Paths and animals in infinite graphs with tempered degree growth

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1. Introduction

Infinite graphs are used in probabilistic combinatorics, image processing, and many other domains. In particular, they serve as the underlying discrete metric spaces for Markov random fields [7,12,16,19–21,27,28]. Various problems of analysis on such graphs are also studied extensively, see, e.g., [9,18]. The structure of infinite graphs is more accessible to study if the vertex degrees are globally bounded. However, in many important applications it is essential to employ unbounded degree graphs. Often, typical realizations of random graphs have this property. For instance, in standard models of continuum percolation [14,25,26], such as the Poisson blob model, Gilbert graph, or the random connection model, the underlying graphs almost surely are of unbounded degree.

For unbounded degree graphs, it is intuitively clear that their global metric properties can be similar to those of the bounded degree graphs if the vertices with large degree are ‘sparse’, see, e.g., Introduction in [28]. In Definition 1 below, we introduce the notion of tempered growth of vertex degrees. As we then show, this property characterizes the so called ‘repulsive graphs’, in which vertices with large degree ‘repel’ each other in the sense of Definition 2. Such graphs naturally appear in the theory of Markov random fields [3,16,19–21], in particular, in the attempts to construct Euclidean quantum fields on curved manifolds [2]. We show that the assumed tempered growth implies exponential upper bounds for the number of connected subgraphs of order N which contain a given vertex, valid for large N. These results allow for obtaining similar estimates also for other metric characteristics, e.g., for the number of vertices in a ball of radius N.

In Section 2, we present necessary notions and notations. Here we introduce the notion of tempered graph by imposing restrictions on the vertex degree growth, see Definition 1. Next, in Definition 2 we introduce two classes of repulsive graphs...
and formulate our main results in Theorems 3–5, and then in Corollaries 6–8. In Section 3, we describe some applications of these results. Among them we note an upper estimate for the generalized Randić index (Proposition 11) and an almost sure sublinear growth of weights of greedy graph animals (Proposition 13). In Section 4, we study the properties of paths and animals in our graphs, which is then used in Section 5 where we prove the statements of Section 2. Particular cases of some results of Section 2 repeat the results obtained for repulsive graphs in [3,21], which is indicated when appropriate.

2. Setup and results

Let $G = (V, E)$ be a countably infinite simple graph. By writing $x \sim y$ we mean that $x, y \in V$ constitute an edge, $(x, y) \in E$. We say that such $x$ and $y$ are adjacent and that they are the endpoints of the edge $(x, y)$. For each $x \in V$, the degree

$$n(x) \defeq \# \{ y \in V : y \sim x \}$$

is assumed to be finite, whereas

$$n_G \defeq \sup_{x \in V} n(x),$$

can be finite or infinite. A finite connected subgraph, $A \subseteq G$, is called an animal (also a polymer, cf. [12,22,27]). By $V(A)$ and $E(A)$ we denote the set of vertices and edges of $A$, respectively. A path, $\vartheta$, is a finite sequence of vertices, $[x_0, x_1, \ldots, x_n]$, not necessarily distinct, such that $x_k \sim x_{k+1}$ for all $k = 0, \ldots, n - 1$. Then $\vartheta$ originates at $x_0$ and terminates at $x_n$. Its length $|\vartheta|$ is set to be $n$. In a simple path, all $x_0, x_1, \ldots, x_{n-1}$ are distinct. By $G_\vartheta$ we denote the graph generated by $\vartheta$. That is, its vertex set $V_\vartheta$ consists of those in $\vartheta$, not counting repeated vertices; the edge set $E_\vartheta$ comprises the edges with both endpoints in $V_\vartheta$. Clearly, each $G_\vartheta$ is an animal.

By $\vartheta(x, y)$ we denote a path such that $x_0 = x$ and $x_n = y$. The path distance $\rho(x, y)$ is set to be the length of the shortest path $\vartheta(x, y)$. A ball $B_N(x)$ (resp., a sphere $S_N(x)$), $N \in \mathbb{N}$ and $x \in V$, is the set of $y \in V$ such that $\rho(x, y) \leq N$ (resp., $\rho(x, y) = N$). For $N \in \mathbb{N}$ and $x \in V$, let $\mathcal{A}_N(x)$ denotes the set of all animals such that $x \in V(A)$ and $|V(A)| = N$. For such $x$ and $N$, let also $\mathcal{S}_N(x)$ be the set of all simple paths of length $N$ originated at $x$. In many applications, see [10–12,17,22,24,27], one needs to estimate the growth of the cardinalities of the mentioned sets as $N \to +\infty$. For a graph $G$ with $n_G < \infty$, there exist positive $q_0, q_G$, and $N_G$ such that the following estimates hold

$$(a) \quad |\mathcal{S}_N(x)| \leq q_G^N, \quad (b) \quad |\mathcal{A}_N(x)| \leq \tilde{q}_G^N,$$

for all $N \geq N_G$. Clearly, $q_0 \leq q_G < 1$. The first estimate can easily be proven to hold with $q_0 = n_G$ and $N_G = 1$, cf. (25) below. The second one is not so immediate, see [23, Chapter 2] where a more general estimate was derived. Note that (b) implies (a). By (a) in (2) one readily gets

$$(a) \quad |\mathcal{S}_N(x)| \leq q_G^N, \quad (b) \quad |B_N(x)| \leq \frac{q_G}{q_G - 1} q_G^N.$$

For graphs with $n_G = +\infty$, the cardinalities in (2) can grow faster than exponentially. Furthermore, if (a) or (b) holds for $N \geq N_k$ with one and the same $N_k$ for all $x \in V$, then $n_k < \infty$. In this work we introduce two classes of graphs with $n_G = +\infty$ for which estimates as in (3) hold for $N$ belonging to infinite sets $\mathcal{K}_k \subseteq \mathbb{N}$. For the first class, $\mathcal{K}_k$ is a strictly increasing sequence $\{N_k\}_{k \in \mathbb{N}}$. The second class comprises graphs with $\mathcal{K}_k = \{N \in \mathbb{N} : N \geq N_k\}$.

For an increasing function $g : \mathbb{N} \to (0, +\infty)$ and an animal $A$, we set

$$G(A; g) = \frac{1}{|V(A)|} \sum_{x \in V(A)} g(n(x)),$$

which can be viewed as the empirical mean value of $g(n)$ on $A$. If $n_G < \infty$, see (1), then $G(A; g) \leq g(n_G)$ for any animal and any function $g$. We say that the vertex degree in $G$ is of tempered growth if

$$\max_{A \in \mathcal{A}_N(x)} G(A; g) \leq \gamma_N,$$

for $N$ belonging to an infinite set $\mathcal{K}_k$. More precisely, we mean the following.

**Definition 1.** The graph $G$ is said to be $g$-tempered (resp. strongly $g$-tempered) if there exists a number $\gamma_N > 0$ such that, for every $x \in V$, there exists a strictly increasing sequence $\{N_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ (resp. there exists $N_k \in \mathbb{N}$) such that (4) holds for all $N = N_k$, $k \in \mathbb{N}$ (resp. for all $N \geq N_k$).

For $x, y \in V$, we set

$$m_+(x, y) = \max\{n(x); n(y)\}, \quad m_-(x, y) = \min\{n(x); n(y)\}.$$

As is shown in Theorem 5 below, the following two families of graphs have the properties just defined.

**Definition 2.** Let $\phi : \mathbb{N} \to (0, +\infty)$ be strictly increasing. By $G_\phi(\phi)$ we denote the family of graphs, for each of which there exists a positive integer $n_\phi$ such that

$$\rho(x, y) \geq \phi(m_+(x, y)), \quad \rho(x, y) \geq \phi(m_-(x, y)),$$

whenever $m_-(x, y) > n_\phi$. No restrictions are imposed if $m_-(x, y) \leq n_\phi$. 

Clearly,
\[ G_+(\phi) \subset G_-(\phi). \]

For \( G \in G_2(\phi) \), by \((5)\) vertices of large degrees ‘repel’ each other. That is why they are called repulsive. For the first time, graphs characterized by the property as in \((5)\) with \( m_- \) appeared in \([3]\), where a spin system with random unbounded interactions on a bounded degree graph was converted into a system with bounded interactions on a graph with such a ‘repulsion’. Graphs as in \((5)\) with \( m_+ \) were introduced in \([21]\) as the underlying graphs for Gibbs random fields.

For a graph as in Definition 2, \( G = (V, E) \), we set
\[
V_+ = \{ x \in V \mid n(x) \leq n_+ \}, \quad V^c_+ = V \setminus V_+.
\]
and consider
\[
K(x) = \{ y \in V \mid \rho(y, x) < \phi(n(x)) \}.
\]

Now let \( G \) be in \( G_+(\phi) \). For \( x \in V_+ \), by \((5)\) we have \( K(x) \cap V_+^c = \{ x \} \) i.e., \( x \) ‘repels’ all vertices \( y \in V_+^c \) from \( K(x) \), cf. \([21, p. 870]\).

For \( G \in G_-(\phi) \), \( x \) ‘repels’ from \( K(x) \) only those \( y \in V_+^c \), for which \( n(y) \geq n(x) \). For the sake of convenience, we shall assume that \( K(x) \) contains the neighborhood of \( x \), which is equivalent to assuming
\[
\phi(n_+ + 1) > 1,
\]
holding for all \( G \in G_+(\phi) \).

**Theorem 3.** For \( g(t) = t \log t, \ t \in \mathbb{N} \), let \( G \) be \( g \)-tempered. Then there exists \( q_G > 1 \) such that, for any \( x \in V \), there exists a strictly increasing sequence \( \{ N_k \}_{k \in \mathbb{N}} \subset \mathbb{N} \) such that the estimate
\[
|A_N(x)| \leq q_G^N
\]
holds for all \( N = N_k, \ k \in \mathbb{N} \). If \( G \) is strongly \( g \)-tempered, then for any \( x \in V \), there exists \( N_x \in \mathbb{N} \) such that \((8)\) holds for all \( N \geq N_x \).

**Theorem 4.** For \( g(t) = \log t, \ t \in \mathbb{N} \), let \( G \) be \( g \)-tempered. Then there exists \( \tilde{q}_G > 1 \) such that, for any \( x \in V \), there exists a strictly increasing sequence \( \{ N_k \}_{k \in \mathbb{N}} \subset \mathbb{N} \) such that the estimate
\[
|\Sigma_N(x)| \leq \tilde{q}_G^N
\]
holds for all \( N = N_k, \ k \in \mathbb{N} \). If \( G \) is strongly \( g \)-tempered, then for any \( x \in V \), there exists \( \bar{N}_x \in \mathbb{N} \) such that the estimate \((9)\) holds for all \( N \geq \bar{N}_x \).

**Theorem 5.** Let the functions \( g \) and \( \phi \) be such that, for some strictly increasing sequence \( \{ t_k \}_{k \in \mathbb{N}} \subset \mathbb{N} \), the following holds
\[
\sum_{k=1}^{\infty} \frac{g(t_{k+1})}{\phi(t_k)} < \infty.
\]
Then any \( G \in G_+(\phi) \) (resp. any \( G \in G_-(\phi) \)) is \( g \)-tempered (resp. strongly \( g \)-tempered).

**Corollary 6** below readily follows from Theorems 3 and 5 with \( g(t) = t \log t \). Likewise, **Corollary 7** follows from **Theorem 4** with \( g(t) = \log t \).

**Corollary 6.** Let \( \phi : \mathbb{N} \to (0, +\infty) \) be such that the following holds
\[
\sum_{k=1}^{\infty} \frac{t_{k+1} \log t_{k+1}}{\phi(t_k)} < \infty,
\]
for some strictly increasing sequence \( \{ t_k \}_{k \in \mathbb{N}} \subset \mathbb{N} \). Then, for each \( G \in G_+(\phi) \), there exists \( \tilde{q}_G > 1 \) such that, for any \( x \in V \), there exists a strictly increasing sequence \( \{ N_k \}_{k \in \mathbb{N}} \subset \mathbb{N} \) such that the estimate
\[
|A_N(x)| \leq \tilde{q}_G^N
\]
holds for all \( N = N_k, \ k \in \mathbb{N} \). If \( G \in G_+(\phi) \), then, for any \( x \in V \), there exists \( \bar{N}_x \in \mathbb{N} \) such that the estimate \((12)\) holds for all \( N \geq \bar{N}_x \).

**Corollary 7.** Let \( \phi : \mathbb{N} \to (0, +\infty) \) be such that the following holds
\[
\sum_{k=1}^{\infty} \frac{\log t_{k+1}}{\phi(t_k)} < \infty,
\]
for some strictly increasing sequence \( \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \). Then, for each \( G \in \mathcal{G}_-(\phi) \), there exists \( q_G > 1 \) such that, for any \( x \in V \), there exists a strictly increasing sequence \( \{N_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \) such that the estimate

\[
|\Sigma_N(x)| \leq q_G^N \tag{14}
\]

holds for all \( N = N_k \), \( k \in \mathbb{N} \). If \( G \in \mathcal{G}_+(\phi) \), then, for each \( x \in V \), there exists \( N_x \in \mathbb{N} \) such that the estimate (14) holds for all \( N \geq N_x \).

Note that, for graphs from \( \mathcal{G}_-(\phi) \), the results of Corollary 7 were obtained in [3, Proposition 1]. An immediate corollary of Theorem 4 is also the following statement.

**Corollary 8.** Let \( G \) be in \( \mathcal{G}_+(\phi) \) with \( \phi \) obeying (13). Let also \( N_x \), \( x \in V \), be as in Corollary 7. Then there exists \( B_x > 0 \) such that, for all \( N > N_x \), the following holds

\[
(a) \ |S_N(x)| \leq q_G^N, \quad (b) \ |B_N(x)| \leq B_x q_G^N. \tag{15}
\]

**Proof.** By the very definition of \( S_N(x) \), we have that \( |S_N(x)| \leq |\Sigma_N(x)| \), which yields (a) in (15), whereas (b) with \( B_x = |B_N(x)|/q_G^N \) follows by (a). \( \square \)

The optimal choice of \( \{t_k\}_{k \in \mathbb{N}} \) in (13) seems to be \( t_k = \exp(e^k) \), for big enough \( k \). Then the choice of \( \phi \) can be \( \phi(t) = v \log t (\log \log t)^{1+\epsilon}, \epsilon > 0 \); cf. [21, Theorem 4].

### 3. Applications

#### 3.1. Percolation

Let \( G = (V, E) \) be as above. For \( E' \subset E \), we set \( G' = (V, E') \). Note that \( G' \) need not be connected. Let now edges \( e \in E' \) be picked at random, independently and with the same probability \( p \) each. This defines a probability measure, \( \mu^p_{\mathcal{E}} \), on the set of all subsets of \( E \). The corresponding subgraph \( G' \) with randomly picked \( e \in E' \) is random as well. The event that it has an infinite connected component (called also cluster) occurs with probability either zero or one, dependent on the value of \( p \). This is the Bernoulli bond percolation model, cf. [15,16].

**Proposition 9.** Let \( \phi \) obey (13) and \( G \) be in \( \mathcal{G}_-(\phi) \), so that (14) holds. Then no cluster appears \( \mu^p_{\mathcal{E}} \)-almost surely whenever \( p < 1/q_G \).

**Proof.** Given \( x \in V \), the probability that there exists at least one simple path of length \( N \) originated at \( x \) does not exceed \( p^N |\Sigma_N(x)| \). Then the proof follows by (14) and the Borel–Cantelli lemma, see, e.g., [15, pages 15, 19]. \( \square \)

Now for \( V' \subset V \), let \( E' \subset E \) comprise the edges with both endpoints in \( V' \). Set \( G' = (V', E') \). Further, suppose that each vertex of \( V' \) is picked at random, independently and with the same probability \( p \) each. This defines a probability measure, \( \mu^p_{\mathcal{V}} \), on the set of all subsets of \( V \). Thereby, the subgraph \( G' \) is random. The event that it has a cluster occurs with probability either zero or one, dependent on \( p \). The appearance of a cluster is called the Bernoulli site percolation, see [15, Chapter 3] or [16].

**Proposition 10.** Let \( \phi \) obey (11) and \( G \) be in \( \mathcal{G}_-(\phi) \), so that (12) holds. Then no cluster appears \( \mu^p_{\mathcal{V}} \)-almost surely whenever \( p < 1/q_G \).

**Proof.** Given \( x \in V \), the probability that there exists at least one connected subgraph of order \( N \), which contains \( x \), does not exceed \( p^N |\Sigma_N(x)| \). Then the proof follows by (12) and the Borel–Cantelli lemma. \( \square \)

Further applications of the above results to models of dependent percolation, e.g., to the random cluster model, can be developed by means of cluster expansion techniques [11,12,22,23,27,28]. Applications in [19–21] to Gibbs random fields on the graphs considered here are based on the estimate in (14).

#### 3.2. Randić index

For a real \( \theta \) and an animal, \( A \), we set

\[
R^\theta(A) = \sum_{(x,y) \in E(A)} |n(x)n(y)|^\theta.
\]

For \( \theta = -1/2 \), it was introduced by M. Randić [29] as a branching index of large molecules considered as graphs. It turns out that its value is closely related to the chemical properties of the corresponding substance, see also [8,30] for further developments. Generalizations of the Randić index are used as numerical characteristics of graphs, see [5,8] and also [4] for a number of other topological indices.
For a vertex $x$ and $N \in \mathbb{N}$, we define
\[ R_N^0(x) = \max_{A \in \mathcal{A}_N(x)} R^0(A). \]
Since $|n(x)n(y)|^k \leq [x^2 + y^2]^k/2$, one has
\[ R^0(A) \leq \sum_{x \in V(A)} [n(x)]^{2^k + 1}, \tag{16} \]
which yields that, for $g(t) = t^{2^k + 1}$, the estimate
\[ R_N^0(x) \leq \gamma N \]
holds for $N = N_k$, $k \in \mathbb{N}$, if $G$ is $g$-tempered, and for $N \geq N_k$ if $G$ is strongly $g$-tempered, where $\gamma > 0$. Let $\gamma_x, \gamma_N$, and $\gamma_k$ be as above. Then there exist $C > 0$ and $N_x \in \mathbb{N}$ such that, for any $x \in V$, there exists $N_x \in \mathbb{N}$ such that
\[ \sum_{k=0}^{\infty} \frac{\mu(\Gamma_k^x \setminus \gamma)}{\mu(\Gamma_k^x \setminus \gamma)} < \infty, \tag{17} \]
for some strictly increasing sequence $\{\gamma_k\}_k \subset \mathbb{N}$. Then, for each $G \in \mathcal{G}_\theta(\phi)$, there exists $\gamma_G$ such that, for any $x \in V$, there exists a strictly increasing sequence $\{N_k\}_k \subset \mathbb{N}$ such that
\[ \sum_{k=0}^{\infty} \frac{\mu(\Gamma_k^x \setminus \gamma)}{\mu(\Gamma_k^x \setminus \gamma)} < \infty, \tag{18} \]
for all $N = N_k$, $k \in \mathbb{N}$. If $G \in \mathcal{G}_\theta(\phi)$, then for any $x \in V$, there exists $N_x \in \mathbb{N}$ such that $\gamma(x)$ holds for all $N \geq N_x$.

The proof of this statement will be given below.

### 3.3. Growth of Aut$(G)$

For a graph $G = (V, E)$, an automorphism, $\gamma$, is a bijection $V \ni x \mapsto xy \in V$ such that $x \sim y$ implies $xy \sim y\gamma$. The automorphisms constitute a group, denoted by Aut$(G)$. Assume that $V$ is given the discrete topology, and let $\mathcal{T}$ be the weakest topology on Aut$(G)$ in which the maps Aut$(G) \ni \gamma \mapsto xy \in V$ are continuous for all $x \in V$. It is known [1] that $(\text{Aut}(G), \mathcal{T})$ is a locally compact Polish group. By the local compactness, there exists a right Haar measure on Aut$(G)$, which we denote by $\mu$. For $x \in V$, the set
\[ \Gamma_x := \{ \gamma \in \text{Aut}(G) : xy = x \} \]
is the stabilizer of $x$. It is compact and open, and thus $0 < \mu(\Gamma_x) < \infty$, for all $x \in V$. Let $\Delta$ stand for a compact neighborhood of the identity of Aut$(G)$. For $n \in \mathbb{N}$, by $\Delta^n$ we denote the set of all products $\gamma_1 \gamma_2 \cdots \gamma_n$ of the elements of $\Delta$.

**Proposition 12.** Let $G$ be in $\mathcal{G}_\phi(\phi)$ with $\phi$ obeying (13), and let $\Delta$ be as above. Then there exist $C > 0$ and $N_x \in \mathbb{N}$ such that, for all $N \geq N_x$, the following holds
\[ \mu(\Delta^N) \leq Cq_0^N, \tag{19} \]
where $q_0$ is the same as in (14).

**Proof.** By Proposition 3.2 in [1], for each $x \in V$, there exists $c > 0$ such that, for all $N \in \mathbb{N}$, the following estimate holds
\[ \mu(\Delta^N) \leq \mu(\Gamma_x) |B_N(x)|. \]
We fix $x$ and apply (b) of (15), which yields (19) with $N_x = N_x$ and $C = B_x \mu(\Gamma_x)q_0^\gamma$. \[ \square \]

### 3.4. Greedy animals

Let $\{Y_x : x \in V\}$ be a family of independent positive random variables (weights). For $N \in \mathbb{N}$ and $x \in V$, we define
\[ S_N(x) = \max_{A \in \mathcal{A}_N(x)} \sum_{x \in V(A)} Y_x. \tag{20} \]
Those $A \in \mathcal{A}_N(x)$, for which the maximum in (20) is attained are called greedy animals, see [10] for motivating examples, applications, and further details.
Lemma 15. For any $x \in V$, obviously, $v_x$ is analytic in some neighborhood of $t = 0$, and hence $w_x(t)/t \to v_x$ as $t \to 0$, where $v_x := E_x$.

$$w_x(t) := \log E_x^{\mathbb{R}_x} < \infty,$$

for a certain $t > 0$. Thus, $w_x$ is analytic in some neighborhood of $t = 0$, and hence $w_x(t)/t \to v_x$ as $t \to 0$, where $v_x := E_x$.

**Proposition 13.** Let $G$ be in $G_{\infty}(\phi)$ with $\phi$ satisfying (11). Suppose also that

$$v_x \leq Cn(x) \log n(x),$$

for some $C > 0$ and each $x \in V$. Then there exists $Y > 0$ such that, for each $x \in V$, there exists an increasing sequence, $\{N_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$, for which

$$\limsup_{k \to +\infty} \frac{1}{N_k} S_{N_k}(x) \leq Y \quad \text{with probability 1.}$$

The proof of this statement will be given below. Let us now make some comments. The lattice $\mathbb{Z}^d$ can be turned into a graph by setting $x \sim y$ if $|x - y| = 1$. The greedy animals on $\mathbb{Z}^d$ were studied in detail in [10,13,17,24]. In those papers, however, the weights are supposed to be identically distributed with law $P_0$ satisfying less restrictive conditions (as compared to (21)), involving the lattice dimension $d$, cf. Theorem 1 in [10] or Theorem 3.3 in [24]. In the statement above, we allow the mean value of $Y_x$ to increase in a controlled way (23), which seems to be quite natural for unbounded degree graphs. In a separate work, we shall study greedy animals in such graphs in more detail. In particular, we plan to relax the exponential integrability assumed in (21).

4. Further properties of paths and animals

4.1. Counting paths

We recall that by $G_\vartheta = (V_\vartheta, E_\vartheta)$ we denote the graph generated by a given path $\vartheta$. For $e \in E$, we say that $\vartheta$ traverses $e$ if $e \in E_\vartheta$. We say that $\vartheta = \{x_0, \ldots, x_n\}$ leaves $x_k$ towards $x_{k+1}$, $k = 0, \ldots, n - 1$. For $x \in V_\vartheta$, let $v_\vartheta(x)$ be the number of times $\vartheta$ leaves $x$. We also set $v_\vartheta(x) = 0$ if $x$ is not in $V_\vartheta$. Then, for a simple path, $v_\vartheta(x) \leq 1$. Recall that $\Sigma_N(x)$ denotes the collection of simple paths of length $N \in \mathbb{N}$ originated at a given $x \in V$. By $\Theta_N(x)$ we denote the collection of paths $\vartheta = \{x, x_1, \ldots, x_N\}$ such that the number of times $\vartheta$ leaves each $y \in V_\vartheta$, towards any $z \in V_\vartheta$, is at most one. Note that this does not mean $v_\vartheta(x) \leq 1$.

**Lemma 14.** For any $x \in V$ and $N \in \mathbb{N}$, it follows that $\Sigma_N(x) \subseteq \Theta_N(x)$. Every $\vartheta \in \Theta_N(x)$ has the properties: (i) each $e \in E_\vartheta$ can be traversed by $\vartheta$ at most twice; (ii) $v_\vartheta(y) \leq n(y)$ for each $y \in V_\vartheta$.

**Proof.** The stated inclusion is immediate, whereas both (i) and (ii) follow from the fact that $\vartheta$ leaves each $x \in V_\vartheta$ towards any $y \sim x$ at most once. □

**Lemma 15.** For any $x \in V$ and $N \in \mathbb{N}$, it follows that

$$|\Theta_N(x)| \leq \max_{\vartheta \in \Theta_N(x)} \exp \left( \sum_{y \in V_\vartheta} n(y) \log n(y) \right).$$

**Proof.** Obviously,

$$|\Theta_N(x)| \leq \sum_{y, y \sim x} |\Theta_{N-1}(y)| \leq \sup_{y \sim x} n(x) |\Theta_{N-1}(y)|,$$

which by the induction in $N$ yields

$$|\Theta_N(x)| \leq \max_{\vartheta \in \Theta_N(x)} n(x_0)n(x_1) \cdots n(x_{N-1})
\leq \max_{\vartheta \in \Theta_N(x)} \exp \left( \sum_{y \in V_\vartheta} v_\vartheta(y) \log n(y) \right)
\leq \max_{\vartheta \in \Theta_N(x)} \exp \left( \sum_{y \in V_\vartheta} n(y) \log n(y) \right),$$

where we have used claim (ii) of Lemma 14. □
Similarly, one proves that, cf. Proposition 1 in [3] and Lemma 5 in [21],
\[ |\Sigma_N(x)| \leq \max_{\vartheta \in \Sigma_N(x)} \exp \left( \sum_{y \in V_\vartheta} \log n(y) \right). \tag{26} \]

4.2. Capacity of animals

For an animal, \( A \subset G \), by \( \rho_A(x, y) \), we denote the length of the shortest path \( \vartheta(x, y) \) in \( A \), i.e., such that \( G_{\vartheta} \subset A \). We shall be interested in estimating the number of vertices in subsets \( B \subset V(A) \), which have the following property.

**Definition 16.** Given \( \lambda > 1 \), a set, \( B \subset V(A) \), is said to be \( \lambda \)-admissible in \( A \) if \( \rho_A(x, y) \geq \lambda \) for each pair of distinct \( x, y \in B \). The quantity
\[ C(A; \lambda) = \max\{|B| : B \text{ is } \lambda \text{-admissible in } A\} \]
is called the \( \lambda \)-capacity of \( A \).

Hence, if \( A' \subset A \) is a connected spanning subgraph, then
\[ C(A; \lambda) \leq C(A'; \lambda). \tag{27} \]

If \( \vartheta \) is a simple path of length \( N \), then
\[ C(G_{\vartheta}; \lambda) \leq 1 + N/\lambda. \]

**Lemma 17.** Let \( A \) be an animal of order \( N \). Then, for any \( \lambda > 0 \),
\[ C(A; \lambda) \leq \max\{1; 2(N - 1)/\lambda\}. \tag{28} \]

**Proof.** Suppose first that \( N \leq \lambda \). As any simple path in \( A \) cannot be longer than \( N - 1 \), one has \( \rho(x, y) \leq N - 1 \) for any \( x, y \in V(A) \). Thus, any \( \lambda \)-admissible set can contain at most one element, and hence (28) holds.

For \( N > \lambda \), we consider the multi-graph \( A \) which has the same vertices as \( \tilde{A} \) but doubled edges. This means that the edge set of \( A \) consists of the pairs \( e, \tilde{e} \), both having the same endpoints, such that \( e \in E(A) \). Then the graph \( \tilde{A} \) is Eulerian, see, e.g., [6, page 51], and hence there exists a path in \( \tilde{A} \), which originates and terminates at the same vertex, enters each \( x \in V(A) \), and traverses each edge of \( \tilde{A} \) exactly once.

Let \( B \) be \( \lambda \)-admissible in \( A \). To estimate \( |B| \) we proceed as follows. Take an Eulerian path in \( \tilde{A} \) starting from a certain \( x_1 \in B \), and number consecutive vertices \( x_i \in B \), \( i = 1, 2, \ldots, M \), along this path. The walk terminates at \( x_M = x_1 \in B \). Then
\[ l_i = \rho_A(x_i, x_{i+1}) \geq \lambda, \quad i = 1, 2, \ldots, M - 1. \]

Since each \( x \in B \) can be visited more than once, we have that \( M \geq |B| \). Then
\[ \lambda |B| \leq \lambda M \leq l_1 + l_2 + \cdots + l_M = 2|E(A)|, \]
which yields \( |B| \leq 2|E(A)|/\lambda \). Now to obtain (28) we use (27) and the fact that, for a spanning tree \( T \subset A \), \( |E(T)| = N - 1 \). \( \square \)

4.3. Balls in repulsive graphs

We recall that \( B_N(x) = \{ y \in V : \rho(x, y) \leq N \} \). \( N \in \mathbb{N} \) denotes the ball in \( G = (V, E) \) of radius \( N \) centered at \( x \), cf. (3) and Corollary 8. Further properties of such sets are described in the following statement.

**Lemma 18.** Let \( G \) be in \( G_-(\phi) \). Then for each \( x \in V \), there exists a strictly increasing sequence \( \{N_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \), such that, for all \( k \in \mathbb{N} \),
\[ \max_{y \in B_{N_k}(x)} n(y) \leq \phi^{-1}(2N_k + 1). \tag{29} \]

If \( G \in G_+(\phi) \), then, for every \( x \in V \), there exists \( N_k \in \mathbb{N} \) such that the estimate
\[ \max_{y \in B_{N_k}(x)} n(y) \leq \phi^{-1}(2N) \tag{30} \]
holds for all \( N \geq N_k \).

**Proof.** First we consider the case of \( G \in G_-(\phi) \). Let \( x_1 \in V^c \) be the closest vertex to \( x \) such that \( n(x_1) > n(x) \). If there are several such vertices, we take the one with the biggest degree. In the same way, we pick \( x_2 \), being the closest vertex to \( x \) such that \( n(x_2) > n(x_1) \). Then we set \( N_1 = \rho(x, x_2) - 1 \), which yields \( n(x_1) = \max_{y \in B_{N_1}(x)} n(y) \). By (5) \( \rho(x_1, x_2) \geq \phi(n(x_1)) \); hence, \( 2N_1 + 1 \geq n(x_2) + \rho(x, x_1) \geq \phi(n(x_1)) \). Thus, (29) holds for \( N = N_1 \). Next we take \( x_3 \) such that \( n(x_3) > n(x_2) \) and set \( N_2 = \rho(x, x_3) - 1 \). In this way, we construct the whole sequence \( \{N_k\}_{k \in \mathbb{N}} \) for which (29) holds.

The part related to \( G_+(\phi) \) was obtained in Lemma 6 in [21]. \( \square \)
5. The proof of the statements

Proof of Theorem 3. Let $A$ be an animal such that $x \in V(A)$. Consider the Eulerian multi-graph $\tilde{A}$, see the proof of Lemma 17. Then there exists a path in $\tilde{A}$, which originates and terminates at $x$, enters each $y \in V(A)$, and traverses each edge of $\tilde{A}$ exactly once. Therefore, $A = G_0$ for some path $\vartheta(x, x) \in \Theta_M(x)$ with $M = 2|E(A)|$. Thus, by (25) we have

$$|\mathcal{A}_N(x)| \leq |\Theta_M(x)| \leq \max_{\vartheta \in \Theta_M(x)} \exp \left( \sum_{y \in V(y)} g(n(y)) \right)$$

$$= \max_{\mathcal{A} \in \mathcal{A}_N(x)} \exp \left( \sum_{y \in V(A)} g(n(y)) \right)$$

$$\leq \max_{A \in \mathcal{A}_N(x)} \exp \left( NG(A; g) \right).$$

Then we set $q_A = e^r$ and obtain (8) from (4). □

In the same way, by means of (26) one proves Theorem 4.

Definition 19. Given $G \in \mathbb{G}_\pm(\phi)$, an $A \subset G$ is said to be a good animal if

$$|V(A)| \geq \phi(n_A)/2, \quad n_A := \max_{x \in V(A)} n(x).$$

By $\mathcal{A}_{\text{good}}$ we denote the set of all good animals, cf. (29) and (30).

Proof of Theorem 5. First we consider the case $G \in \mathbb{G}_-(\phi)$. Let the sequence in (10) be such that $t_1 = n_*$. The proof will be done by showing that: (a) the upper bound as in (4) holds for any good animal; (b) for each $x \in V$, one can pick $\{N_k\}_{k \in \mathbb{N}}$ such that each $A \in \mathcal{A}_N(x)$, $k \in \mathbb{N}$, is a good animal.

For $A \in \mathcal{A}_{\text{good}}$, we set

$$M_k(A) = \{x \in V(A) : n(x) \in (t_k, t_{k+1}]\}, \quad k = 1, \ldots, l,$$

$$m_k(A) = |M_k(A)|,$$

where $l \in \mathbb{N}$ is the smallest number such that $n_A \leq t_{l+1}$, see (31). By (5) we then get $\rho(x, y) \geq \phi(t_k)$ for each $x, y \in M_k(A)$. Hence, by Lemma 17 we have

$$m_k(A) \leq C(A, \phi(t_k)) \leq 2|V(A)|/\phi(t_k),$$

which leads to the following estimate, cf. (4),

$$G(A; g) \leq \frac{1}{|V(A)|} \sum_{k=1}^{l} g(t_{k+1}) m_k(A) \leq 2 \sum_{k=1}^{\infty} \frac{g(t_{k+1})}{\phi(t_k)} \overset{\text{def}}{=} \gamma(g, \phi).$$

(32)

Let $x$ be an arbitrary vertex. For this $x$, let $\{N_k\}_{k \in \mathbb{N}}$ be the sequence as in Lemma 18. Then, for any $A$ such that $x \in V(A)$ and $|V(A)| = N_1$, we have $V(A) \subset B_{N_1-1}(x)$. Then by (29)

$$2N_1 > 1 + 2(N_1 - 1) \geq \phi \left( \max_{y \in V(A)} n(y) \right),$$

which yields $A \in \mathcal{A}_{\text{good}}$. Hence, (32) holds for any $A \in \mathcal{A}_N(x)$. Then we repeat the same procedure with $N_2$, $N_3$, and so on. For $G \in \mathbb{G}_-(\phi)$, the proof follows along the same line of arguments, with the only difference that by (30) we show that $\mathcal{A}_N(x) \subset \mathcal{A}_{\text{good}}$ whenever $N \geq N_*$. □

Proof of Proposition 11. By (17) and Theorem 5, for $g = t^{\theta+1}$ the graph $G \in \mathbb{G}_-(\phi)$ is $g$-tempered and strongly $g$-tempered, respectively. On the other hand, by the assumption in (7), at most one of two adjacent vertices can be in $V'_c$, see (6). Hence, cf. (16),

$$R^\theta(A) \leq n_*^\theta \sum_{x \in V(A)} [p(x)]^{\theta+1}.$$

Then, the proof of (18) follows by (4) with $\gamma_A = n_*^\theta \gamma_A$. □

Proof of Proposition 13. We proceed as follows. Set

$$S(A) = \sum_{x \in V(A)} Y_x,$$
Then, for $Y > 0$ and $t > 0$, we have, cf. (21),
\[
\mathbb{P}(S(A) \geq Y | V(A)) \leq \exp(-tY|V(A)|) \mathbb{E} \left( t \sum_{x \in V(A)} Y_x \right)
\]
\[
= \exp \left( -t \sum_{x \in V(A)} (Y - w_x(t)/t) \right)
\]
\[
\leq \exp \left( -tY|V(A)| + tC \sum_{x \in V(A)} n(x) \log n(x) \right),
\]
which holds for small enough $t > 0$, see (22) and (23). For $\phi$ satisfying (11), the graph in question is $g$-tempered with $g(t) = t \log t$, see Theorem 5. Given $x \in V$, let $\{N_k\}_{k \in \mathbb{N}}$ be the sequence as in Definition 1. Then, for $A \in \mathcal{A}_{N_k}(x)$, by (4) and the latter estimate we obtain
\[
\mathbb{P}(S_{N_k}(x) \geq Y N_k) \leq \exp(-tN_k(Y - \gamma C)).
\]
Now take $Y > \gamma C$ and obtain (24) by applying the Borel–Cantelli lemma.

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