+p} = \frac{1}{R^2} \left( \frac{D_t}{24\Delta \varphi^2} + \text{sgn}(p) \frac{M_t}{24\Delta \varphi} \right) \text{ for } p = \pm 2, \]
\[ M_t = \frac{-D_{t+2} + 8D_{t+1} - 8D_{t-1} + D_{t-2}}{12\Delta \varphi}, \]

Using the bicyclic splitting (28)–(31), we also achieve the second approximation order in time.

Analysis of the properties of the fourth-order schemes in space is more cumbersome than that done above for the second order.

3. Numerical experiments

To verify the developed schemes we have simulated a few diffusion phenomena. We used grids \( 2^4 \times 2^4, 4^4 \times 4^4 \) and \( 6^4 \times 6^4 \); below, if not explicitly mentioned, the latter grid is assumed. Throughout the experiments we employed the Strang splitting that provides second-order accuracy in time.

3.1. Experiment 1—The simplest diffusion problem

First of all we considered the simplest case—linear diffusion \((a = 0)\) with a constant factor \(\mu\) in the absence of sources \((f = 0)\). As the initial condition we took a smooth function in the shape of a spot located at \(\lambda = \frac{\pi}{6}, \varphi = \frac{\pi}{4}\). The diameter of

![Fig. 5. Experiment 2: profile of the diffusion amplification factor (42) (North pole view).](image-url)
the spot was chosen such that the latter did not cover the North pole. The aim of this experiment was to demonstrate the correctness of the method in the sense of accurate simulation of the diffusion process near the pole: since the initial condition is symmetric and no external forcing is present, it is evident to expect a symmetric spread of the spot over the sphere.
This would prove that the presence of the pole singularities caused by the convergence of meridians in the lat-log coordinate system does not affect the diffusion process near the poles and that the splitting on the sphere and the map swap (11) and (12) are mathematically correct.

In Fig. 3 we plot the temporal graph of the \( L_2(\mathbb{S}^{(1)}_{\lambda,A_\phi}) \)-norm of the second-order numerical solution, while in Fig. 4 we show the numerical solution at a few time moments (the exact solution was expanded into a series of spherical harmonics [30]; it looked quite similar to the numerical solution, so we omit it). As it is seen, the solution’s \( L_2 \)-norm is monotonically decaying, according to (22) and (23), whereas the spot is propagating uniformly all over the sphere and no any nonphysical effects (e.g., the mass accumulation near the pole, etc.) are observed.

3.2. Experiment 2—Variable diffusion coefficient

Consider a more complicated problem. Let

\[
\mu(\lambda, \phi) \sim \sin^4 2\lambda \sin^2 \phi, \quad (42)
\]

while the initial condition be the same spot placed exactly on the North pole. Since all the directions from the North pole are equivalent, the spot is expected to spread according to the diffusion amplification factor’s profile (Fig. 5). In [20] a similar linear problem was concerned briefly, while in the current paper the process is complicated by introducing a nonlinearity with \( \alpha = 1 \).

The fourth-order (in space) numerical solution is shown in Fig. 6 (we used a nonuniform colour map for better visualisation). One can see that the initial spot is propagating as it should be doing, consistent with the diffusion coefficient’s profile (42). The temporal graph of the solution’s \( L_2 \)-norm shown in Fig. 7 demonstrates the property of the solution’s dissipation.

3.3. Experiment 3—Large gradients near the poles

This experiment was performed to test the schemes on a function with large gradients in order to make sure that the developed method allows accurate modelling of phenomena which imply nonsmooth solutions. The schemes were tested on a series of grids \( 6 \times 6, 4 \times 4 \) and \( 2 \times 2 \).

The numerical solution was compared with the analytical one using the relative error

\[
\delta^\alpha \equiv \delta(t_n) = \frac{|| T^{\text{num}}(t_n) - T^{\text{exact}}(t_n) ||_{L_2(\mathbb{S}^{(1)}_{\lambda,A_\phi})}}{|| T^{\text{exact}}(t_n) ||_{L_2(\mathbb{S}^{(1)}_{\lambda,A_\phi})}}. \quad (43)
\]

As the exact solution we chose the function

\[
T(\lambda, \phi, t) = c_1 \sin \xi \cos \phi \cos^2 t + c_2, \quad (44)
\]

where

\[
\xi \equiv \omega(\lambda - \vartheta_1 \tan \kappa_1 \phi + \vartheta_2 \tan \kappa_2 \phi \sin \gamma t.
\]

The source function resulted to be

\[
f(\lambda, \phi, t) = \frac{\partial T}{\partial t} - \frac{\mu T_{x,1}^{-1} R}{R \sin \phi} \left[ \frac{1}{R \sin \phi} \left( \frac{\partial T}{\partial \lambda} \right)^2 + \frac{\cos \phi}{R} \left( T \left( \frac{\partial^2 T}{\partial \phi^2} - \tan \phi \frac{\partial T}{\partial \phi} \right) + \omega \left( \frac{\partial T}{\partial \phi} \right)^2 \right) \right]. \quad (46)
\]

where

\[
\frac{\partial T}{\partial \lambda} = c_1 \cos \phi (\vartheta_2 \gamma \cos \xi \tan \kappa_2 \phi \cos \gamma t \cos^2 t - \sin \xi \sin 2t),
\]

\[
\frac{\partial T}{\partial \phi} = c_1 \omega \cos \phi \cos^2 t \cos \xi,
\]

\[
\frac{\partial^2 T}{\partial \lambda^2} = -c_1 \omega^2 \cos \phi \cos^2 t \sin \xi,
\]

\[
\frac{\partial^2 T}{\partial \phi^2} = c_1 \cos^2 t \left( \cos \phi \cos \xi \frac{\partial \xi}{\partial \phi} - \sin \xi \sin \phi \right),
\]

\[
\frac{\partial^2 T}{\partial \phi \partial \xi} = c_1 \cos^2 t \left[ -2 \sin \phi \cos \xi \frac{\partial \xi}{\partial \phi} - \sin \xi \cos \phi \left( 1 + \left( \frac{\partial \xi}{\partial \phi} \right)^2 \right) 
\]

\[
+ 2 \cos \xi \cos \phi \left( -\omega \vartheta_1 \kappa_1^2 \sin \kappa_1 \phi \cos^3 \kappa_1 \phi + \vartheta_2 \kappa_2^2 \sin \gamma t \frac{\sin \kappa_2 \phi}{\cos^3 \kappa_2 \phi} \right) \right),
\]

\[
\frac{\partial \xi}{\partial \phi} = -\omega \frac{\vartheta_1 \kappa_1 \sin \kappa_1 \phi}{\cos^2 \kappa_1 \phi} + \sin \gamma t \frac{\vartheta_2 \kappa_2 \sin \kappa_2 \phi}{\cos^2 \kappa_2 \phi}.
\]
Solution (44) was chosen because its spatial structures convey spiral spherical waves interesting for a variety of applications [31–33]. A smooth solution of a similar form was studied in [20], while now we substantially complicate the problem introducing in (45) the term $\sin n/\sqrt{C_2^2 \tan j^2 u}$. That term has the intention to simulate rather large gradients in the

Fig. 7. Experiment 2: graph of the $L_2$-norm of the fourth-order solution.

Fig. 8. Experiment 3: graphs of the function $s(\varphi) = \sin(\varphi \tan \kappa_2 \varphi)$ at $\varphi_2 = 15. \kappa_2 = 0.1, 0.3, 0.5, 0.7, 0.9$.

Fig. 9. Experiment 3: graph of the relative error ($\text{grid } 2 \times 2$).
solution at high latitudes. Examples of such a behaviour at $\phi_2 = 15$ and $\kappa_2 = 0.1, 0.3, 0.5, 0.7, 0.9$ are given in Fig. 8. As it is seen, from $\kappa_2 = 0.1$ to 0.9 the solution is getting more oscillating, with sharp gradients.

In our experiment we assumed $c_1 = 10, c_2 = 100, \omega = 7, \phi_1 = 1, \kappa_1 = 0.5, \phi_2 = 15, \kappa_2 = 0.7, \gamma = 4, \mu = 1 \cdot 10^{-7}, x = 2$. 

Fig. 10. Experiment 3: numerical solution at several time moments.
where it is nearly constant. This perfectly matches the behaviour of the "integrated sources" (cf. (34)).

and

We took where $6$ is now equal to $2$ (clockwise); later, when the time comes to the sinusoid begins travelling to the West (clockwise) at the low latitudes, while at the high ones it is going to the East (anti-clockwise). The different behaviours of the solution are determined by the steady-state term $\sin \gamma x \cos \phi t$ instead of $\sin \gamma t \cos \phi t$. However, the dependence $\varphi = \varphi(x)$ results in a forcing function that produces two totally different behaviours of the solution $T$: the case $x = 0$ corresponds to the linear diffusion equation and a wave solution determined by the steady-state term $\sin \omega \lambda$, while the case $x \neq 0$ corresponds to various nonlinear diffusion problems with nonstationary wave solutions of the form $\sin (\omega \lambda + 5 \pi \cos \kappa \phi \sin t)$.

In Fig. 11 there are the graphs of the "mass" $T^{(1)}_k = \sum_{\lambda \in \Lambda} T^{(1)}_{\lambda \lambda \Lambda \phi}$ and the "integrated sources" $f^{(1)}_k = \sum_{\lambda \in \Lambda} f^{(1)}_{\lambda \lambda \Lambda \phi}$ in time for $x = 3$. The "mass" looks as a decaying function of time almost everywhere, except, roughly, $t \in [1.3, 1.7]$ and $t \in [4.5, 4.9]$, where it is nearly constant. This perfectly matches the behaviour of the "integrated sources" (cf. (34)).

In Fig. 12 we show the fourth-order numerical solution at $x = 3$ (note that unlike the previous experiment the periodicity is now equal to $2\pi$, not $\pi$, due to $\sin t$ in (49)). The solution is behaving in accordance with the term $\sin \xi$: as the time grows, the sinusoid begins travelling to the West (clockwise) at the low latitudes, while at the high ones it is going to the East (anti-clockwise); later, when the time comes to $t = \frac{\pi}{2}$, the direction of rotation is changed to the opposite one, etc.

Numerical solutions related to different $x$'s are shown in Fig. 13. As it is seen, the higher the degree of nonlinearity, the more complicated structure of the solution, especially when the phase $\varphi \cos \kappa \phi \sin t$ in (49) achieves the maximum at $t \approx \frac{\pi}{2}$ and $\frac{3\pi}{2}$. In fact, in the strongly nonlinear cases ($x = 2, 3$) the interaction between the three basic mechanisms—the nonlinearity, forcing and dissipation—generates solutions with periodic breakups and renewals of complex structures via self-organisation.

In Table 1 we summarise the maximum relative errors $\delta^{(n)}_{\text{max}}$'s, corresponding to different values of $x$. As expected, the fourth-order schemes yield more precise solutions.
3.5. Experiment 5—Modelling of combustion

The phenomenon of combustion was studied in [8] on the real line as a Cauchy problem described by the nonlinear diffusion equation with an appropriate parameter \( \alpha \) and a source function \( f \). We tested the developed method having simulated this phenomenon within a bounded area on a sphere. This problem has an interesting application in metal flaming and burning under the laser action [34].

For the initial condition we took a spot with the centre at \( \lambda = \varphi = \frac{\pi}{4} \). We also had \( \mu = 1 \cdot 10^{-3} \), \( \alpha = 1 \) and \( f = cT^\beta \), where \( c = 4.1 \). The parameter \( \beta > 0 \) introduces an additional, aside from \( \alpha \), non-linearity, now with respect to the sources, and determines two different regimes of combustion: the case \( \beta < \alpha + 1 \) corresponds to the HS-regime—the area of combustion is getting expanded while burning; the case \( \beta > \alpha + 1 \) corresponds to the LS-regime—the area of combustion is getting narrow [8].

![Fig. 12. Experiment 4: fourth-order numerical solution at several time moments (\( \alpha = 3 \)).](image-url)
The point is that in both cases the given source leads to an infinite increase of the temperature $T$ while the time grows, that is a blow-up occurs [8].

In Fig. 14 we present the numerical solutions related to $\beta = 1$ (HS-regime) and $\beta = 4$ (LS-regime). In Fig. 15 we also plot the $L_2$-norms of the second-order numerical solutions. The experiment confirms the theory: in the HS-regime we observe the increase of the area of combustion, while in the LS-regime the area is reducing to a singularity. The graphs of the energy prove the solutions tend to infinity, that is a blow-up happens. Further analysis allows to conclude that in the HS-regime the entire character of the blow-up asymptotics is slower than that in the LS-regime—the curve of the solution’s $L_2$-norm at $\beta = 1$ is growing bit by bit all the time from $t = 0$ to $t = 1$, while at $\beta = 4$ it is going up slowly only up to $t \approx 0.1$, and

![Fig. 13. Experiment 4: fourth-order numerical solutions at several time moments: $\alpha = 0$ (top), $\alpha = 1$ (middle), $\alpha = 2$ (bottom).](image)

![Table 1](table)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta_{\text{max}}$</th>
<th>2nd order</th>
<th>4th order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$4.650 \cdot 10^{-3}$</td>
<td>$6.587 \cdot 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$3.639 \cdot 10^{-3}$</td>
<td>$5.397 \cdot 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1.314 \cdot 10^{-2}$</td>
<td>$7.939 \cdot 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$3.496 \cdot 10^{-2}$</td>
<td>$2.591 \cdot 10^{-2}$</td>
<td></td>
</tr>
</tbody>
</table>
it rapidly changes the growth rate after that. In its turn, the growth of the combustion area observed in Fig. 14 is smooth at $\beta = 1$ all the time, whereas it is much sharper at $\beta = 4$ after $t \approx 0.1$ (cf., e.g., $t = 0.09$ and $t = 0.11$).

Fig. 14. Experiment 5: second-order numerical solutions at several time moments for HS-regime (first two rows) and LS-regime (last two rows).
4. Conclusions

A new method for the numerical solution of nonlinear diffusion equations on a sphere has been developed. Due to the splitting of the differential operator by coordinates and the use of different coordinate maps for the sphere at two split time intervals, both one-dimensional split problems obey periodic boundary conditions. Thus, although the sphere is not a doubly periodic manifold, our method allows to construct a numerical algorithm for the sphere in the same way as for a torus. Hence, the method does not require to impose any artificial boundary conditions at the poles (which, in addition, must be physically and mathematically correct)—and it must be so indeed, since the sphere has no boundaries and preferred points, and so all directions on the sphere are equivalent. This resulted in constructing second- and fourth-order finite difference schemes in space without using computationally cumbersome numerical procedures for computing the numerical solution at the poles. The schemes keep the properties required by the original differential problem: they appear balanced and dissipative, thus providing physically adequate numerical solutions. The schemes are also computationally inexpensive and can be solved by direct band linear solvers. The numerical experiments proved the approach to allow accurate simulating diverse diffusion phenomena with constant and variable diffusion coefficients on the entire sphere, including those that imply high nonlinearity, nonsmoothness and infinite growth of the solutions. The accuracy of the method was evaluated by comparing the numerical results with the analytical solutions obtained under forcings of special types. The convergence of the numerical solution to the analytics was verified on a series of spatial grids 6/4, 4/4, 2/2. As an interesting nonlinear effect, it was also constructed an example showing that the competition of the three basic mechanisms of the model—the nonlinear interaction, external forcing and dissipation—generates a wave solution, whose spatial structure varies periodically due to the alternating influence of the processes of self-organisation and self-destruction.

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References


Fig. 15. Experiment 5: graphs of the $L_2$-norm of the second-order solutions in time for HS-regime (left) and LS-regime (right).