A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT METHODS FOR REISSNER-MINDLIN PLATES

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ABSTRACT. This paper establishes a very general theory for a posteriori error analysis of finite element methods of the Reissner-Mindlin plate problem in the literature. The theory assures reliability of explicit residual error estimates. The conclusion of this theory is sparsity in the mathematical research of uniform a posteriori error control. Indeed, the a posteriori error estimate for various finite element methods of the Reissner-Mindlin plate problem is reduced to three parts: (1) Check the four conditions (H1)-(H4); (2) Design free functions $\phi_h$, $\bar{w}_h$, and $\Gamma_h$ and the free parameter $\alpha$; (3) Estimate the last six terms of the abstract estimator $\eta_h$.

As examples, we apply the present theory to four classes of finite element methods for the Reissner-Mindlin plate problem: the methods based on the linked technique, the Arnold-Falk type methods, the MITC methods, and the discontinuous Galerkin methods. For all these methods, it is proved that the error can be estimated by a computable error estimator from above and below up to multiplicative constants that are independent of both the meshsize and the plate thickness. The error is bounded in norms that are analyzed in the a priori error analysis for all the methods under consideration.

Among aforementioned methods, the first class of methods has been analyzed in literature under the saturation assumption. The estimator of this paper improves that result by abandoning that constraint. For the second class of methods, only the Arnold-Falk element has been estimated under the condition $t \lesssim h_K$ for any element $K$ of the mesh $T_h$ with the element diameter $h_K$ and the plate thickness $t$. However, this assumption is removed by the present theory. For other methods of the Arnold-Falk type, there is no a posteriori analysis in literature. For the MITC methods, our theory recovers the results in literature. For the discontinuous Galerkin methods, no a posteriori analysis can be found. It is stressed that, for all the methods which have already been analyzed, the frameworks of the analysis herein are completely different from those used for them in the literature.

1. INTRODUCTION AND MAIN RESULTS

The Reissner-Mindlin plate problem. This paper is devoted to a posteriori error analysis of finite element methods for the Reissner-Mindlin plate problem:

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Given $g \in L^2(\Omega)$ find $(w, \phi) \in W \times \Theta := H^1_0(\Omega) \times H^1_0(\Omega)^2$ with
\begin{equation}
(1.1) \quad a(\phi, \psi) + (\gamma, \nabla v - \psi)_{L^2(\Omega)} = (g, v)_{L^2(\Omega)} \quad \text{for all } (v, \psi) \in W \times \Theta,
\end{equation}
and the shear force
\begin{equation}
(1.2) \quad \gamma = \lambda t^{-2}(\nabla w - \phi).
\end{equation}

Here and throughout this paper, $t$ denotes the plate thickness with the shear modulus $\lambda = Ek/2(1 + \nu)$, the Young modulus $E$, the Poisson ratio $\nu$, and the shear correction factor $\kappa$. Given $\phi \in \Theta$, the linear Green strain $\varepsilon(\phi) = 1/2[\nabla \phi + \nabla \phi^T]$ is the symmetric part of gradient field $\nabla \phi$. For all $2 \times 2$ symmetric matrices the linear operator $C$ is defined by
\begin{equation}
C\tau := \frac{E}{12(1-\nu^2)}[(1-\nu)\tau + \nu \text{tr}(\tau)I].
\end{equation}
The bilinear form $a(\cdot, \cdot)$ models the linear elastic energy defined as
\begin{equation}
(1.3) \quad a(\phi, \psi) = (C\varepsilon(\phi), \varepsilon(\psi))_{L^2(\Omega)}, \quad \text{for any } \phi, \psi \in \Theta := H^1_0(\Omega)^2
\end{equation}
which gives rise to the energy norm
\begin{equation}
(1.4) \quad \|\psi\|^2_C := a(\psi, \psi) \quad \text{for any } \psi \in \Theta,
\end{equation}
while $\|\cdot\|_{C_h}$ denotes the broken version with the piecewise defined operator $\varepsilon_h$ taking place of $\varepsilon$, and $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the $L^2$ scalar product.

**Locking effects and a priori error analysis of finite element methods.** This Reissner-Mindlin plate theory is widely used by engineers and mathematicians to describe the behavior of an elastic plate loaded by a transverse force. The main feature of this theory is that it takes into account the shear deformations, and thus allows one to consider both thin and moderately thin plate. However, a direct lower-order finite element approximation often yields poor results because of the shear locking phenomenon, namely the numerical solution is much smaller than the exact one. In order to weaken or overcome this problem, a number of principles or methods have already been proposed, and are being proposed. The common idea of all methods to weaken or overcome this drawback is to use the so-called *Reduction Integration Method* to reduce the shear influence.

In the a priori error analysis of the Reissner-Mindlin plate problem, we now know how to analyze the stability of discrete schemes since their stability is more or less related to a stable discretization of an equivalent mixed problem; which is proposed by Brezzi and Fortin [20] by employing the Helmholtz decomposition through introducing some auxiliary variables. This mixed formulation is made of a system of equations with two decoupled Poisson problems and a Stokes-like problem. The main task to analyze the discrete problem reduces to prove that there exist some discrete Helmholtz decomposition and consequently some discrete equivalent mixed problem. A similar guideline of a priori error analysis can also be found in [46].
Known a posteriori error results. Although there are a vast number of papers dealing with a priori estimates of the Reissner-Mindlin plate problem, there are only few concerning a posteriori estimates. As far as we know, only three works [38, 35, 57] in this direction estimate convergent norms without unnatural assumptions. The first one [38] works on the Arnold-Brezzi type method initialized in [4, 39]. The argument therein is based on the equivalence between the energy norm of the error and the dual norm of the residual. Note that such a direct equivalence is only valid for this class of finite element methods for the Reissner-Mindlin plate problem; which can not be generalized to other methods. The other two [35, 57] present a posteriori analysis of MITC elements without stabilization. The main tools therein are the regular decomposition of functions in $H_0(\text{rot}, \Omega)$, a well-tailored error representation formula, and a mesh dependent norm. Also, the $H_0(\text{rot}, \Omega)$ regularity of the discrete shear force can not happen to other schemes.

Other a posteriori error estimators for the finite element methods of the Reissner-Mindlin plate problem can be found in literature. They seem for us unsatisfactory. An a posteriori error estimate is proposed for the DL element of [48] in [71]. However, the term $\|\gamma - \gamma_h\|_{H^{-1}(\text{div}, \Omega)}$ is involved in that estimator ($\gamma_h$ denotes the discrete shear force). It is unsatisfactory since the a priori convergence of $\|\gamma - \gamma_h\|_{H^{-1}(\text{div}, \Omega)}$ is open for MITC methods. The paper [33] concerns the nonconforming Arnold-Falk element [7] under the assumption $t \lesssim h_K$ for each element $K$ with the diameter $h_K$. This assumption restricts however the refinement at singular points of the solution. In [77], the authors discuss the a posteriori error estimator for the schemes based on the linked interpolation technique. We note that the result therein is derived under the saturation assumption which is abandoned by the a posteriori error estimators of the Poisson problem.

Difficulties faced with a posteriori error analysis. With the aforementioned knowledge of the a priori error analysis, one may wonder what happens to the a posteriori analysis and why it is so difficult. In our opinion, there are at least three difficulties. First, since most methods are implemented with the primary variables, the discrete counterparts of those auxiliary variables introduced by the equivalent discrete mixed problem are only for the a priori error analysis but not available for the a posteriori error analysis. Therefore this mixed formulation is inappropriate for the a posteriori error analysis. Second, it is obvious that the primary formulation (1.1) is a very bad choice for the a posteriori error analysis. Otherwise, one will suffer from the deterioration of the efficiency when the plate thickness $t$ tends to zero. So far, there is no theory for how to choose the formulation of a posteriori analysis for various finite element methods in literature. Third, it is still unclear which norm should be analyzed for various discrete schemes. But, it is clear that the norm used should converge with the same rate of the primary variables in the $H^1$ norm, and that their convergence should be independent of the plate thickness $t$. These difficulties are partially reflected from the analysis in [38, 35, 57]. One feeling from these three papers is that the a posteriori error analysis for the Reissner-Mindlin plate problem is very tricky and technical, and that one needs a very delicate balance between the norm analyzed and the estimator obtained. Another feeling is that different methods seem to need completely different analysis.
Moreover, we think those unsatisfactory results of [71, 33, 77] can be attributed to these difficulties.

The general theory and main results. One most natural question is that: Are there some general guidelines for the a posteriori analysis of the finite element methods of the Reissner-Mindlin plate problem? This paper aims at answering this question by providing a unified framework for the a posteriori error analysis of finite element methods of the Reissner-Mindlin plate problem. This framework can be used to analyze most of popular locking free discrete schemes with robust estimators in the sense that the reliability and efficiency constants independent of the plate thickness \( t \). We shall accomplish this in two steps.

First, we introduce a very sparse mixed formulation proposed in [4]. Given some positive function \( \alpha \in L^\infty(\Omega) \) with \( |\alpha|^2_{L^\infty(\Omega)} < \frac{\lambda}{t^2} \), we define a positive function \( \beta \in L^\infty(\Omega) \) by

\[
\frac{1}{\beta^2} = \frac{\lambda}{t^2} - \alpha^2.
\]

Usually \( \alpha \) is a function with respect to both \( t \) and the local meshsize in the sense \( \alpha_K := \alpha|_K = \alpha(t, h_K) \) for any \( K \in \mathcal{T}_h \), Also, we let \( \alpha_E = \alpha(t, h_E) \) for any \( E \in \mathcal{E} \). Then, Problem (1.1) is equivalent to the following mixed problem: Given \( g \in L^2(\Omega) \) find \( (w, \phi, \gamma') \in W \times \Theta \times Q := H^1_0(\Omega) \times H^1_0(\Omega)^2 \times L^2(\Omega)^2 \) with

\[
A(\phi, w, \gamma'; \psi, v, \delta) = (g, v)_{L^2(\Omega)} \quad \text{for all } (v, \psi, \delta) \in W \times \Theta \times Q,
\]

with the bilinear form \( A \) defined by

\[
A(\phi, w, \gamma'; \psi, v, \delta) := a(\phi, \psi) + (\alpha^2(\nabla w - \phi, \nabla v - \psi)_{L^2(\Omega)} + (\nabla w - \phi, \delta)_{L^2(\Omega)} + (\nabla v - \psi, \gamma')_{L^2(\Omega)} - (\beta^2 \gamma', \delta)_{L^2(\Omega)}.
\]

We use the idea of [4, 38, 35, 57] to define the following norm on the space \( W \times \Theta \):

\[
\| \phi, w \| := \| \phi \|_C + \| \alpha(\nabla w - \phi) \|_{L^2(\Omega)}, \quad \text{for any } (w, \phi) \in W \times \Theta.
\]

Define the following mesh dependent norm for the space \( Q \):

\[
\| \delta \|_Q := \sup_{0 \neq (\psi, v) \in H^1_0(\Omega)^3} \frac{(\nabla v - \psi, \delta)_{L^2(\Omega)}}{\| \psi, v \|} + \| \beta \delta \|_{L^2(\Omega)}, \quad \text{for any } \delta \in Q.
\]

Since \( a(\phi, \phi) + (\alpha^2(\nabla w - \phi, \nabla w - \phi)_{L^2(\Omega)} = \| \phi, w \|^2 \), it is a direct consequence of [24, Theorem 2] that the bilinear form \( A \) is an isomorphism between \( W \times \Theta \times Q \) and its dual. This means

\[
\| \phi, w \| + \| \gamma' \|_Q \approx \sup_{0 \neq (\psi, v, \delta) \in H^1_0(\Omega)^3 \times Q} \frac{A(\phi, w, \gamma'; \psi, v, \delta)}{\| \psi, v \| + \| \delta \|_Q}.
\]

This result is also proved in Theorem 4.2 of [38] with a different proof. Here and throughout, an inequality \( a \lesssim b \) replaces \( a \leq Cb \) with some multiplicative mesh-size independent constant \( C > 0 \) that depends only on the domain \( \Omega \) and the shape (e.g., through the aspect ratio) of elements \( (C > 0 \text{ is also independent of the crucial parameter } t ) \). Finally, \( a \approx b \) abbreviates \( a \lesssim b \lesssim a \).
Second, suppose \((\tilde{w}_h, \tilde{\phi}_h, \tilde{\gamma}'_h) \in W \times \Theta \times Q\) is some approximation to \((w, \phi, \gamma')\) over some regular partition \(T_h\) and define

\[
\begin{align*}
\text{Res}_Q(\delta) &:= (\nabla \tilde{w}_h - \tilde{\phi}_h, \delta)_{L^2(\Omega)} - (\beta^2 \tilde{\gamma}'_h, \delta)_{L^2(\Omega)}, \\
\text{Res}_W(v) &:= (gv, v)_{L^2(\Omega)} - (\alpha^2 (\nabla \tilde{w}_h - \tilde{\phi}_h) + \tilde{\gamma}'_h, \nabla v)_{L^2(\Omega)}, \\
\text{Res}_\Theta(\psi) &:= -a(\tilde{\phi}_h, \psi) + (\alpha^2 (\nabla \tilde{w}_h - \tilde{\phi}_h) + \tilde{\gamma}'_h, \psi)_{L^2(\Omega)}.
\end{align*}
\]

for all \(\delta \in Q, v \in W\) and \(\psi \in \Theta\). Here and throughout, \(\tilde{w}_h, \tilde{\phi}_h\) and \(\tilde{\gamma}'_h\) are not necessarily discrete functions. However, the subindex \(h\) refers to the fact that they might be closely related to some discrete functions \(w_h, \phi_h\) and \(\gamma_h\) and they are on our disposal. They are essentially designed to deal with the nonconformity which may arise from the discrete spaces, the reduction integration, and the discrete formulation. Another very important function of them is to cope with different classes of methods through different choices of these functions or parameters. Then it follows from (1.9) that

\[
\|\phi - \tilde{\phi}_h, w - \tilde{w}_h\| + \|\gamma' - \tilde{\gamma}'_h\|_Q \\
\approx \sup_{0 \neq (\phi, v) \in H_h^1(\Omega)^3} \frac{\text{Res}_W(v) + \text{Res}_\Theta(\psi)}{\|\phi, v\|} + \sup_{0 \neq \delta \in Q} \frac{\text{Res}_Q(\delta)}{\|\delta\|_Q}.
\]

For methods based on (1.5), one can simply take \(\tilde{\phi}_h = \phi_h, \tilde{w}_h = w_h\), and \(\tilde{\gamma}'_h = \gamma_h\), the finite element approximations to \(\phi, w\), and \(\gamma'\) respectively, in (1.13). This actually recovers the analysis in [38] for such a class of finite element methods. Doubtlessly, the difficulties lie in how to apply this formulation to schemes based on other principles than (1.5); which is main goal of this paper. In particular, we plan to prove that (1.13) applies to various finite element schemes of the Reissner-Mindlin plate problem by appropriate choices of \(\alpha, \beta, \gamma_h, \tilde{w}_h\), and \(\tilde{\gamma}'_h\). To this end, let \(W_h, \Theta_h, \) and \(\Gamma_h\) be the displacement, the rotation, and the shear force spaces, respectively, and let \((w_h, \phi_h, \gamma_h) \in W_h \times \Theta_h \times \Gamma_h\) be some finite element approximation to \((w, \phi, \gamma)\) with the reduction integration operator \(R_h\) in the context of locking. Define

\[
\tilde{\eta}_h := \left( \sum_{K \in T_h} \alpha_K^2 h_K^2 \| \nabla \gamma_h + g \|_{L^2(K)}^2 \right)^{1/2} + \left( \sum_{E \in \mathcal{E}(\Omega)} \alpha_E^2 h_E \| [\gamma_h] \cdot \nu_E \|_{L^2(E)}^2 \right)^{1/2} + \left( \sum_{K \in T_h} h_K \| \text{div} C \varepsilon_h(\phi_h) + \gamma_h \|_{L^2(K)}^2 \right)^{1/2} + \left( \sum_{E \in \mathcal{E}(\Omega)} h_E \| C \varepsilon_h(\phi_h) \cdot \nu_E \|_{L^2(E)}^2 \right)^{1/2} + \mu_h(\gamma_h) + \left( \sum_{E \in \mathcal{E}} h_E^{-1} \| [\phi_h] \|_{L^2(E)}^2 \right)^{1/2} + \sup_{0 \neq p \in H^1(\Omega)} \frac{\| R_{\tilde{h}} \cdot \text{Curl} J p \|_{L^2(\Omega)}}{\| p \|_{L^2(\Omega)} + \| t \nabla p \|_{L^2(\Omega)}} + \left( \sum_{E \in \mathcal{E}} \min \left( \frac{1}{t}, \frac{h_E}{t^2} \right) \| [\tau_E, \tilde{\tau}_E] \|_{L^2(E)}^2 \right)^{1/2} + \left( \sum_{K \in T_h} \min \left( 1, \frac{h_K}{t} \right)^2 \| \text{rot} \tilde{\tau}_h \|_{L^2(K)}^2 \right)^{1/2} + \| \alpha \tilde{\tau}_h \|_{L^2(\Omega)} + \| 1/\alpha (\gamma_h - \alpha^2 (\nabla \tilde{w}_h - \tilde{\phi}_h) - \tilde{\gamma}'_h) \|_{L^2(\Omega)} + \| \varepsilon_h(\phi_h - \tilde{\phi}_h) \|_{L^2(\Omega)},
\]

This actually recovers the analysis in [38] for such a class of finite element methods.
for any $\tilde{w}_h \in W$, $\tilde{\phi}_h \in \Theta$, and $\tilde{\gamma}'_h \in Q$. Herein, $\gamma' = \beta^2(\nabla w - \phi)$, and $\tilde{r}_h = \beta^2\tilde{\gamma}'_h - (\nabla \tilde{w}_h - \tilde{\phi}_h)$. The term $\mu_h(\gamma_h)$ reads

$$
\mu_h(\gamma_h) = \sup_{0 \neq (v, \psi) \in W \times \Theta} \frac{(\gamma_h, (R_h - I)(\nabla (J v + L J \psi) - J \psi))_{L^2(\Omega)}}{||\psi, v||}
$$

In this paper $L$ is a suitable linear operator which will be specified for various methods and $J$ is the usual Clément interpolation operator. We refer the next section for the definitions of other notations and more details.

Then, we prove that the error $||\phi - \tilde{\phi}_h, w - \tilde{w}_h|| + ||\gamma' - \tilde{\gamma}'_h||_Q$ can be estimated by $\tilde{\eta}_h$ in the sense:

**Theorem 1.1.** There holds

$$\tag{1.14} ||\phi - \tilde{\phi}_h, w - \tilde{w}_h|| + ||\gamma' - \tilde{\gamma}'_h||_Q \lesssim \tilde{\eta}_h,$$

provided that Hypothesis (H1)-(H4) proposed in Section 3 are valid. (Theorem 1.1 will be proved in Section 3 below).

Now let $(w_h, \phi_h, \gamma'_h) \in W_h \times \Theta_h \times Q$ be some finite element approximation with (H1)-(H4) to $(w, \phi, \gamma') \in W \times \Theta \times Q$. One important consequence of Theorem 1.1 which will be frequently used is:

**Theorem 1.2.** Let Hypothesis (H1)-(H4) of Section 3 hold. Then,

$$\tag{1.15} ||\phi - \tilde{\phi}_h, w - w_h|| + ||\gamma' - \tilde{\gamma}'_h||_Q \lesssim \tilde{\eta}_h + ||\phi_h - \tilde{\phi}_h, w_h - \tilde{w}_h||_h + ||\gamma_h - \tilde{\gamma}'_h||_Q.$$

Herein and throughout this paper, $||\cdot||_h$ is the broken version of $||\cdot||$ with elementwise defined $\varepsilon_h$ and $\nabla_h$ taking the place of $\varepsilon$ and $\nabla$ respectively.

With Theorems 1.1-1.2 at hand, the remaining tasks to achieve a reliable estimator for various finite elements are to choose $\tilde{\phi}_h$, $\tilde{w}_h$, $\tilde{\gamma}'_h$ and the piecewise function $\alpha$, and then bound the last six terms of $\tilde{\eta}_h$ and the last two terms on the right-hand side of (1.15) based on these choices.

**Remark 1.3.** For the efficiency of the estimator, the choice for $\alpha$ is usually method-dependent for various finite element methods of the Reissner-Mindlin plate problem. Our experience herein shows the following rule:

- If $||\gamma - \gamma_h||_{H^{-1}(\text{div}, \Omega)}$ converges, one can take $\alpha$ as a global constant independent of $h$ the mesh size and $t$ the plate thickness;
- If $||\gamma - \gamma_h||_{H^{-1}(\Omega)}$ converges, one can choose $\alpha$ as $\alpha^2|_K = \frac{1}{t^2 + h_K^2}$ for any $K \in T_h$.

Moreover, we believe that the discrete shear force $\gamma_h$ should converge with respect to either the norm $||\gamma - \gamma_h||_{H^{-1}(\text{div}, \Omega)}$ or the norm $||\gamma - \gamma_h||_{H^{-1}(\Omega)}$ for all locking free schemes of the Reissner-Mindlin plate problem.

**Applications.** As applications for the present theory, we shall apply Theorem 1.1-1.2 to: the methods based on the linked technique [12, 99, 102, 94, 13, 14, 49, 75, 78, 11, 77] in Section 4, the Arnold-Falk type methods [7, 100, 31, 46, 76, 61, 62] in Section 5, the MITC methods [9, 18, 19, 21, 25, 20, 30, 48, 50, 56, 58, 60, 82, 92, 83, 81] in Section 6, and the discontinuous Galerkin methods [5, 6] in Section 7. We stress that they are the most popular methods of the Reissner-Mindlin
plate problem in the literature. Let $\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h)$ denote the energy norm error analyzed in the a priori analysis for these methods. We shall prove the reliability and efficiency of the estimator $\eta_h$ for these methods in the sense that

\begin{equation}
\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h) \approx \eta_h + \text{osc}(g)
\end{equation}

for various popular finite element methods of the Reissner-Mindlin plate problems. The estimator $\eta_h$ is defined as

$$\eta_h^2 := \sum_{K \in T_h} h_K^2 (\| \text{div} C\varepsilon(\phi_h) + \gamma_h \|^2_{L^2(K)} + \alpha_K^{-2} \| \text{div} \gamma_h + g \|^2_{L^2(K)}) + \sum_{E \in \mathcal{E}(\Omega)} h_E (\| C\varepsilon(\phi_h) \cdot \nu_E \|^2_{L^2(E)} + \alpha_E^{-2} \| \gamma_h \cdot \nu_E \|^2_{L^2(E)}) + \| \alpha r_{1,h} \|^2_{L^2(\Omega)}$$

$$+ \| r_{2,h} \|^2_{L^2(\Omega)} + \| \text{rot} h r_{2,h} \|^2_{L^2(\Omega)} + \sum_{K \in T_h} \min(1, \frac{h_K}{t})^2 \| \text{rot} r_{1,h} \|^2_{L^2(K)}$$

$$+ \sum_{E \in \mathcal{E}} \min(1, \frac{h_E}{t^2}) (\| R_h(\nabla_h w_h - \phi_h) \cdot \tau_E \|^2_{L^2(E)} + \mu_h(\gamma_h)^2)$$

$$+ \sum_{E \in \mathcal{E}} \alpha_E^2 h_E \| \nabla_h w_h \cdot \tau_E \|^2_{L^2(E)} + \sum_{E \in \mathcal{E}} \alpha_E^2 h_E^{-1} \| \phi_h \|^2_{L^2(E)}.$$

Here and throughout this paper, $r_{1,h}$ and $r_{2,h}$ denote some residuals which will be specified for various classes of finite element methods for the Reissner-Mindlin plate problem. The oscillation of $g$ reads

\begin{equation}
\text{osc}^2(g) := \sum_{K \in T_h} \alpha^2 h_K^2 \min_{g_k \in Q_h(\omega)} \| g - g_k \|^2_{L^2(K)}.
\end{equation}

Given a non negative integer $k$, the space $Q_k(\omega)$ consists of polynomials of total degree at most $k$ defined over $\omega$ in the case $\omega = K$ is a triangle whereas it denotes polynomials of degree at most $k$ in each variable in the case $K$ is a quadrilateral.

**Remark 1.4.** The energy norm $\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h)$ usually involves other terms than $|||\phi - \phi_h, w - w_h|||_h$ and $|||\gamma - \gamma_h|||_Q$. As we shall see in these example methods in Sections 4-8, the estimator of these terms can be derived from their definitions and the estimator of $|||\phi - \phi_h, w - w_h|||_h$ and $|||\gamma - \gamma_h|||_Q$.

**Remark 1.5.** The terms with the residual $r_{1,h}$ (resp. $r_{2,h}$) concern the consistency error from the reduction integration; they vanish for $R_h = I$. Hence the results below applies to any finite element methods without reduction integration.

**Remark 1.6.** $\mu_h(\gamma_h)$ vanishes for the cases where $R_h$ is a $L^2$ projection operator from $Q$ onto $\Gamma_h$. For the MITC methods, $\mu_h(\gamma_h) = 0$ for high-order schemes [35]; this term can be bounded by

\begin{equation}
\mu_h(\gamma_h)^2 \lesssim \sum_{E \in \mathcal{E}(\Omega)} h_E (t^2 + h_E^2) \| \gamma_h \cdot \nu_E \|^2_{L^2(E)}$$

$$+ \sum_{K \in T_h} h_K^2 (t^2 + h_K^2) \| g + \text{div} \gamma_h \|^2_{L^2(K)}.$$

for the lower order schemes, up to some computable high-order term [35].
Remark 1.7. The term \( \sum_{E \in \mathcal{E}} \min \left( \frac{1}{T}, \frac{1}{T^2} \right) \| \mathbf{R}_h(\nabla w_h - \phi_h) \|_{L^2(E)} \) disappears for the MITC methods.

Remark 1.8. The last two terms concern the nonconformity of \( w_h \) and \( \phi_h \), which will vanish for the conforming methods.

Note that the estimator \( \eta_h \) is robust in the sense that the reliability and efficiency constants are independent of the plate thickness \( t \). In particular, this result recovers the estimators of [35, 57] and improves the estimator of [77] by abandoning the saturation assumption. For the nonconforming Arnold-Falk element [7], this estimator is different from that in [33]. For many methods other than the aforementioned ones, the estimator is new.

Finally, we conjecture that Theorem 1.1 and Theorem 1.2 apply to all locking-free finite element methods of the Reissner-Mindlin plate problem in literature. This idea is also expected to be instrumental in a posteriori analysis for other parameter dependent problems.

2. Notations

This section introduces necessary notations for the Sobolev space, differential operators, triangulations, discrete spaces, and the Clément interpolation. Also, some preliminary results will be presented in the last subsection.

2.1. Sobolev Spaces and Differential Operators. We use the standard differential operators:

\[ \nabla r = (\partial r/\partial x, \partial r/\partial y), \quad \text{Curl } p = (\partial p/\partial y, -\partial p/\partial x). \]

Given any 2D vector function \( \psi = (\psi_1, \psi_2) \), its divergence reads \( \text{div } \psi = \partial \psi_1/\partial x + \partial \psi_2/\partial y \). With the differential operator \( \text{rot } \psi = \partial \psi_2/\partial x - \partial \psi_1/\partial y \) for a vector function \( \psi = (\psi_1, \psi_2) \), the space \( H_0(\text{rot}, \Omega) \) is defined as

\[ H_0(\text{rot}, \Omega) := \{ v \in L^2(\Omega)^2, \text{rot } v \in L^2(\Omega) \text{ and } v \cdot \tau = 0 \text{ on } \partial \Omega \} \]

endowed with the norm

\[ \| v \|_{H(\text{rot}, \Omega)} := \left( \| v \|_{L^2(\Omega)}^2 + \| \text{rot } v \|_{L^2(\Omega)}^2 \right)^{1/2}. \]

The dual space for \( H_0(\text{rot}, \Omega) \) reads

\[ H^{-1}(\text{div}, \Omega) := \{ v \in H^{-1}(\Omega)^2, \text{div } v \in H^{-1}(\Omega) \}, \]

with the norm

\[ \| v \|_{H^{-1}(\text{div}, \Omega)} := \left( \| v \|_{H^{-1}(\Omega)}^2 + \| \text{div } v \|_{H^{-1}(\Omega)}^2 \right)^{1/2}. \]

The piecewisely defined gradient operator is denoted by \( \nabla_h \), and \( \varepsilon_h \) is the piecewise counterpart of \( \varepsilon \) for elements in \( \Theta_h \). The broken \( H^1 \) norm \( \| \cdot \|_{1,h} \) is defined as

\[ \| v_h \|_{1,h} := \left( \sum_{K \in \mathcal{T}_h} \| v_h \|_{H^1(K)}^2 \right)^{1/2}, \quad \text{for all } v_h \in W + W_h. \]

Here and throughout this paper, \( \Theta_h \subset L^2(\Omega)^2 \) and \( W_h \subset L^2(\Omega) \) denote some finite element spaces over some regular partition \( \mathcal{T}_h \) while \( \mathbf{R}_h \) denotes the reduction.
introduction operator in the context of shear locking with values in the discrete shear force space \( \mathbf{\Gamma}_h \).

### 2.2. Triangulations and Discrete Spaces.

Suppose that the closure \( \overline{\Omega} \) is covered exactly by a regular triangulation \( \mathcal{T}_h \) of \( \overline{\Omega} \) into (closed) triangles or quadrilaterals in 2D or other unions of simplices, that is

\[
\overline{\Omega} = \bigcup \mathcal{T}_h \quad \text{and} \quad |K_1 \cap K_2| = 0 \quad \text{for} \quad K_1, K_2 \in \mathcal{T}_h \quad \text{with} \quad K_1 \neq K_2, 
\]

where \(| \cdot |\) denotes the volume (as well as the length of an edge and the modulus of a vector etc, when there is no real risk of confusion). Let \( \mathcal{E} \) denote the set of all edges in \( \mathcal{T}_h \) with \( \mathcal{E}(\Omega) \) the set of interior edges, and \( \mathcal{N}(\Omega) \) the set of interior nodes. The set of edges of the element \( K \) is denoted by \( \mathcal{E}(K) \). By \( h_K \) we denote the diameter of the element \( K \in \mathcal{T}_h \). Also, we denote by \( \omega_K \) the union of elements \( K' \in \mathcal{T}_h \) that share an edge with \( K \), and by \( \omega_E \) the union of elements having in common the edge \( E \). Given any edge \( E \in \mathcal{E}(\Omega) \) with length \( h_E = |E| \) we assign one fixed unit normal \( \nu_E := (\nu_1, \nu_2) \) and tangential vector \( \tau_E := (-\nu_2, \nu_1) \). For \( E \) on the boundary we choose \( \nu_E = \nu \) the unit outward normal to \( \Omega \). Once \( \nu_E \) and \( \tau_E \) have been fixed on \( E \), in relation to \( \nu_E \) one defines the elements \( K_- \in \mathcal{T}_h \) and \( K_+ \in \mathcal{T}_h \), with \( E = K_+ \cap K_- \) and \( \omega_E = K_+ \cup K_- \). Given \( E \in \mathcal{E}(\Omega) \) and some \( \mathbb{R}^d \)-valued function \( v \) defined in \( \Omega \), with \( d = 1, 2 \), we denote by \( [v] := (v|_{K_+})|_E - (v|_{K_-})|_E \) the jump of \( v \) across \( E \).

Let \( \hat{K} \) be a reference element. In the case of triangles \( \hat{K} := \{ (\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1 - \xi \} \), and quadrilaterals \( \hat{K} := [-1, 1]^2 \). The invertible linear (resp. bilinear) transformation \( \hat{K} \rightarrow K \) is denoted by \( F_K \) for any triangle (resp. quadrilateral) \( K \in \mathcal{T}_h \) with the Jacobian matrix \( DF_K \) and its determinant \( J_K \).

Let \( S^1_0(\mathcal{T}_h) \) denote the lowest order conforming finite element space over \( \mathcal{T}_h \) which reads

\[
S^1_0(\mathcal{T}_h) := \{ v \in H^1_0(\Omega), v|_K \circ F_K \in Q_1(\hat{K}), \forall K \in \mathcal{T}_h \}. 
\]

Given a non negative integer \( k \), the space \( Q_k(\omega) \) consists of polynomials of total degree at most \( k \) defined over \( \omega \) in the case \( \omega = K \) is a triangle whereas it denotes polynomials of degree at most \( k \) in each variable in the case \( K \) is a quadrilateral.

With the first order conforming finite element space \( S^1_0(\mathcal{T}_h) \), we consider the Clément-type interpolation operator or any other regularized conforming finite element approximation operator \( \mathcal{J} : H^1_0(\Omega) \mapsto S^1_0(\mathcal{T}_h) \) with the properties

\[
\| \nabla \varphi \|_{L^2(K)} + \| h_K^{-1}(\varphi - \mathcal{J} \varphi) \|_{L^2(K)} \lesssim \| \nabla \varphi \|_{L^2(\omega_K)} \quad \text{and} 
\]

\[
\| h_E^{-1/2}(\varphi - \mathcal{J} \varphi) \|_{L^2(E)} \lesssim \| \nabla \varphi \|_{L^2(\omega_E)} 
\]

for all \( K \in \mathcal{T}_h, E \in \mathcal{E} \), and \( \varphi \in H^1_0(\Omega) \). The existence of such operators is guaranteed, for instance, in [44, 90, 32, 22].

Given \( g \in L^2(\Omega) \), let \( g_h \in Q_k(\mathcal{T}_h) \) denote its projection on the (possibly discontinuous) piecewise polynomial space of degree \( k \) with respect to \( \mathcal{T}_h \). We refer to \( \text{osc}(g) \) as the oscillation of \( g \)

\[
\text{osc}^2(g) := \sum_{K \in \mathcal{T}_h} \alpha^2 h_K^2 \min_{g_h \in Q_k(\mathcal{K})} \| g - g_h \|_{L^2(K)}^2. 
\]
Here and throughout this paper, $\omega_K$ denotes the element patch defined as

$$\omega_K := \{ T \in \mathcal{T}_h : \bar{T} \cap \bar{K} \neq \emptyset \}.$$ 

2.3. Some preliminary results. By the Helmholtz decomposition, there exist $p \in H^1(\Omega) = H^1(\Omega) \cap L^2_0(\Omega)$ and $r \in H^1_0(\Omega)$ with

$$\delta = \nabla r + \text{Curl} \, p,$$

for any $\delta \in L^2(\Omega)^2$. We have the following result

**Lemma 2.1.** There exist $p \in \dot{H}^1(\Omega)$ and $\sigma \in L^2(\Omega)^2$ such that

$$\sup_{0 \neq (\psi, v) \in H^1_0(\Omega)^3} \frac{(\nabla v - \psi, \delta)_{L^2(\Omega)}}{||\psi, v||} = ||\text{Curl} \, p||_{H^{-1}(\Omega)} + \frac{||\sigma||_{L^2(\Omega)}}{\alpha},$$

for any $\delta \in L^2(\Omega)^2$ with $\delta = \sigma + \text{Curl} \, p$.

**Remark 2.2.** Up to a Stokes Theorem, this result is actually hidden in the proof of [38, Lemma 4.1]. Here we give a direct proof based on the aforementioned Helmholtz decomposition.

**Proof.** Given $\delta \in Q\setminus\{0\}$, we follow the idea of [38, Lemma 4.1] to let $(\psi, v) \in H^1_0(\Omega)^3$ solve, for any $(z, s) \in H^1_0(\Omega)^3$,

$$\begin{align*}
(\nabla \psi, \nabla z)_{L^2(\Omega)} + (\psi, z)_{L^2(\Omega)} + (\alpha^2(\nabla v - \psi), \nabla s - z)_{L^2(\Omega)} \\
= (\nabla s - z, \delta)_{L^2(\Omega)} = (\nabla s - z, \nabla r + \text{Curl} \, p_1)_{L^2(\Omega)}.
\end{align*}
$$

In the second equation, we use the Helmholtz decomposition. Taking $z = 0$, one finds that there exists $q \in \dot{H}^1(\Omega)$ with

$$\nabla r = \alpha^2(\nabla v - \psi) + \text{Curl} \, q.$$

Let $\sigma = \alpha^2(\nabla v - \psi)$ and $p = p_1 + q$, we have

$$\delta = \sigma + \text{Curl} \, p.$$

With $s = 0$, we deduce

$$\begin{align*}
||\text{Curl} \, p||_{H^{-1}(\Omega)} &= \sup_{z \in H^1_0(\Omega)^3} \frac{(-z, \text{Curl} \, p)_{L^2(\Omega)}}{||z||_{H^1(\Omega)}} \\
&= \sup_{z \in H^1_0(\Omega)^3} \frac{(\nabla \psi, \nabla z)_{L^2(\Omega)} + (\psi, z)_{L^2(\Omega)}}{||z||_{H^1(\Omega)}} \\
&\leq ||\psi||_{H^1(\Omega)},
\end{align*}$$

which implies that

$$\begin{align*}
||\text{Curl} \, p||_{H^{-1}(\Omega)} + \frac{||\sigma||_{L^2(\Omega)}}{\alpha} &\leq ||\psi||_{H^1(\Omega)} + ||\alpha(\nabla v - \psi)||_{L^2(\Omega)} \\
&\leq \sup_{0 \neq (\psi, v) \in H^1_0(\Omega)^3} \frac{(\nabla v - \psi, \delta)_{L^2(\Omega)}}{||\psi, v||}.
\end{align*}$$
On the other hand, we have

\[
\sup_{0 \neq (\psi, v) \in H_0^1(\Omega)^3} \frac{(\nabla v - \psi, \delta)_{L^2(\Omega)}}{\|\psi, v\|} = \sup_{0 \neq (\psi, v) \in H_0^1(\Omega)^3} \frac{(\nabla v - \psi, \sigma + \text{Curl} \, p)_{L^2(\Omega)}}{\|\psi, v\|} \leq \|\text{Curl} \, p\|_{H^{-1}(\Omega)} + \|\frac{\sigma}{\alpha}\|_{L^2(\Omega)},
\]

which ends the proof. 
\[\square\]

**Remark 2.3.** If \(\alpha\) is a global constant, one can prove that the norm \(\sup_{0 \neq (\psi, v) \in H_0^1(\Omega)^3} \frac{(\nabla v - \psi, \delta)_{L^2(\Omega)}}{\|\psi, v\|}\) is equivalent to the \(H^{-1}(\text{div}, \Omega)\) norm for any \(\delta \in Q\).

**Lemma 2.4.** [38, 35] Given any \(v \in H_0^1(\Omega)\), set \(v_h = J \, v\). Then for any \(K \in T_h\) and \(\psi \in H_0^1(\Omega)^2\),

\[
(2.13) \quad h_K^{-1}\|v - v_h\|_{L^2(K)} + h_E^{-1/2}\|v - v_h\|_{L^2(E)} \lesssim \|\nabla v - \psi\|_{L^2(\partial K)} + h_K\|\nabla \psi\|_{L^2(\partial K)},
\]

for any \(E \subset \partial K\).

**Lemma 2.5.** [36, Theorem 3.1] There holds

\[
(2.14) \quad \inf_{\phi_h \in \Theta} \|\nabla_h (\phi_h - \tilde{\phi}_h)\|_{L^2(\Omega)} \lesssim \left( \sum_{E \in \mathcal{E}} h_E^{-1} \|\phi_h\|_{L^2(E)}^2 \right)^{1/2},
\]

\[
(2.15) \quad \inf_{w_h \in W} \|\nabla_h (w_h - \tilde{w}_h)\|_{L^2(\Omega)} \lesssim \left( \sum_{E \in \mathcal{E}} h_E^{-1} \|w_h\|_{L^2(E)}^2 \right)^{1/2}.
\]

**Lemma 2.6.** There holds

\[
(2.16) \quad \inf_{\phi_h \in \Theta} \sum_{K \in T_h} \min\left(1, \frac{1}{h_K} \frac{1}{t}\right)^2 \|\phi_h - \tilde{\phi}_h\|_{L^2(K)}^2 \lesssim \sum_{E \in \mathcal{E}} h_E^{-1} \|\phi_h\|_{L^2(E)}^2,
\]

\[
(2.17) \quad \inf_{\phi_h \in \Theta} \|\alpha (\phi_h - \tilde{\phi}_h)\|_{L^2(\Omega)} \lesssim \left( \sum_{E \in \mathcal{E}} \alpha_E h_E \|\phi_h\|_{L^2(E)}^2 \right)^{1/2}.
\]

**Proof.** The result can be proved by proceeding along the same line of [27] with an appropriate modification. For brevity, we omit the details. \(\square\)

### 3. The Reliability of the Unifying Estimator for Finite Elements of the Reissner-Mindlin Plate Problem

This section presents a unifying analysis for finite elements of the Reissner-Mindlin plate problem. Let \(a_h(\cdot, \cdot)\) be the discrete bilinear form which will be specified for various finite element methods with a suitable linear operator \(L\) (the so-called linking operator). The unifying estimator is based on the following hypothesis:

(H1) \( (g, J \, v + L \, J \psi)_{L^2(\Omega)} - (\gamma_h, R_h(\nabla_h (J \, v + L \, J \psi) - J \psi))_{L^2(\Omega)} - a_h(\phi_h, J \psi) = 0 \), for any \((v, \psi) \in W \times \Theta\).

(H2) \( |a_h(\phi_h, \psi) - (C \, \varepsilon(\phi_h), \varepsilon(\psi))| \lesssim \left( \sum_{E \in \mathcal{E}} h_E^{-1} \|\phi_h\|_{L^2(E)}^2 \right)^{1/2}\|\psi\|_{H^1(\Omega)}, \)
for any $\psi \in \Theta$.

\begin{equation}
(H3) \quad \| \nabla L\psi_h \|_{L^2(K)} \lesssim h_K |\psi_h|_{H^1(K)}, \quad \text{for any } \psi_h \in \Theta_h.
\end{equation}

Let $W_h$ and $\Theta_h$ be the displacement and the rotation spaces respectively. We assume the following inclusion

\begin{equation}
(H4) \quad S_0^1(T_h)^3 \subset W_h \times \Theta_h.
\end{equation}

**Remark 3.1.** The first hypothesis ($H1$) is on the discrete problem and it holds actually for all methods. The second hypothesis ($H2$) is used to deal with the non-conformity in the discontinuous methods. The third hypothesis ($H3$) is from the methods based on the linked technique. For other methods, it holds with $L = 0$.

The fourth hypothesis ($H4$) implies that the finite element space $W_h \times \Theta_h$ contains the lowest order $H^1$ conforming finite element space. We propose this condition only for simplicity of the presentation since it can be removed with the theory of [36].

**Proof.** of Theorem 1.1. We use the equivalence (1.9) to show this result. The third term on the right-hand side of (1.9) is estimated by Lemma 3.2 below. Now we bound the first two terms on the right-hand side of (1.9). With the Clément interpolations $Jv$ and $J\psi$, we have the following decomposition:

\[
\text{Res}_W(v) + \text{Res}_\Theta(\psi) = (g, v - Jv)_{L^2(\Omega)} - (\gamma_h, \nabla(v - Jv))_{L^2(\Omega)} + (\gamma_h, \psi - J\psi)_{L^2(\Omega)} - (C\varepsilon_h(\phi_h), \psi - J\psi)_{L^2(\Omega)}
\]
\[
+ \varepsilon(\phi_h, J\psi)_{L^2(\Omega)} - a_h(\phi_h, J\psi) + \varepsilon(\phi_h, J\psi)_{L^2(\Omega)} - a_h(\phi_h, J\psi)
\]
\[
- (g, L J\psi)_{L^2(\Omega)} + (\gamma_h, \nabla L J\psi)_{L^2(\Omega)} + a_h(\phi_h, \psi)_{L^2(\Omega)} - (C\varepsilon_h(\phi_h), \varepsilon(\psi))_{L^2(\Omega)}
\]
\[
+ (C\varepsilon_h(\phi_h - \phi_h), \varepsilon(\psi))_{L^2(\Omega)} + (\gamma_h - \alpha^2(\nabla \tilde{w}_h - \phi_h) - \gamma'_h, \nabla v - \psi)_{L^2(\Omega)}
\]
\[
+ (\gamma_h, (R_h - I)(\nabla (J v + L J \psi) - J \psi))_{L^2(\Omega)} = I_1 + I_2 + \cdots + I_{10}.
\]

Integrating by parts and using (2.13) can estimate the first term $I_1$ by

\[
|I_1| \lesssim \left( \sum_{K \in T_h} \alpha_K^2 h_K^2 \| \text{div} \gamma_h + g \|_{L^2(K)}^2 \right)^{1/2} ||\psi, v||
\]
\[
+ \left( \sum_{E \in \mathcal{E}(\Omega)} \alpha_E^2 h_E \| [\gamma_h] \cdot \nu_E \|_{L^2(E)}^2 \right)^{1/2} ||\psi, v||.
\]
\[
(3.1)
\]
An integration by parts together with (2.2)-(2.3) proves
\[ |I_2| \lesssim \left( \sum_{K \in \mathcal{T}_h} h_K^2 \| \text{div} \mathcal{C} \varepsilon_h(\phi_h) + \gamma_h \|_{L^2(K)}^2 \right)^{1/2} \| \psi \|_{H^1(\Omega)} \]
\[ + \left( \sum_{E \in \mathcal{E}(\Omega)} h_E \| [\mathcal{C} \varepsilon_h(\phi_h)] \cdot \nu_E \|_{L^2(E)}^2 \right)^{1/2} \| \phi_h \|_{H^1(\Omega)}. \]
(3.2)

The third term \( I_3 \) and \( I_7 \) is bounded by Assumption (H2), which reads
\[ |I_3| + |I_7| \lesssim \left( \sum_{E \in \mathcal{E}} h_E^{-1} \| \phi_h \|_{L^2(E)}^2 \right)^{1/2} \| \psi \|_{H^1(\Omega)}. \]
(3.3)

It follows from the discrete problem that
\[ I_4 + I_5 = 0. \]
(3.4)

Taking \( v_h = \mathcal{J} L \mathcal{J} \psi \) and \( \psi_h = 0 \) in (H1) and applying Assumption (H3) and (2.2)-(2.3) gives
\[ |I_6| = |(g, \mathcal{J} L \mathcal{J} \psi - \mathcal{J} L \mathcal{J} \psi)_L^2(\Omega) - (\gamma_h, \nabla (\mathcal{J} L \mathcal{J} \psi - \mathcal{J} L \mathcal{J} \psi))_L^2(\Omega)| \]
\[ \lesssim \sum_{K \in \mathcal{T}_h} h_K \| g + \text{div} \beta_h \|_{L^2(K)} \| \nabla \mathcal{J} L \mathcal{J} \psi \|_{L^2(K)} \]
\[ + \sum_{E \in \mathcal{E}(\Omega)} h_E^{1/2} \| [\beta_h] \cdot \nu_E \|_{L^2(E)} \| \nabla \mathcal{J} L \mathcal{J} \psi \|_{L^2(\omega E)} \]
\[ \lesssim \left( \sum_{K \in \mathcal{T}_h} h_K^4 \| g + \text{div} \beta_h \|_{L^2(K)}^2 \right)^{1/2} \| \psi \|_{H^1(\Omega)} \]
\[ + \left( \sum_{E \in \mathcal{E}(\Omega)} h_E^3 \| [\beta_h] \cdot \nu_E \|_{L^2(E)}^2 \right)^{1/2} \| \psi \|_{H^1(\Omega)}. \]
(3.5)

Finally, Cauchy-Schwarz inequalities yield
\[ |I_8| \lesssim \| \varepsilon_h(\phi_h - \tilde{\phi}_h) \|_{L^2(\Omega)} \| \psi \|_{H^1(\Omega)}, \]
(3.6)
\[ |I_9| \lesssim \| 1/\alpha (\gamma_h - \alpha^2 (\nabla \tilde{w}_h - \tilde{\phi}_h) - \gamma'_h) \|_{L^2(\Omega)} \| \psi, v \|, \]
(3.7)
\[ |I_{10}| \lesssim \mu_h(\gamma_h) \| \psi, v \|. \]
(3.8)

This completes the proof. \[ \Box \]

Define
\[ \tilde{\eta}_Q = \| \alpha \tilde{\tau}_h \|_{L^2(\Omega)} + \left( \sum_{K \in \mathcal{T}_h} \min(1, \frac{h_K}{t})^2 \| \text{rot} \tilde{\tau}_h \|_{L^2(K)}^2 \right)^{1/2} \]
\[ + \left( \sum_{E \in \mathcal{E}} \min(1, \frac{h_E}{t^2}) \| [\tilde{\tau}_h] \cdot \tau_E \|_{L^2(E)}^2 \right)^{1/2} + \sup_{0 \neq p \in H^1(\Omega)} \frac{(\tilde{\tau}_h, \text{Curl} \mathcal{J} p)_{L^2(\Omega)}}{\| p \|_{L^2(\Omega)} + \| \nabla p \|_{L^2(\Omega)}}. \]
Lemma 3.2. Let \( \tilde{r}_h = \beta^2 \tilde{\gamma}_h - (\nabla \tilde{w}_h - \tilde{\phi}_h) \). Then

\[
(3.9) \quad \sup_{0 \neq \delta \in Q} \frac{(\tilde{r}_h, \delta)}{\|\delta\|_Q} \lesssim \tilde{\eta}_Q.
\]

Proof. First we state that

\[
(3.10) \quad \sup_{\delta \in Q} \frac{(\tilde{r}_h, \delta)}{\|\delta\|_Q} \lesssim \|\alpha \tilde{r}_h\|_{L^2(\Omega)} + \sup_{0 \neq p \in H^1(\Omega)} \frac{(\tilde{r}_h, \text{Curl} p)}{\|p\|_{L^2(\Omega)} + \|t \nabla p\|_{L^2(\Omega)}}.
\]

(The proof of this statement is postponed to the end of this lemma). Let \( Jp \) be the Clément interpolation of \( p \). Then,

\[
(\tilde{r}_h, \text{Curl} p)_{L^2(\Omega)} = (\tilde{r}_h, \text{Curl}(p - Jp))_{L^2(\Omega)} + (\tilde{r}_h, \text{Curl} Jp)_{L^2(\Omega)}
\]

\[
(3.11) \quad = \sum_{K \in T_h} \langle \text{rot} \tilde{r}_h, p - Jp \rangle_{L^2(K)} + \sum_{E \in \mathcal{E}} \int_E [\tilde{r}_h] \cdot \tau_E (p - Jp) \, ds
\]

\[
+ (\tilde{r}_h, \text{Curl} Jp)_{L^2(\Omega)}.
\]

By the trace theorem, we have

\[
(3.12) \quad \|p - Jp\|_{L^2(\mathcal{E})} \lesssim \|p - Jp\|_{L^2(\Omega_E)} + \|p - Jp\|_{L^2(\Omega_E)}^{1/2} \|\nabla (p - Jp)\|_{L^2(\Omega_E)}^{1/2}.
\]

With \( \|p - Jp\|_{L^2(K)} \lesssim \min(\|p\|_{L^2(\omega_K)}, h_K \|\nabla p\|_{L^2(\omega_K)}) \) and the inverse estimate, we deduce the desired conclusion as

\[
(3.13) \quad \sup_{0 \neq p \in H^1(\Omega)} \frac{(\tilde{r}_h, \text{Curl} p)_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)} + \|t \nabla p\|_{L^2(\Omega)}} \lesssim \left( \sum_{K \in T_h} \min(1, \frac{h_K}{t})^2 \|\text{rot} \tilde{r}_h\|_{L^2(K)}^2 \right)^{1/2}
\]

\[
+ \left( \sum_{E \in \mathcal{E}} \min\left(\frac{1}{t}, \frac{h_E}{t^2}\right) \|\tilde{r}_h\|_{L^2(\mathcal{E})} \right)^{1/2} + \sup_{0 \neq p \in H^1(\Omega)} \frac{(\tilde{r}_h, \text{Curl} Jp)_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)} + \|t \nabla p\|_{L^2(\Omega)}}.
\]

This ends the proof. \( \square \)

When \( \tilde{r}_h \) is a piecewise polynomial, the statement (3.10) can be found in [38, Lemma 6.1]. Here we give a proof for the general case and it is slightly different from that of [38, Lemma 6.1]. Given \( \delta \in Q \), it follows from Lemma 2.1 that there exist \( p \in H^1(\Omega) \) and \( \sigma \in Q \) with \( \delta = \text{Curl} p + \sigma \) and

\[
(3.14) \quad \sup_{0 \neq (\psi, v) \in H^1_0(\Omega)^3} \frac{(\nabla v - \psi, \delta)}{\|\psi, v\|} = \|\text{Curl} p\|_{H^{-1}(\Omega)} + \|\frac{\sigma}{\alpha}\|_{L^2(\Omega)}.
\]

Since \( \int_{\Omega} p \, dx \, dy = 0 \), one can show [38, Lemma 6.1]

\[
(3.15) \quad \|p\|_{L^2(\Omega)} \approx \|\text{Curl} p\|_{H^{-1}(\Omega)}.
\]

This consideration and (3.14) lead to

\[
(3.16) \quad \sup_{\delta \in Q} \frac{(\tilde{r}_h, \delta)}{\|\delta\|_Q} \lesssim \sup_{0 \neq (\sigma, p) \in (Q, H^1(\Omega))^3} \frac{(\tilde{r}_h, \text{Curl} p + \sigma)}{\|p\|_{L^2(\Omega)} + \|\sigma/\alpha\|_{L^2(\Omega)} + \|\beta(\text{Curl} p + \sigma)\|_{L^2(\Omega)}}.
\]
Since $\alpha^{-1} > t$ and $\beta > t$, we deduce it as
\[
\|p\|_{L^2(\Omega)} + \|\frac{\sigma}{\alpha}\|_{L^2(\Omega)} + t \|\text{Curl} p\|_{L^2(\Omega)}
\]
\[
\leq \|p\|_{L^2(\Omega)} + \frac{\sigma}{\alpha}\|\|_{L^2(\Omega)} + t\|\text{Curl} (p + \sigma)\|_{L^2(\Omega)} + \|t\sigma\|_{L^2(\Omega)}
\]
\[
\leq \|p\|_{L^2(\Omega)} + 2\|\frac{\sigma}{\alpha}\|_{L^2(\Omega)} + \|\text{Curl} (p + \sigma)\|_{L^2(\Omega)}.
\]

One can use this estimate in (3.16) to prove (3.10).

**Remark 3.3.** One can show that the term $\|\alpha \tilde{r}_h\|_{L^2(\Omega)}$ can be improved to $\|\frac{1}{1/\alpha + \beta} \tilde{r}_h\|_{L^2(\Omega)}$; cf. [38] for details.

We give a corollary of the previous lemma to end this section. We first propose some hypothesis: Assume that
\[
(H5) \quad (\nabla_h w_h, \text{Curl } Jp)_{L^2(\Omega)} = 0, \text{ for any } p \in \hat{H}^1(\Omega) := H^1(\Omega) \cap L^2_0(\Omega),
\]
\[
(H6) \quad \text{Curl } Jp \in \Gamma_h, \text{ for any } p \in \hat{H},
\]
\[
(H7) \quad (R_h \phi_h - \phi_h, \text{Curl } Jp)_{L^2(\Omega)} = 0, \text{ for any } p \in \hat{H}^1(\Omega).
\]

**Lemma 3.4.** Let $\tilde{r}_h = R_h((\nabla_h w_h - \phi_h) - (\nabla \tilde{w}_h - \tilde{\phi}_h)$ and $(H5)$-$(H7)$ hold. Then,
\[
\tilde{\eta}_Q \leq \|\alpha(I - R_h)((\nabla_h w_h - \phi_h)\|_{L^2(\Omega)}
\]
\[+ \|\alpha \nabla_h (w_h - \tilde{w}_h)\|_{L^2(\Omega)} + \|\alpha (\phi_h - \tilde{\phi}_h)\|_{L^2(\Omega)}
\]
\[+ \frac{\min(1, \frac{h}{K})}{t} \text{rot}(R_h((\nabla_h w_h - \phi_h) + \phi_h)\|_{L^2(K)})^{1/2}
\]
\[
+ \frac{\min(1, \frac{h}{E})}{t} \text{rot}(R_h((\nabla_h w_h - \phi_h)\|_{L^2(E)})^{1/2}
\]
\[
+ \frac{\min(1, \frac{h}{K})}{t} \|\phi_h - \tilde{\phi}_h\|^2_{L^2(K)}
\]
\[
+ \frac{\min(1, \frac{h}{K})}{t} \|\text{rot}(\phi_h - \tilde{\phi}_h)\|^2_{L^2(K)}
\]
\[
(3.18)
\]

**Proof.** This is a direct consequence of Lemma 3.2 with $\tilde{r}_h = R_h((\nabla_h w_h - \phi_h) - (\nabla \tilde{w}_h - \tilde{\phi}_h)$ and Conditions $(H5)$-$(H6)$.

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4. **Finite elements based on linked interpolation technique**

This section presents a posteriori analysis of finite element methods for the Reissner-Mindlin plate problem based on the linked interpolation technique [12, 99, 94, 13, 102, 14, 49, 75, 78, 11, 77].
4.1. The linked interpolation technique. We first follow [77] to describe the linking interpolation technique. Given the finite element spaces $\Theta_h \subset \Theta$, $W_h \subset W$ and $\Gamma_h \subset Q$ with a suitable linear operator (the so-called linking operator) the norm herein also converges.

(4.1) \[ L : \Theta_h \rightarrow W, \]

the linked interpolation scheme for the Reissner-Mindlin plate problem reads: Find $(w_h, \phi_h, \gamma_h) \in (W_h, \Theta_h, \Gamma_h)$ with

\[
\alpha_h^2 = \alpha^2 |_K := \frac{\lambda}{t^2 + h_K^2}, \quad \beta_h^2 = \beta^2 |_K := \frac{t^2(h_h^2 + t^2)}{\lambda h_K^2},
\]

and take $\tilde{w}_h = w_h\phi_h$, $\tilde{\phi}_h = \phi_h$, and $\tilde{\gamma}_h = \gamma_h = \beta^{-2} R_h(\nabla w_h - \phi)$ in Theorem 1.1 and Theorem 1.2. Then $\tilde{r}_h = (R_h - I)(\nabla w_h - \phi)$.

\[
\mathcal{E}(\phi - \phi, w - w_h, \gamma - \gamma_h) = \|\phi - \phi_h, w - w_h\| + \|\gamma - \gamma_h\|_Q + t\|\gamma - \gamma_h\|_{L^2(\Omega)} + \|\gamma - \gamma_h\|_{H^{-1}(\Omega)}.
\]

Theorem 4.1. Under Hypothesis (H1)-(H7),

\[
\mathcal{E}(\phi - \phi, w - w_h, \gamma - \gamma_h) \lesssim \eta_h.
\]

Theorem 4.2. Under the hypothesis (H1)-(H7),

\[
\eta_h \lesssim \mathcal{E}(\phi - \phi, w - w_h, \gamma - \gamma_h) + \text{osc}(g).
\]

Remark 4.3. Since $w_h \in W$ and $\phi_h \in \Theta$, the last two terms of $\eta_h$ vanish for this case. Moreover since $R_h$ is a projection operator $\mu_h(\gamma_h) = 0$.

Remark 4.4. The estimator $\eta_h$ herein is different from that of [77] by the sixth and seventh terms. More important, we abandon the saturation assumption used in that paper.

Remark 4.5. The norm analyzed herein is slightly different from that used in [77]. Since

\[
\|\gamma' - \gamma_h'\|_Q \lesssim \|\gamma' - \gamma_h'\|_{L^2(\Omega)} + \|\beta(\gamma' - \gamma_h')\|_{L^2(\Omega)} \lesssim (t + h)\|\gamma - \gamma_h\|_{L^2(\Omega)},
\]

the norm herein also converges.

Remark 4.6. The assumptions supposed in Theorem 4.1 and Theorem 4.2 are valid for all finite element methods in [12, 99, 94, 13, 102, 14, 49, 75, 78, 11, 77]. Before showing these two theorems, we first establish a mediate result.
Lemma 4.7. Under the hypothesis (H1)-(H7),

\[ \| \phi - \phi_h, w - w_h^* \| + \| \gamma' - \gamma'_h \|_Q \lesssim \eta_h. \]  

**Proof.** By the unifying theory developed in Theorem 1.1 and Theorem 1.2, we only need to estimate the last six terms of \( \tilde{\eta}_h \) and the last two terms on the right-hand side of (1.15). Since \( \tilde{\phi}_h = \phi_h, \tilde{\omega}_h = w_h \), and \( \tilde{\gamma}'_h = \gamma'_h \), the last two terms on the right-hand side of (1.15) vanish. The other terms except the seventh term can directly be bounded by inserting \( \tilde{r}_h = (R_h - I)(\nabla w_h^* - \phi_h) \), \( \tilde{r}'_h = \gamma'_h \), \( \tilde{\omega}_h = w_h^* \), and \( \tilde{\phi}_h = \phi_h \), into \( \tilde{\eta}_h \). Thanks to (H5)-(H7), we obtain

\[ \sup_{0 \neq p \in \mathcal{H}_1(\Omega)} \frac{\langle \tilde{r}_h, \text{Curl} \mathcal{J} p \rangle_{L^2(\Omega)}}{\| p \|_{L^2(\Omega)} + \| t \nabla p \|_{L^2(\Omega)}} = 0. \]

This ends the proof. \( \square \)

**Proof.** of Theorem 4.1

\[ \sum_{K \in T_h} \| \alpha_K(\nabla w - \phi - R_h(\nabla w_h^* - \phi_h)) \|_{L^2(K)}^2 \]

\[ \leq \sum_{K \in T_h} 2\| \alpha_K(\nabla w - \nabla w_h^* + \phi_h - \phi) \|_{L^2(K)}^2 \]

\[ + \sum_{K \in T_h} 2\| \alpha_K(I - R_h)(\nabla w_h^* - \phi_h) \|_{L^2(K)}^2. \]

This, together with Lemma 4.7, implies that

\[ \| \phi - \phi_h, w - w_h^* \| + \| \gamma' - \gamma'_h \|_Q + t\| \gamma - \gamma_h \|_{L^2(\Omega)} \lesssim \eta_h. \]

We remain to bound \( \| \gamma - \gamma_h \|_{H^{-1}(\Omega)} \). Given \( \psi \in \Theta \), we have

\[ (\gamma - \gamma_h, \psi) = a(\phi - \phi_h, \psi) + a(\phi_h, \psi) - (\gamma_h, \psi). \]

A similar argument for the term \( a(\phi_h, \psi) - (\gamma_h, \psi) \) in the previous lemma leads to

\[ \| \gamma - \gamma_h \|_{H^{-1}(\Omega)} \lesssim \eta_h. \]

This ends the proof. \( \square \)

Now we prove the uniform efficiency of \( \eta_h \).

**Proof.** of Theorem 4.6. Since all terms except the last two terms are shown in [77], we only need to bound them. Define \( a = b_K \text{rot} \mathbf{r}_{1,h} \), we have

\[ (\text{rot} \mathbf{r}_{1,h}, \text{rot} \mathbf{r}_{1,h})_{L^2(K)} \approx (\text{rot} \mathbf{r}_{1,h}, a)_{L^2(K)} \]

\[ = (\text{rot}(\nabla w_h^* - \phi_h - \lambda^{-1}t^2 \gamma_h), a)_{L^2(K)} \]

\[ = (\text{rot} (\phi - \phi_h + \lambda^{-1}t^2 (\gamma - \gamma_h)), a)_{L^2(K)} \]

\[ \lesssim \| \text{rot} (\phi - \phi_h) \|_{L^2(K)} \| a \|_{L^2(K)} + t^2 \| \gamma - \gamma_h \|_{L^2(K)} \| \text{Curl} a \|_{L^2(K)} \]

\[ \lesssim \| \text{rot} (\phi - \phi_h) \|_{L^2(K)} \| a \|_{L^2(K)} + \frac{t^2}{\eta_K} \| \gamma - \gamma_h \|_{L^2(K)} \| a \|_{L^2(K)} \].
This proves
\begin{equation}
\min(1, \frac{h_E}{t}) \| \text{rot} \, \mathbf{r}_{1,h} \|_{L^2(K)} \lesssim \| \text{rot}(\mathbf{\phi} - \mathbf{\phi}_h) \|_{L^2(K)} + t \| \mathbf{\gamma} - \mathbf{\gamma}_h \|_{L^2(K)}.
\end{equation}

Given any $E \in \mathcal{E}$, let $\delta_E = \min(1, \frac{1}{h_E})$. To prove the efficiency of the ninth term, we follow the idea of [38] to introduce $v_E \in H_0^1(\omega_E)$ with
\[ (\text{rot}_h \, \mathbf{r}_{1,h}, v_E)_{L^2(\omega_E)} = 0, \]
\begin{align}
\| v_E \|_{L^2(\omega_E)} & \lesssim \delta_E h_E^{1/2} \| [\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E \|_{L^2(\omega_E)}, \\
\| \nabla v_E \|_{L^2(\omega_E)} & \lesssim \sqrt{\delta_E + 1/\delta_E} h_E^{-1/2} \| [\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E \|_{L^2(\omega_E)}. \end{align}

We refer the reader to [38, Theorem 5.3 and Theorem 6.3] for the construction of functions with these properties. Set $\rho_E^2 = \min(1, \frac{h_E}{t})$ and let $v = \sum_{E \in \mathcal{E}} \rho_E^2 v_E$, integrating by parts and using (4.16) yields
\begin{equation}
\sum_{E \in \mathcal{E}} \rho_E^2 \| [\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E \|_{L^2(E)}^2 \approx \sum_{E \in \mathcal{E}} \rho_E^2 ([\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E, v_E)_{L^2(E)}
\end{equation}
\begin{equation}
\leq \| \mathbf{r}_{1,h} \|_{Q^*} \| p \|_{L^2(\Omega)} + t \| \nabla p \|_{L^2(\Omega)},
\end{equation}
with $\| \mathbf{r}_{1,h} \|_{Q^*} = \sup_{\delta \in Q} \frac{\langle \delta \mathbf{r}_{1,h} \rangle_{L^2(\Omega)}}{\| \delta \|_{Q}}$. In what follows we bound the term $\| p \|_{L^2(\Omega)} + t \| \nabla p \|_{L^2(\Omega)}$.
\begin{equation}
\| p \|_{L^2(\Omega)}^2 + t^2 \| \nabla p \|_{L^2(\Omega)}^2 \lesssim \sum_{E \in \mathcal{E}} ([p]_{L^2(\omega_E)}^2 + t^2 \| \nabla p \|_{L^2(\omega_E)}^2)
\end{equation}
\begin{equation}
\lesssim \sum_{E \in \mathcal{E}} \rho_E^4 (h_E \delta_E + h_E^{-1} (\delta_E + 1/\delta_E) t^2) \| [\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E \|_{L^2(\omega_E)}^2
\end{equation}
\begin{equation}
\lesssim \sum_{E \in \mathcal{E}} \rho_E^2 \| [\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E \|_{L^2(E)}^2.
\end{equation}

This and (1.13) show that
\begin{equation}
\left( \sum_{E \in \mathcal{E}} \rho_E^2 \| [\mathbf{r}_{1,h}] \cdot \mathbf{\tau}_E \|_{L^2(E)}^2 \right)^{1/2} \lesssim \| \mathbf{r}_{1,h} \|_{Q^*} \leq \mathcal{E}(\mathbf{\phi} - \mathbf{\phi}_h, w - w_h^*, \mathbf{\gamma} - \mathbf{\gamma}_h),
\end{equation}
which completes the proof. \qed

**Remark 4.8.** We remark that the a posteriori analysis presented herein can be straightforwardly extended to the schemes taking advantage of the linked technique in connection with the partial selective reduced integration technique [10].

4.3. **Some lower order examples.** This subsection gives two examples for this class of methods with (H1)-(H7); cf. [12, 99, 94, 13, 102, 14, 49, 75, 78, 11, 77] for other examples of this class of methods. These two elements are summarized in Table 1.
4.3.1. The triangular linear element. This element is described by the finite element spaces

\begin{align}
\Theta_h &= \{\psi \in \Theta, \psi|_K \in (Q_1(K) \oplus B_3(K))^2, \forall K \in \mathcal{T}_h\}, \\
W_h &= \{v \in W, v|_K \in Q_1(K), \forall K \in \mathcal{T}_h\}, \\
\Gamma_h &= \{\sigma \in Q, \sigma|_K \in (Q_0(K))^2, \forall K \in \mathcal{T}_h\},
\end{align}

where \(B_3(K)\) denotes the cubic bubble function space over \(K\). For each element \(K \in \mathcal{T}_h\), we set

\begin{equation}
\phi_i = \lambda_j \lambda_k \text{ and } EB_2(K) = \text{span}\{\phi_i\}_{i=1}^3
\end{equation}

with \(\{\phi_i\}_{i=1}^3\) the barycentric coordinates of \(K\). The indices \((i, j, k)\) form a permutation of the set \((1, 2, 3)\). Then, the linked operator is locally defined as [77]

\begin{equation}
L \psi_h|_K = \sum_{i=1}^3 c_i \phi_i \in EB_2(K),
\end{equation}

for any \(\psi_h \in \Theta_h\). The interpolation coefficients \(c_i, i = 1, 2, 3\), are determined by

\begin{equation}
(\nabla L \psi_h - \psi_h) \cdot \tau_E \text{ is constant on each } E.
\end{equation}

For this case, (H1)-(H2) and (H4)-(H7) hold. The proof of (H3) can be found in [75, 78].

4.3.2. The triangular quadratic element. For this element, the finite element spaces are chosen as

\begin{align}
\Theta_h &= \{\psi \in \Theta, \psi|_K \in (Q_2(K))^2 \oplus ((Q_0(K))^2 \oplus \nabla B_3(K))b_K, \forall K \in \mathcal{T}_h\}, \\
W_h &= \{v \in W, v|_K \in Q_2(K) \oplus B_3(K), \forall K \in \mathcal{T}_h\}, \\
\Gamma_h &= \{\sigma \in Q, \sigma|_K \in (Q_1(K))^2 \oplus \nabla B_3(K), \forall K \in \mathcal{T}_h\},
\end{align}
with \( b_K = 27 \lambda_1 \lambda_2 \lambda_3 \). For any \( K \in \mathcal{T}_h \), we set
\[
(4.29) \quad \phi_i = \lambda_j \lambda_k (\lambda_k - \lambda_j) \text{ and } EB_3 = \text{span}\{\phi_i\}_{i=1}^3,
\]
for any \( \psi_h \in \Theta_h \). The linked operator \( L \) is locally defined as [77]
\[
(4.30) \quad L\psi_h|_K = \sum_{i=1}^3 c_i \phi_i \in EB_3(K),
\]
with the coefficients \( c_i \)'s determined by
\[
(4.31) \quad (\nabla L\psi_h - \psi_h) \cdot \tau_E \text{ is linear on } E.
\]
Also, there hold (H1)-(H2) and (H4)-(H7) for this element. This proof of (H3) is presented in [11].

5. A posterior error analysis for Arnold-Falk type finite elements

This section presents a posteriori error analysis for Arnold-Falk type finite elements for the Reissner-Mindlin plate problem [7, 100, 31, 46, 76, 61, 62]. For these methods, we have a priori convergence in the norm \( \|\gamma - \gamma_h\|_{H^{-1}(\text{div})} \) for the shear force. As we have already mentioned in the introduction, only the Arnold-Falk element [7] was analyzed in [33] under the assumption that \( h_K \lesssim t \) for any \( K \in \mathcal{T}_h \). Note that this assumption is usually invalid for the singular solution in which the adaptive finite element methods are specially interested.

5.1. Arnold-Falk type finite elements. Let \( W_h \subset L^2(\Omega) \), \( \Theta_h \subset L^2(\Omega)^2 \) and \( \Gamma_h \subset Q \) be some approximation spaces for the displacement, rotation and shear force respectively. The discrete spaces \( W_h \) and \( \Theta_h \) may be nonconforming. In those cases, the gradient operator \( \nabla \) and the symmetric gradient operator \( \varepsilon \) are defined element by element, denoted by \( \nabla h \) and \( \varepsilon_h \) respectively. We assume the reduction operator \( R_h \) satisfies (H3) defined at the beginning of Section 3.

To deal with the discontinuity of \( \Theta_h \), the usual way is to define a modified bilinear form \( a_h(\cdot, \cdot) \) as follows. Given any vector-valued function \( \psi \in \Theta_h \), we define the symmetric jump across the edge \( E \in \mathcal{E}(\Omega) \) as
\[
[\psi]_S = (\psi^+ \otimes \nu_E^+)_S + (\psi^- \otimes \nu_E^-)_S,
\]
where \((\psi \otimes \nu_E)_S\) denotes the symmetric part of the tensor, and \( \nu_E^+ \) (resp. \( \nu_E^- \)) is the outward unit normal to \( E \subset \partial K^+ \) (resp. \( \partial K^- \)). On the boundary edge, we define the jump as \( [\psi]_S = (\psi \otimes \nu_E)_S \) with \( \nu_E \) the outward unit normal to \( \partial \Omega \).

Moreover, we introduce the following discrete bilinear form with a penalty term
\[
(5.1) \quad a_h(\phi_h, \psi_h) = \sum_{K \in \mathcal{T}_h} (C\varepsilon(\phi_h), \varepsilon(\psi_h))_{L^2(K)} + \sum_{E \in \mathcal{E}_h} \frac{\gamma_E}{h_E} \int_E [\phi_h]_S : [\psi_h]_S ds
\]
with \( \gamma_E \) some constant to be chosen.

The Arnold-Falk type finite element methods read: Find \( (w_h, \phi_h, \gamma_h) \in W_h \times \Theta_h \times \Gamma_h \) with
\[
(5.2) \quad a_h(\phi_h, \psi_h) + (\gamma_h, R_h(\nabla w_h - \phi_h))_{L^2(\Omega)} = (g, v_h), \quad \text{for any } \psi_h \in \Theta_h,
\]
\[
(R_h(\nabla w_h - \phi_h), \sigma)_{L^2(\Omega)} - \lambda^{-1} \eta^2(\gamma_h, \sigma)_{L^2(\Omega)} = 0, \quad \text{for any } \sigma \in \Gamma_h.
\]
It follows that \( L = 0 \).
5.2. A posteriori error analysis. To use the notation in the previous section, we define
\begin{equation}
\gamma_h' = \beta^{-2} R_h(\nabla w_h - \phi_h), \quad \text{and} \quad \gamma' = \beta^{-2}(\nabla w - \phi).
\end{equation}
These methods converge optimally in the following norm [7, 100, 61, 62]
\begin{equation}
\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h) = \|\phi - \phi_h, w - w_h\|_h + \|\gamma' - \gamma_h'\|_Q
\end{equation}
\begin{equation}
+ \left( \sum_{E \in \mathcal{E}} h^{-1}_E \|\phi\|_{L^2(E)}^2 \right)^{1/2},
\end{equation}
with $\alpha$ a global positive constant independent of $h$ and $t$ in the spirit of [4]. Therefore the global positive constant $\beta$ follows
\begin{equation}
\beta^{-2} = \frac{\lambda}{t^2} - \alpha^2.
\end{equation}
We choose $\tilde{\gamma}_h = \gamma_h'$, and $\tilde{w}_h \in W$ and $\tilde{\phi}_h$ are arbitrary in Theorem 1.1 and Theorem 1.2. Then, $\tilde{r}_h = \beta^2 \gamma_h' - (\nabla \tilde{w}_h - \tilde{\phi}_h)$. For this case, $r_{1,h} = (I - R_h)(\nabla_h w_h - \phi_h)$ and $r_{2,h} = 0$ in $\eta_h$.

For this class of methods, we can assume that
\begin{equation}
\text{(H8)} \quad \int_E [v] ds = 0, \text{ for any } E \in \mathcal{E}, \text{ for any } v \in W_h.
\end{equation}

**Theorem 5.1.** Under Hypothesis (H1)-(H8),
\begin{equation}
\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h) \lesssim \eta_h.
\end{equation}

**Theorem 5.2.** Under Hypothesis (H1)-(H8),
\begin{equation}
\eta_h \lesssim \mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h) + \text{osc}(g).
\end{equation}

**Remark 5.3.** $\mu_h(\gamma_h) = 0$ for this class of methods.

**Remark 5.4.** We assume in Theorem 5.1 and Theorem 5.2 that $W_h$ contains a conforming subspace $S^1_h(T_h)$. This assumption does not hold for the methods of [100, 61]. However, this difficulty can be coped with by the theory of [37, 36].

**Proof.** of Theorem 5.1. We use the unifying theory in Theorem 1.2. We first bound the last two terms on the right-hand side of (1.15). Since $\tilde{\gamma}_h = \gamma_h'$, $\|\tilde{\gamma}_h' - \gamma_h'\|_Q = 0$. Applying Lemma 2.5 and Lemma 2.6 and the Poincare inequality (due to Hypothesis (H8)), we obtain
\begin{equation}
\inf_{(\tilde{\omega}_h, \tilde{\phi}_h) \in W \times \mathcal{E}} \|\phi_h - \tilde{\phi}_h, w_h - \tilde{w}_h\|_h
\end{equation}
\begin{equation}
\lesssim \left( \sum_{E \in \mathcal{E}} \alpha_E^2 h_E \|\nabla_h w_h\|_{L^2(E)} \right)^{1/2} + \left( \sum_{E \in \mathcal{E}} \alpha_E^2 h^{-1}_E \|\phi_h\|_{L^2(E)}^2 \right)^{1/2}.
\end{equation}
The seventh to the tenth terms of $\tilde{\eta}_h$ are bounded by Lemma 3.4 and Lemmas 2.5-2.6.

$$
\|1/\alpha (\gamma_h - \alpha^2 (\nabla \bar{w}_h - \bar{\phi}_h) - \gamma_h')\|_{L^2(\Omega)} \\
\leq \| \alpha (R_h (\nabla_h w_h - \phi_h) - (\nabla \bar{w}_h - \bar{\phi}_h))\|_{L^2(\Omega)} \\
\leq \| \alpha (R_h - I) (\nabla_h w_h - \phi_h)\|_{L^2(\Omega)} \\
+ \| \alpha (\nabla_h w_h - \phi_h) - \alpha (\nabla \bar{w}_h - \bar{\phi}_h))\|_{L^2(\Omega)}.
$$

(5.8)

Applications of Lemmas 2.5-2.6 give the upper bounds of the last term of the above inequality and the last term of $\tilde{\eta}_h$, which ends the proof.

Proof. of Theorem 5.2. The efficiency of the terms $\sum_{K \in T_h} h_K^2 \| \text{div} \mathcal{C} \varepsilon (\phi_h) + \gamma_h \|_{L^2(K)}^2$ and $\sum_{E \in \mathcal{E}(\Omega)} h_E \| [\mathcal{C} \varepsilon_h (\phi_h)] \cdot \nu_E \|_{L^2(E)}^2$ can be shown with an argument similar to that in [35]. The efficiency of $\sum_{E \in \mathcal{E}} \alpha^2 h^{-1}_E \| [\phi_h] \|_{L^2(E)}^2$ is straightforward. The proof for the efficiency of $\sum_{K \in T_h} \min(1, b_h \frac{h^2}{t}) \| \text{rot} r_{1,h} \|_{L^2(K)}^2$ is the same as that in Theorem 4.2.

For any piecewise polynomial functions $\bar{w}_h \in W$ and $\tilde{\phi}_h \in \Theta$, a similar argument of Theorem 4.2 gives

$$
\sum_{E \in \mathcal{E}} \min\left(1, \frac{h_E}{t^2}\right) \| [R_h (\nabla_h w_h - \phi_h)] \cdot \tau_E \|_{L^2(E)}^2 \lesssim \| \text{Res}_Q \|_Q^2.
$$

(5.9)

$$
\lesssim \| \phi - \tilde{\phi}_h, w - \bar{w}_h \| + \| \gamma' - \gamma_h' \|_Q \\
\lesssim \| \phi - \tilde{\phi}_h, w - w_h \| + \| \gamma' - \gamma_h' \|_Q + \| \tilde{\phi}_h - \phi_h, \bar{w}_h - w_h \|.
$$

In the theory of [27], we can choose piecewise polynomial functions $\bar{w}_h \in W$ and $\tilde{\phi}_h \in \Theta$ such that

$$
\| \phi_h - \phi_h, \bar{w}_h - w_h \| \lesssim \sum_{E} \alpha^2 h^{-1}_E \| [\phi_h] \|_{L^2(E)}^2 + h^2_E \| [\nabla_h w_h] \cdot \tau_E \|_{L^2(E)}^2.
$$

(5.10)

The efficiency of the second term on the right-hand side (5.10) is shown in (5.13). Inserting this estimate into (5.9) proves the efficiency of $\sum_{E \in \mathcal{E}} \min\left(1, \frac{h_E}{t^2}\right) \| [R_h (\nabla_h w_h - \phi_h)] \cdot \tau_E \|_{L^2(E)}^2$. It remains to show the efficiency of the rest two terms.

$$
\| \alpha R_{1,h} \|_{L^2(\Omega)} = \| \alpha (\nabla_h w_h - \phi_h - \nabla w + \phi_h) + \alpha \lambda^{-1} t^2 (\gamma - \gamma_h) \|_{L^2(\Omega)} \\
\lesssim \| \phi - \gamma_h \|_{L^2(\Omega)} + \| \gamma - \gamma_h \|_{L^2(\Omega)}.
$$

(5.11)

To bound the term $\sum_{E \in \mathcal{E}} \alpha^2 h_E \| [\nabla_h w_h] \cdot \tau_E \|_{L^2(E)}^2$, we define $v_E = b_E [\nabla_h w_h] \cdot \tau_E$. Thus,

$$
\| [\nabla_h w_h] \cdot \tau_E \|_{L^2(E)}^2 \lesssim \| [\nabla_h w_h] \cdot \tau_E, w_h \|_{L^2(E)} = (\nabla_h w_h, \text{Curl} v_E)_{L^2(\omega_E)} \\
= (\nabla_h (w_h - w), \text{Curl} v_E)_{L^2(\omega_E)} \leq \| \alpha \nabla_h (w_h - w) \|_{L^2(\omega_E)} \alpha^{-1} \| \text{Curl} v_E \|_{L^2(\omega_E)} \\
\lesssim \| \alpha \nabla_h (w_h - w) \|_{L^2(\omega_E)} \alpha^{-1} h^{-1/2}_E \| [\nabla_h w_h] \cdot \tau_E \|_{L^2(\omega_E)}.
$$

(5.12)
Table 2. Lower Order Examples of Finite Element Spaces $W_h$, $\Theta_h$ and $\Gamma_h$ of the Arnold-Falk type.

This shows that

\begin{equation}
\sum_{E \in \mathcal{E}} \alpha^2 h_E \| [\nabla w_h] \cdot \tau_E \|_{L^2(E)}^2 \lesssim \| \alpha \nabla (w_h - w) \|_{L^2(\Omega)}^2,
\end{equation}

which completes the proof \(\square\)

5.3. Examples. This subsection presents some examples in literature for this class of schemes with (H1)-(H8). Some lower order examples are displayed in Table 2.
Remark 5.5. The schemes proposed in [65] can also be analyzed by the theory in this section.

5.3.1. The Arnold-Falk element. Based on the regular triangulation $T_h$ into simplices, the set of midpoints $\mathcal{M}$ of edges, the nonconforming Crouzeix-Raviart finite element space reads

$$W_h^{CR} := \{ v \in L^2(\Omega), v|_K \in Q_1(K), \int_E [v]ds = 0 \text{ on } E \in \mathcal{E}(\Omega),$$

$$\text{and } \int_E vds = 0 \text{ on } E \in \mathcal{E} \cap \partial \Omega \}.$$ (5.14)

In the Arnold-Falk element [7], the Crouzeix-Raviart finite element space $W_h^{CR}$ is chosen as the displacement space $W_h$, the space $\Theta_h$ defined in Subsubsection 4.3.1 is taken as the rotation space. The discrete shear force space $\Gamma_h$ is the piecewise constant space $(Q_0)^2$. Therefore, the reduction operator $R_h$ is the piecewise constant projection operator from $(L^2(\Omega))^2$ onto $(Q_0)^2$.

5.3.2. The Ye element. This is a rectangular scheme. Let $NR(\hat{K}) := \text{span}\{1, \xi, \eta, \xi^2 - \eta^2\}$, the nonconforming rotated $Q_1$ space reads [87]

$$W_h^{NR} := \{ v \in L^2(\Omega), (v|_K) \circ F_K \in NR(\hat{K}), \int_E [v]ds = 0 \text{ on } E \in \mathcal{E}(\Omega)$$

$$\text{and } \int_E vds = 0 \text{ on } E \in \mathcal{E} \cap \partial \Omega \}.$$ (5.15)

In the Ye element [100], this nonconforming rotated $Q_1$ element space is taken as the displacement space. The rotation and the shear force spaces read, respectively,

$$\Theta_h := \{ \psi \in \Theta, (\psi|_K) \circ F_K \in (Q_2(K))^2, \text{ for any } K \in T_h \},$$

$$\Gamma_h := \left\{ v \in (L^2(\Omega))^2, v|_K = \left( \frac{b_K + d_Kx}{c_K - d_Ky} \right), \forall K \in T_h \right\},$$

with $b_K$, $c_K$, and $d_K$, some free parameters. The reduction operator is the $L^2$ projection operator from $(L^2(\Omega))^2$ onto $\Gamma_h$.

Remark 5.6. For this element, the discrete problem (5.2) is slightly different from that of [100]; which reads: Find $(w_h, \phi_h) \in W_h \times \Theta_h$ with

$$a_h(\phi_h, \psi_h) + (\gamma_h, \nabla_h w_h - R_h \psi_h) = (g, v_h), \text{ for any } \psi_h \in \Theta_h,$$

$$\gamma_h = \lambda t^{-2} (\nabla_h w_h - R_h \phi_h).$$ (5.18)

As pointed out in [61], the analysis for Problem (5.18) in [100] is only valid for the uniform square mesh. Following the argument of [61], one can prove the uniformly optimal error estimates for Problem (5.2) on general rectangular mesh.

Remark 5.7. For this element, one can easily show that

$$\| \sigma \|_{H^{-1}(\text{div}, \Omega)} \lesssim \sup_{(v, \psi) \in W_h \times \Theta_h} \frac{(\sigma, \nabla_h v - \psi)}{\| \nabla_h v \|_{L^2(\Omega)} + \| \psi \|_{H^1(\Omega)}}, \text{ for any } \sigma \in \Gamma_h.$$ (5.19)

This guarantees the optimal a priori convergence rate in the norm $\| \gamma - \gamma_h \|_{H^{-1}(\text{div}, \Omega)}$ provided that the solution is smooth enough.
5.3.3. The Brezzi-Marini element. This first order nonconforming triangular element is proposed in [31]. Given any element $K$, define the nonconforming bubble function

$$B_{nc}^K := 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2,$$

where $\lambda_i$, $i = 1, 2, 3$, are as before the barycentric coordinates of $K$. The displacement and the shear force spaces read, respectively,

$$W_h := \{ v \in L^2(\Omega), v|_K \in Q_1(K) \oplus \text{span}\{B_{nc}^K\}, \int_E [v]ds = 0 \text{ on } E \in \mathcal{E}(\Omega) \},$$

$$\Gamma_h := \{ \sigma \in (L^2(\Omega))^2, \sigma|_K \in (Q_0(K))^2 \oplus (\text{span}\{\nabla B_{nc}^K\})^2, K \in T_h \}. $$

The rotation space is $\Theta_h := W_h \times W_h$, and the reduction integration operator $R_h$ is the $L^2$ projection operator from $Q$ onto $\Gamma_h$.

Following the argument of [76], one can prove the following inf-sup condition

$$||\sigma||_{H^{-1}(\text{div},\Omega)} \lesssim \sup_{(v,\psi)\in W_h \times \Theta_h} \frac{\langle \sigma, \nabla_h v - \psi \rangle}{\|\nabla_h v\|_{L^2(\Omega)} + \sqrt{a_h(\psi,\psi)}}, \text{ for any } \sigma \in \Gamma_h.$$  

This implies the optimal convergence of $||\gamma - \gamma_h||_{H^{-1}(\text{div},\Omega)}$.

5.3.4. The Lovadina element. This triangular method [76] employs the nonconforming Crouzeix-Raviart element to approximate both the displacement and the rotation with the piecewise constant shear force space $\Theta_h$. The reduction integration operator $R_h$ is the $L^2$ projection operator from $(L^2(\Omega))^2$ onto $\Gamma_h$. Since both components of the rotation is nonconforming, the bilinear form defined in (5.1) is used to guarantee the existence and uniqueness of the discrete problem.

5.3.5. The Hu-Shi elements. There are two quadrilateral methods from [61, 62]. For ease of presentation, we only describe their rectangular versions in [61]; we refer the interested reader to [62] for the details of the quadrilateral elements. First, we set

$$b(\xi, \eta) = (1 + \xi + \eta)(1 - \xi^2)(1 - \eta^2).$$

Define

$$V_h^c := \{ v \in H^1_0(\Omega), v|_K \circ F_K \in Q_1(\hat{K}) \oplus \text{span}(b), \forall K \in T_h \}.$$ 

Denote by $MNR(\hat{K})$ the modified nonconforming rotated $Q_1$ element space defined by

$$MNR(\hat{K}) = \text{span}\{1, \xi, \eta, \xi^2 - \eta^2, 1 - \frac{3}{4}(\xi^2 + \eta^2) \}.$$ 

The modified nonconforming rotated $Q_1$ element space $V_h^{nc}$ is then defined as [54, 87, 72]

$$V_h^{nc} := \{ v \in L^2(\Omega), (v|_K) \circ F_K \in MNR(\hat{K}), \int_E [v]ds = 0 \text{ on } E \in \mathcal{E}(\Omega) \},$$

$$\text{and } \int_E vds = 0 \text{ on } E \in \mathcal{E} \cap \partial \Omega \}.$$
The shear force space reads [61]:

\[
\Gamma_h = \left\{ \mathbf{v} \in (L^2(\Omega))^2, \mathbf{v} \mid_{K} = \left( \frac{b_K + d_K x}{c_K + e_K y} \right), \forall K \in T_h \right\},
\]

with \(b_K, c_K, d_K, e_K\) some free parameters.

In both methods of [61], \(W_h = V_h^{nc}\) and the reduction integration \(R_h\) is the \(L^2\) projection operator from \((L^2(\Omega))^2\) onto \(\Gamma_h\).

In the first method, the rotation space reads

\[
(5.27) \quad \Theta_h := V_h^c \times V_h^c.
\]

In the second method, the rotation space reads

\[
(5.28) \quad \Theta_h := V_h^{nc} \times V_h^{nc}.
\]

Since both components of the rotation in the second method is nonconforming, the bilinear form defined in (5.1) is used to guarantee the existence and uniqueness of the discrete problem.

**Remark 5.8.** For any node \(Z\), let \(\omega_Z\) be the node patch defined in Section 2. Choose \(\omega_Z\) as the macroelement, one can use the macroelement technique of [76] to prove

\[
(5.29) \quad \| \sigma \|_{H^{-1}(\text{div}, \Omega)} \lesssim \sup_{(v, \psi) \in W_h \times \Theta_h} \frac{\langle \nabla_h v - \psi \rangle}{\| \nabla_h v \|_{L^2(\Omega)} + \sqrt{a_h(\psi, \psi)}}, \quad \text{for any } \sigma \in \Gamma_h,
\]

for both methods under consideration. This implies that they converge optimally in the norm \(\| \gamma - \gamma_h \|_{H^{-1}(\text{div}, \Omega)}\).

5.3.6. *High order Arnold-Falk elements.* This family of high order Arnold-Falk elements is proposed in [6]. Let \(\Pi_{k-1}\) denote the \(L^2\) projection on the space of polynomials degree \(k - 1\) on \(E\), the displacement, the rotation, and the shear force spaces, read, respectively,

\[
(5.30) \quad W_h := \left\{ v_h \in L^2(\Omega), v \mid_{K} \in Q_k(K), \int_E \Pi_{k-1}[v] ds = 0 \text{ on } E \in \mathcal{E}(\Omega), \right. \\
\quad \text{and } \int_E \Pi_{k-1}v ds = 0 \text{ on } E \in \mathcal{E} \cap \partial \Omega, \}
\]

\[
(5.31) \quad \Theta_h := \left\{ \psi \in \Theta, \psi \mid_{K} \in \Theta(K), \text{ for any } K \in T_h \right\},
\]

\[
(5.32) \quad \Gamma_h := \left\{ \sigma \in Q, \sigma \mid_{K} \in Q_{k-1}(K), \text{ for any } K \in T_h \right\},
\]

where \(\Theta(K) = (Q_h(K))^2 + b_K \Gamma_{k-1}^*(K)\) for any odd integer \(k > 1\). The bubble function \(b_K\) is defined as before in Section 4.3.1, and

\[
(5.33) \quad \Gamma_{k-1}^*(K) := \left\{ \sigma + \text{Curl}(b_K v), \sigma \in (Q_{k-1}(K))^2, \right. \\
\quad \text{and } \text{div } \sigma \in Q_{k-3}(K), v \in Q_{k-2}(K) \}
\]

Define

\[
(5.34) \quad \Gamma_h^* := \left\{ \sigma \in Q, \sigma \mid_{K} \in \Gamma_{k-1}^*(K), \text{ for any } K \in T_h \right\}.
\]
The projection operator $R_h : Q \rightarrow \Gamma_h$ is defined as

$$ (\sigma - R_h \sigma, \delta)_{L^2(\Omega)} = 0, \quad \text{for any } \delta \in \Gamma_h^*, $$

for any $\sigma \in Q$.

**Remark 5.9.** The convergence of $\|\gamma - \gamma_h\|_{H^{-1}(\text{div},\Omega)}$ can be proved by the similar way for the Arnold-Falk element [47].

### 6. A posteriori analysis for MITC methods

The MITC methods are very popular in the numerical analysis of the Reissner-Mindlin plate problem. There are a lot of MITC methods in literature [9, 18, 19, 21, 25, 20, 48, 50, 56, 60, 82, 92, 83, 81]. A posteriori error analysis for this class of schemes appear in [71] for a MITC method from [48]. However the energy norm analyzed therein involves $\|\gamma - \gamma_h\|_{H^{-1}(\text{div}),\Omega}$, while the a priori convergence of $\|\gamma - \gamma_h\|_{H^{-1}(\text{div},\Omega)}$ is still open for the MITC methods in the literature. In recent papers [35, 57], we give a posteriori analysis based on a new formulation of the residuals and a mesh dependent energy norm. The estimator only involves $\|\gamma - \gamma_h\|_{H^{-1}(\text{div},\Omega)}$ instead. For the MITC methods, $\|\gamma - \gamma_h\|_{H^{-1}(\text{div},\Omega)}$ converges optimally.

The purpose of this section is to use the unifying theory to analyze the MITC element methods. As we shall see later, this analysis will recover results of [35, 57].

#### 6.1. MITC finite element methods

Let $\Theta_h \subset (L^2(\Omega))^2$ and $W_h \subset W$ be some finite element spaces over some regular partition $T_h$ while $R_h$ denotes the reduction integration operator in the context of shear locking with values in the discrete shear force space $\Gamma_h \subset H_0(\text{rot}, \Omega)$. As usual, we assume that $\nabla W_h \subset \Gamma_h$.

The discrete problem reads: Find $(w_h, \phi_h) \in W_h \times \Theta_h$ with

$$ a_h(\phi_h, \psi_h) + (\gamma_h, \nabla v - R_h \psi_h)_{L^2(\Omega)} = (g, v)_{L^2(\Omega)}, $$

for all $(v, \psi_h) \in W_h \times \Theta_h$ and the discrete shear force

$$ \gamma_h = \lambda t^{-2}(\nabla w_h - R_h \phi_h). $$

For this case, $L = 0$.

#### 6.2. A posteriori error analysis

For this case, we take

$$ \alpha^2|_K := \frac{\lambda}{t^2 + h^2_K}, \quad \beta^2|_K := \frac{t^2(h^2_K + t^2)}{\lambda h^2_K}. $$

Since $R_h \phi_h \in H_0(\text{rot}, \Omega)$, there exist $\omega \in H^1_0(\Omega)$ and $\beta \in H^1_0(\Omega)^2$ with

$$ R_h \phi_h - \Psi = \nabla \omega - \beta $$

and

$$ \|\omega\|_{H^1(\Omega)} + \|\beta\|_{H^1(\Omega)} \lesssim \|R_h \phi_h - \Psi\|_{H(\text{rot},\Omega)}, $$

for any $\Psi \in \Theta$. The proof of (6.4)-(6.5) can be found in Lemma 3.2 of Page 298 of [29].

Given any $\Psi \in \Theta$, we define

$$ \gamma'_h = \beta^{-2}(\nabla (w_h - \omega) - \Psi + \beta), \quad \gamma' = \beta^{-2}(\nabla w - \phi). $$

Taking \( \bar{w}_h = w_h - \omega, \bar{\phi}_h = \Psi - \beta \), and \( \tilde{\gamma}_h' = \gamma_h' \). Thus, \( \bar{r}_h = 0 \). For this case, \( r_{1,h} = 0 \), and \( r_{2,h} = (I - R_h)(\nabla w_h - \phi_h) \). Assume that (H1)-(H4) hold. It follows from Theorem 1.1 with \( \bar{r}_h = 0 \) and (6.4)-(6.5) that

\[
(6.7) \quad \| \phi - \bar{\phi}_h, w - \bar{w}_h \| + \| \gamma' - \bar{\gamma}_h \|_Q \lesssim \eta_h + \| \phi_h - \Psi \|_{1,h}.
\]

Triangle inequalities and Lemma 2.6 show that

\[
(6.8) \quad \| \phi - \phi_h \|_{\mathcal{C}_h} + \| \nabla w - \nabla w_h \|_{L^2(\Omega)} + t \| \gamma - \gamma_h \|_{L^2(\Omega)} \lesssim \eta_h.
\]

By the definitions of \( \gamma \) an \( \gamma_h \), we get for any \( \Psi \in \Theta \),

\[
(6.9) \quad t^2 \| \text{rot}(\gamma - \gamma_h) \|_{L^2(\Omega)} \lesssim \| \text{rot}_h(\phi - \phi_h) \|_{L^2(\Omega)}
\]

\[
\leq \| \text{rot}(\phi - \Psi) \|_{L^2(\Omega)} + \| \text{rot}_h(\Psi - \phi_h) \|_{L^2(\Omega)}
\]

\[
\lesssim \eta_h + \| \Psi - \phi_h \|_{1,h}.
\]

This proves

\[
(6.10) \quad t^2 \| \text{rot}(\gamma - \gamma_h) \|_{L^2(\Omega)} \lesssim \eta_h.
\]

Given any \( \psi \in \Theta \), we have

\[
(6.11) \quad (\gamma - \gamma_h, \psi)_{L^2(\Omega)} = -(C\varepsilon_h(\phi_h - \phi), \varepsilon(\psi))_{L^2(\Omega)} - (\gamma_h, \psi_h - R_h\psi_h)_{L^2(\Omega)}
\]

\[
- (\gamma_h, \psi - \psi_h)_{L^2(\Omega)} + (C\varepsilon_h(\phi_h), \varepsilon(\psi - \psi_h))_{L^2(\Omega)}.
\]

This implies that

\[
(6.12) \quad \| \gamma - \gamma_h \|_{H^{-1}(\Omega)} \lesssim \eta_h.
\]

Let

\[
(6.13) \quad E(\phi - \phi_h, w - w_h, \gamma - \gamma_h)^2 = a_h(\phi - \phi_h, \phi - \phi_h) + \| \nabla w - \nabla w_h \|_{L^2(\Omega)}^2
\]

\[
+ t^2 \| \gamma - \gamma_h \|_{L^2(\Omega)}^2 + t^4 \| \text{rot}(\gamma - \gamma_h) \|_{L^2(\Omega)}^2 + \| \gamma - \gamma_h \|_{H^{-1}(\Omega)}^2
\]

A summary of these estimates proves

**Theorem 6.1.** Assume (H1)-(H4) hold. Then,

\[
(6.14) \quad E(\phi - \phi_h, w - w_h, \gamma - \gamma_h) \lesssim \eta_h,
\]

with \( r_{1,h} = 0 \) and \( r_{2,h} = (I - R_h)(\nabla w_h - \phi_h) \).

This result recovers that from [35, 57] for conforming and nonconforming MITC methods.

**Remark 6.2.** In Theorem 6.1, we assume that \((S^1_0(T_h))^2 \subseteq \Theta_h \) which is not valid for the methods from [80]. However, this difficulty can be removed by the method in [36].

**Remark 6.3.** If the discrete Korn inequality holds for the rotation space \( \Theta_h \) [80], one can use \((C\varepsilon_h(\phi_h), \varepsilon(\psi_h))_{L^2(\Omega)} \) instead of \( a_h(\phi_h, \psi_h) \). The analysis and results in Theorem 6.1 and Theorem 6.4 hold equally; c.f [57] for details.

**Theorem 6.4.** There holds

\[
(6.15) \quad \eta_h \lesssim E(\phi - \phi_h, w - w_h, \gamma - \gamma_h) + \text{osc}(g).
\]

**Proof.** We refer the interested readers to [35, 57] for further details. \( \square \)
6.3. **Examples.** This subsection gives some example (conforming or nonconforming) MITC elements with (H1)-(H4) of the Reissner-Mindlin plate problem. Their lower order versions are depicted in Table 3.

<table>
<thead>
<tr>
<th>$W_h$</th>
<th>$\Theta_h$</th>
<th>$\Gamma_h$</th>
<th>Name, Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td>DL, §6.3.1</td>
</tr>
<tr>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
<td>DHHLR, §6.3.3</td>
</tr>
<tr>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td><img src="image9" alt="Diagram" /></td>
<td>MOZF, §6.3.2</td>
</tr>
<tr>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
<td>M-S 1, §6.3.4</td>
</tr>
<tr>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
<td><img src="image15" alt="Diagram" /></td>
<td>MITC7, §6.3.5</td>
</tr>
<tr>
<td><img src="image16" alt="Diagram" /></td>
<td><img src="image17" alt="Diagram" /></td>
<td><img src="image18" alt="Diagram" /></td>
<td>MITC9, §6.3.6</td>
</tr>
<tr>
<td><img src="image19" alt="Diagram" /></td>
<td><img src="image20" alt="Diagram" /></td>
<td><img src="image21" alt="Diagram" /></td>
<td>MITC12, §6.3.6</td>
</tr>
</tbody>
</table>

**Table 3.** Lower Order Examples of Finite Element Spaces $W_h$, $\Theta_h$ and $\Gamma_h$ of MITC methods.
6.3.1. *The Durán and Liberman element.* This triangular method is proposed in [48]. The displacement space \( W_h \) is the conforming linear element space defined in Section 4.3.1. The rotation space reads
\[
\Theta_h := (W_h)^2 \oplus B_h.
\]
Herein, the edge bubble function space \( B_h \) is defined as following. Define
\[
\psi_1 := \phi_1 \tau_1, \psi_2 := \phi_2 \tau_2, \psi_3 := \phi_3 \tau_3,
\]
where the edge bubble functions \( \phi_i, i = 1, 2, 3 \), are as before defined in Section 4.3.1. Then,
\[
B_h := \{ \beta \in H_0^1(\Omega)^2, \beta|_K \in \text{span}\{\psi_1, \psi_2, \psi_3\}, K \in T_h \}.
\]
For a triangulation into triangular elements, let
\[
\Gamma_h := \{ \sigma \in H_0(\text{rot}, \Omega), \sigma|_K \in (Q_0(K))^2 \oplus Q_0(K)(y, -x)^T, K \in T_h \}
\]
and define the reduction operator \( R_h \) by
\[
\int_E (R_h \sigma - \sigma) \cdot \tau_E ds = 0 \text{ for every edge } E \text{ of } K.
\]

6.3.2. *The MOZF element.* This element adopts all notations \( W_h, \Gamma_h, R_h \) from the previous Section 6.3.1. The only difference is the rotation space. In this case, the rotation space reads instead
\[
(6.16) \quad \Theta_h := W_h \times W_h^{\text{CR}},
\]
with \( W_h^{\text{CR}} \) the nonconforming triangular Crouzeix-Raviart element space defined in Section 5.3.1.

6.3.3. *The rectangular DHHLR element.* In this case, we restrict ourselves to the rectangular mesh, and define
\[
\Gamma_h := \{ \sigma \in H_0(\text{rot}, \Omega), \sigma|_K \in (Q_0(K))^2 \oplus Q_0(K)(y, 0)^T \oplus Q_0(K)(0, x)^T, K \in T_h \},
\]
with the reduction operator \( R_h \) defined as, for any \( K \in T_h \),
\[
\int_E (R_h \sigma - \sigma) \cdot \tau_E ds = 0 \text{ for every edge } E \text{ of } K.
\]
The rotation space is defined as
\[
\Theta_h := (S_0^1(\Omega))^2 \oplus B_h.
\]
The edge bubble function space \( B_h \) is defined as following. For each edge \( E_i \) of \( K \), \( i = 1, 2, 3, 4 \), let \( \hat{\psi}_i \) denote the cubic functions vanishing on \( E_j \) when \( j \neq i \). Then we define \( \psi_i = \hat{\psi}_i \circ F_K^{-1} \tau_{E_i} \) and set
\[
B_h := \{ \beta \in H_0^1(\Omega)^2, \beta|_K \in \text{span}\{\psi_1, \psi_2, \psi_3, \psi_4\}, K \in T_h \}.
\]
The displacement space
\[
(6.17) \quad W_h := S_0^1(\Omega)^2.
\]
6.3.4. The rectangular Ming-Shi 1 element. This element employs the same displacement and the shear force spaces as the rectangular DHHLR element of Section 6.3.3. The only difference is the rotation space, which reads instead
\begin{equation}
\Theta_h := S_h^0(\Omega) \times W_h^{NR},
\end{equation}
where the nonconforming rotated \( Q_1 \) space \( W_h^{NR} \) is as before defined in Section 5.3.2.

6.3.5. High order triangle MITC elements. This subsection presents three families of high order triangle MITC elements. These elements are proposed and analyzed in [30]. The lowest element among them is the usual MITC7 element appeared in [20].

Family I. We define for \( k \geq 2 \)
\[
W_h := \{ v \in H_0^1(\Omega), v|_K \in Q_k(K), K \in T_h \},
\]
\[
\Theta_h := \{ \beta \in H_0^1(\Omega)^2, \beta|_K \in \Theta_k(K), K \in T_h \},
\]
where
\[
\Theta_k(K) := \begin{cases} (Q_k(K))^2 & \text{for } k \geq 4, \\ (S_k(K))^2 & \text{for } k = 2, 3, \end{cases}
\]
and
\[
S_k(K) := \{ w \in Q_{k+1}(K), w|_E \in Q_k(E) \text{ for every edge } E \text{ of } K \}.
\]
For the shear force space \( \Gamma_h \) we take the rotated Raviart-Thomas space
\[
\Gamma_h := \{ \sigma \in H_0(\text{rot}, \Omega), \sigma \in (Q_{k-1}(K))^2 \oplus Q_{k-1}(K)(y,-x)^T, K \in T_h \},
\]
with the reduction operator \( R_h \) defined through
\[
\int_E (R_h \sigma - \sigma) \cdot \tau_E v ds = 0, v \in Q_{k-1}(E) \text{ for every edge } E \text{ of every } K \in T_h,
\]
\[
\int_K (R_h \sigma - \sigma) \cdot \beta dx dy = 0 \text{ for every } \beta \in (Q_{k-2}(K))^2.
\]

Family II. Let \( S_k \) be as in Family I, and define
\[
W_h := \{ v \in H_0^1(\Omega), v|_K \in S_h(K), K \in T_h \}.
\]
We choose the BDFM space
\[
\Gamma_h := \{ \sigma \in H_0(\text{rot}, \Omega), \sigma|_K \in (Q_k(K))^2, \sigma \cdot \tau_E \in Q_{k-1}(E) \text{ for every edge } E \text{ of } K \in T_h \}
\]
as the shear force space \( \Gamma_h \) with the reduction operator defined as
\[
\int_E (R_h \sigma - \sigma) \cdot \tau_E v ds = 0, v \in Q_{k-1}(E) \text{ for every edge } E \text{ of } K,
\]
\[
\int_K (R_h \sigma - \sigma) \cdot \beta dx = 0 \text{ for every } \beta \in (Q_{k-2}(K))^2,
\]
\[
\int_K (R_h \sigma - \sigma) \cdot \nabla \psi_j dx = 0 \text{ for every } j = 0, \cdots, k-2,
\]
where \( \psi_0, \cdots, \psi_{k-2} \) are arbitrary polynomials in \( Q_k(K) \), chosen once and for all, with \( \nabla \psi_j = x^j y^{k-j-2}, j = 0, \cdots, k-2 \). The rotation space is the same as in Family
I. Family III. In this case let

\[ W_h := \{ v \in H^1_0(\Omega), v|_K \in Q_{k+1}(K), K \in \mathcal{T}_h \} \]

and use the BDM space

\[ \Gamma_h := \{ \sigma \in H_0(\text{rot}, \Omega), \sigma|_K \in (Q_k(K))^2, K \in \mathcal{T}_h \} . \]

The reduction operator \( R_h \) is defined through

\[
\int_E (R_h \sigma - \sigma) \cdot \tau_E v \, ds = 0, \quad v \in Q_k(E) \quad \text{for every edge } E \text{ of } K, \\
\int_K (R_h \sigma - \sigma) \cdot \beta \, dx \, dy = 0 \quad \text{for every } \beta \in (Q_{k-2}(K))^2, \\
\int_K (R_h \sigma - \sigma) \cdot \nabla \psi_j \, dx \, dy = 0 \quad \text{for every } j = 0, \ldots, k-2 ,
\]

with \( \psi_j \) and the rotation space \( \Theta_h \) from Family II.

6.3.6. High order quadrilateral MITC elements. For \( S \subset \mathbb{R}^2 \), we let \( P_k(S) \) denote the set of polynomials of total degree \( \leq k \), and \( Q'_k(S) \) denote the “trunk” or “serendipity” space of polynomials. The spaces \( W_h \) and \( \Theta_h \) are defined as

\[
W_h := \{ v \in H^1_0(\Omega), v|_K = \hat{v} \circ F_K^{-1}, \hat{v} \in W_h(\hat{K}), K \in \mathcal{T}_h \}, \\
\Theta_h := \{ \beta \in (H^1_0(\Omega))^2, \beta|_K = \hat{\beta} \circ F_K^{-1}, \hat{\beta} \in \Theta_h(\hat{K}), K \in \mathcal{T}_h \},
\]

where \( W_h(\hat{K}) \) and \( \Theta_h(\hat{K}) \), which will be specified in the sequel, are polynomial spaces on the reference element \( \hat{K} \). The space \( \Gamma_h \) is defined differently by

\[
\Gamma_h := \{ \sigma \in H_0(\text{rot}, \Omega), \sigma|_K \in \Gamma_h(K), K \in \mathcal{T}_h \}
\]

for the space \( \Gamma_h(K) \) defined from the space \( \Gamma_h(\hat{K}) \) on the reference square through the following Piola transformation for the operator \( \text{rot} \),

\[
\Gamma_h(K) := \{ \sigma, \sigma = DF_K^{-T} \tilde{\sigma} \circ F_K^{-1}, \tilde{\sigma} \in \Gamma_h(\hat{K}) \}.
\]

The reduction operator \( R_h \) is also defined locally on each element from the reduction operator \( \hat{R}_h \) defined on the reference element with the same transformation:

\[
R_h \sigma|_K = DF_K^{-T} \hat{R}_h \tilde{\sigma} \circ F_K^{-1}, \quad \text{with } \tilde{\sigma} = DF_K^T \sigma = DF_K^T \hat{\sigma} \circ F_K.
\]

We consider four families of quadrilateral MITC elements for the Reissner-Mindlin plate problem. The rectangular version of these elements are proposed and analyzed in [92]. We refer to [56, 60] for the a priori error analysis of the general case.

Family I. In this family, the displacement and the rotation spaces read, respectively,

\[
W_h(\hat{K}) := Q_{\hat{K}} \cap P_{k+1}(\hat{K}), \\
\Theta_h(\hat{K}) := [Q_{\hat{K}} \cap P_{k+2}(\hat{K})]^2.
\]

For \( \Gamma_h \), we choose \( \Gamma_h(\hat{K}) \) as the following BDFM space [29]

\[
\Gamma_h(\hat{K}) := \{ \sigma \mid \sigma \in [P_{\hat{K}}(\hat{K}) \setminus \{z^k\}] \times P_{\hat{K}}(\hat{K}) \setminus \{y^k\} \}.
\]
The reduction operator $\mathbf{R}_K$ is defined by

\[
\int_{\hat{E}} [(\mathbf{R}_K \sigma - \sigma) \cdot \hat{\tau}] \hat{w} \hat{d}s = 0 \quad \text{for all } \hat{w} \in P_{k-1}(\hat{E}) \quad \text{and for every edge } \hat{E} \text{ of } \hat{K},
\]

\[
\int_{\hat{K}} (\mathbf{R}_K \sigma - \sigma) \cdot \hat{v} \hat{d}\hat{x}\hat{d}\hat{y} = 0, \quad \forall \hat{v} \in [P_{k-2}(\hat{K})]^2.
\]

**Family II.** In this method, $W_h$, and $\Gamma_h$ are the same as in Family I with the different choice of the rotation space, which reads as

\[
\Theta_k(\hat{K}) = [Q_k(\hat{K})]^2.
\]

**Family III.** The spaces for the rotation are chosen as in Family I. However, we take the following BDM space [29]

\[
\Gamma_k(\hat{K}) = \{ \sigma \mid \sigma \in [P_k(\hat{K})]^2 \oplus \hat{\nabla}(\hat{x}^{k+1}) \oplus \hat{\nabla}(\hat{y}^{k+1}) \},
\]

as the shear force space with the reduction operator defined by

\[
\int_{\hat{E}} [(\mathbf{R}_K \sigma - \sigma) \cdot \hat{\tau}] \hat{w} \hat{d}s = 0, \quad \forall \hat{w} \in P_k(\hat{E}) \text{ for every edge } \hat{E} \text{ of } \hat{K},
\]

\[
\int_{\hat{K}} (\mathbf{R}_K \sigma - \sigma) \cdot \hat{v} \hat{d}\hat{x}\hat{d}\hat{y} = 0, \quad \forall \hat{v} \in [P_{k-2}(\hat{K})]^2.
\]

Therefore, the deflection space has to be selected as

\[
W_k(\hat{K}) = Q'_{k+1}(\hat{K}).
\]

**Family IV.** The rotation space reads

\[
\Theta_k(\hat{K}) = \{ \psi \in [Q_{k+1}(\hat{K})]^2 \mid \psi |_{\hat{E}} \in [P_k(\hat{K})]^2 \text{ for every edge } \hat{E} \text{ of } \hat{K} \}.
\]

The corresponding space for the shear is the following rotated Raviart-Thomas space over quadrilaterals,

\[
\Gamma_k(\hat{K}) = \{ \sigma \mid \sigma \in Q_{k-1,k}(\hat{K}) \times Q_{k,k-1}(\hat{K}) \},
\]

with the reduction operator defined by

\[
\int_{\hat{E}} [(\mathbf{R}_K \sigma - \sigma) \cdot \hat{\tau}] \hat{w} \hat{d}s = 0, \quad \forall \hat{w} \in P_{k-1}(\hat{E}) \text{ for every edge } \hat{E} \text{ of } \hat{K},
\]

\[
\int_{\hat{K}} (\mathbf{R}_K \sigma - \sigma) \cdot \hat{v} \hat{d}\hat{x}\hat{d}\hat{y} = 0, \quad \forall \hat{v} \in Q_{k-1,k-2}(\hat{K}) \times Q_{k-2,k-1}(\hat{K}).
\]

The space for the deflection is selected as

\[
W_k(\hat{K}) = Q_k(\hat{K}).
\]
7. A posteriori error analysis for discontinuous Galerkin methods

This section considers the extension of the unifying theory to the discontinuous Galerkin methods for the Reissner-Mindlin plate. We first introduce some notations. Define

\begin{equation}
\mathbb{H}^k(T_h) := \{ v \in L^2(\Omega), v|_K \in H^k(K), \text{ for any } K \in T_h \}, k = 1, 2,
\end{equation}

\begin{equation}
\mathbb{H}^2(T_h) := H^2(T_h) \times H^2(T_h).
\end{equation}

If $\psi \in H^1(T_h)$ (or possibly the vector- or tensor-valued analogue), we define the average $\{\psi\}$ on $E \in \mathcal{E}(\Omega)$ as usual:

\begin{equation}
\{\psi\} = \frac{\psi^+ + \psi^-}{2},
\end{equation}

with $\psi^+ = \psi|_{K^+}$ and $\psi^- = \psi|_{K^-}$. On the boundary $E \subset \partial \Omega$, the average $\{\psi\}$ is defined simply as the trace of $\psi$. Given $\phi_h \in \mathbb{H}^2(T_h)$ and $\psi_h \in \mathbb{H}^2(T_h)$, define the following bilinear form:

\begin{equation}
\mathfrak{A}_h(\phi_h, \psi_h) := (C\varepsilon_h(\phi_h), \varepsilon_h(\psi_h))_{L^2(\Omega)} - \sum_{E \in \mathcal{E}} \{ (C\varepsilon_h(\phi_h), [\psi_h])_{L^2(E)} - \sum_{E \in \mathcal{E}} (\{C\varepsilon_h(\psi_h)\}, [\phi_h])_{L^2(E)} + \sum_{E \in \mathcal{E}} \frac{\gamma_E}{h_E} (\{\phi_h\}, [\psi_h])_{L^2(E)},
\end{equation}

where $\gamma_E$ are some constants to ensure the stability of the discrete problem. For any $u \in W_h$ and $v \in W_h$, we define the following penalty term

\begin{equation}
P_W(u, v) = \sum_{E \in \mathcal{E}} \frac{\beta_E}{h_E} \int_E [u][v] ds,
\end{equation}

Let $W_h \subset H^2(T_h)$, $\Theta_h \subset \mathbb{H}^2(T_h)$, and $\Gamma_h \subset \mathbb{H}^1(T_h)$ be some piecewise polynomial spaces which will be specified for various discontinuous Galerkin schemes of the Reissner-Mindlin plate problem. Let $\mathbf{R}_h$ be the associated reduction integration operator. In this section, we shall consider three families of discontinuous Galerkin methods. For all of them, their discrete problems read: Find $(w_h, \phi_h, \gamma_h) \in W_h \times \Theta_h \times \Gamma_h$ with

\begin{equation}
\mathfrak{A}_h(\phi_h, \psi_h) + P_W(w_h, v_h) + (\gamma_h, R_h(\nabla_h v_h - \psi_h))_{L^2(\Omega)}
\end{equation}

\begin{equation}
- \sum_{E \in \mathcal{E}} \int_E \{\gamma_h\} \cdot [v_h]\nu_E ds = (g, v_h)_{L^2(\Omega)}, \text{ for any } (v_h, \psi_h) \in W_h \times \Theta_h,
\end{equation}

\begin{equation}
(R_h(\nabla_h w_h - \phi_h), \sigma)_{L^2(\Omega)} - \sum_{E \in T_h} \int_E [w_h] \nu_E \cdot \{\sigma\} ds - \lambda^{-1} \mu^2 (\gamma_h, \sigma)_{L^2(\Omega)} = 0,
\end{equation}

for any $\sigma \in \Gamma_h$.

We define norms

\begin{equation}
\|\psi\|_{\Theta}^2 := \|\psi\|_{L^2}^2 + \sum_{E \in \mathcal{E}} \frac{1}{h_E} \|[\psi]\|_{L^2(E)}^2 + h_E \{C\varepsilon_h(\psi)\}_{L^2(E)}^2, \psi \in \mathbb{H}^2(T_h),
\end{equation}
### Table 4. Lower Order Examples of Finite Element Spaces $W_h$, $\Theta_h$ and $\Gamma_h$ of the discontinuous Galerkin methods.

<table>
<thead>
<tr>
<th>$W_h$</th>
<th>$\Theta_h$</th>
<th>$\Gamma_h$</th>
<th>Name, Description</th>
</tr>
</thead>
</table>

\[
\begin{align*}
\|v\|_{\tilde{W}}^2 &= \|v\|_{1,h}^2 + \sum_{E \in \mathcal{E}} \frac{1}{h_E} \|\nabla v\|_{L^2(E)}^2, \quad v \in H^1(T_h), \\
\|\sigma\|_{\tilde{\Gamma}}^2 &= \|\sigma\|_{L^2(\Omega)}^2 + \sum_{E \in \mathcal{E}} h_E \|\sigma\|_{L^2(E)}^2, \quad \sigma \in H^1(T_h).
\end{align*}
\]

For all methods considered in this section $L = 0$.

#### 7.1. The Arnold-Brezzi-Falk-Marini elements I

Based on the regular triangular partition of $\Omega$, this class of methods takes \cite{5}

\[
W_h := \{v \in H_0^1(\Omega), v|_K \in Q_k(K), \text{ for any } K \in T_h\},
\]

for any integer $k \geq 2$. The shear force and the rotation spaces read

\[
\Theta_h = \Gamma_h = \text{BDM}_{k-1},
\]

where BDM$_{k-1}$ denotes the Brezzi-Douglas-Marini space of degree $k - 1$, i.e., the space of all piecewise polynomial vector fields at most subject to interelement continuity of the tangential components \cite{29}. Since $\Theta_h = \Gamma_h$, no reduction integration is used in this family of methods, namely, $R_h = I$.

Let the energy norm $\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h)$ is defined in (6.13). It is easy to see that there holds (H1)-(H4). Since $\gamma_h \subset H_0(\text{rot}, \Omega)$ and $W_h \subset H_0^1(\Omega)$, the arguments of Section 6 prove that

\[
\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h) \approx \eta_h + \text{osc}(g).
\]
To show that (7.12) makes sense, we only need to establish the uniform convergence of \( \| \gamma - \gamma_h \|_{H^{-1}(\Omega)} + t^2 \| \text{rot}(\gamma - \gamma_h) \|_{L^2(\Omega)} \). To this end, one can use the same arguments of MITC methods to prove:

**Lemma 7.1.** For any \( q \in \Gamma_h \), there exist unique \( r \in W_h \), \( p \in Q_{k-2} \cap L_0^2(\Omega) \), and \( \alpha \in \Gamma_h \) such that

\[
q = \nabla r + \alpha, \\
\langle \alpha, \sigma \rangle = \langle \text{rot} \sigma, p \rangle, \forall \sigma \in \Gamma_h.
\]

Here and in the sequel, the pressure space \( Q_{k-2} \) is defined as

\[
Q_{k-2} := \{ q \in L^2(\Omega), q|_K \in Q_{k-2}(K), \text{ for any } K \in T_h \}.
\]

With some abusing use of the notation, herein we use \( R_h \) to denote the associated interpolation operator of the Brezzi-Douglas-Marini elements \([29]\). Then, we have the following property:

\[
\langle \text{rot}(I - R_h)\psi, q \rangle = 0, \text{ for any } q \in Q_{k-2},
\]

\[
\mathcal{A}_h(R_h\psi, R_h\psi) \lesssim \| \psi \|_{H^1(\Omega)},
\]

for any \( \psi \in \Theta \). Then, one can use the same argument for the MITC methods to show the convergence of \( \| \gamma - \gamma_h \|_{H^{-1}(\Omega)} + t^2 \| \text{rot}(\gamma - \gamma_h) \|_{L^2(\Omega)} \).

### 7.2. The Arnold-Brezzzi-Falk-Marini elements II.

This is a family of triangular elements where all variables are approximated by completely discontinuous polynomials \([5]\). The displacement, the rotation, and the shear force spaces read, respectively,

\[
W_h := \{ v \in L^2(\Omega), v|_K \in Q_h(K), \text{ for any } K \in T_h \},
\]

\[
\Theta_h := \{ \psi \in (L^2(\Omega))^2, \psi|_K \in (Q_{k-1}(K))^2, \text{ for any } K \in T_h \},
\]

\[
\Gamma_h := \{ \psi \in (L^2(\Omega))^2, \psi|_K \in (Q_{k-1}(K))^2, \text{ for any } K \in T_h \},
\]

for some integer \( K \geq 2 \). Again, no reduction integration is used in these methods, namely, \( R_h = I \). It is obvious that (H1)-(H4) hold for this case.

Defining a lifting operator \( \mathcal{P} : H^1(T_h) \rightarrow \Gamma_h \) by the equation

\[
\langle \mathcal{P}(v), \delta \rangle = \sum_{E \in E} \int_E [v]_E \cdot \{ \delta \} ds, \text{ for any } \delta \in \Gamma_h.
\]

Then the discrete shear force \( \gamma_h \) reads

\[
\gamma_h = \lambda t^{-2}(\nabla_h w_h - \phi_h - \mathcal{P}(w_h)).
\]

For this class of schemes, one can show the following inf-sup condition:

\[
\sup_{(v,\psi) \in W_h \times \Theta_h} \frac{\langle \nabla_h v - \psi, \sigma \rangle}{\| \psi \|_H + \| v \|_W} \geq \| \sigma \|_{H^{-1}(\text{div},\Omega)}, \text{ for any } \psi \in \Gamma_h.
\]

Since \( \nabla_h W_h \subset \Gamma_h = \Theta_h \), one can use the mixed theory to show the optimal convergence rate of \( \| \gamma - \gamma_h \|_{H^{-1}(\text{div},\Omega)} \) provided that the solution is smooth enough.
With these preparations, we choose $\alpha$ as a global constant independent of the meshsize and the plate thickness as in Section 5. We take

$$(7.22) \quad \tilde{\nabla}_h \phi = \gamma_h^n = \beta^{-2} (\nabla_h w_h - \phi_h - \mathcal{P}(w_h)).$$

Since (H5) is not valid for this case, we have to bound the seventh term of $\tilde{\eta}_h$. Since $\tilde{\tau}_h = \nabla_h w_h - \phi_h - \mathcal{P}(w_h) - (\nabla \tilde{w}_h - \tilde{\phi}_h)$, we deduce it as

$$(7.23) \quad (\tilde{\tau}_h, \text{Curl } \mathcal{J} p)_{L^2(\Omega)} = (\nabla_h w_h - \phi_h - \mathcal{P}(w_h) - (\nabla \tilde{w}_h - \tilde{\phi}_h), \text{Curl } \mathcal{J} p)_{L^2(\Omega)}$$

Integrating by parts and applying the definition of $\mathcal{P}(w_h)$ leads to

$$(7.24) \quad (\nabla_h w_h - \mathcal{P}(w_h), \text{Curl } \mathcal{J} p)_{L^2(\Omega)} = 0.$$

This and the inverse estimate together with $\|p - \mathcal{J} p\|_{L^2(\Omega)} \lesssim \min(\|p\|_{L^2(\omega_K)}, h_K \|\nabla p\|_{L^2(\omega_K)})$ yield

$$(7.25) \quad \sup_{0 \neq p \in H^1(\Omega)} \frac{(\tilde{\tau}_h, \text{Curl } \mathcal{J} p)_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)}} \lesssim \left( \sum_{K \in \mathcal{T}_h} \min(\frac{1}{h_K^2}, \frac{1}{l^2}) \right) \|\phi_h - \tilde{\phi}_h\|_{L^2(\Omega)}^{1/2}.\$$

Since there hold (H1)-(H4) and (H6)-(H7), a similar procedure of Section 5 proves that

$$\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h)^2 + \sum_{E \in \mathcal{E}} h_E^{-1} \|w_h\|_{L^2(E)}^2 \approx \eta_h^2 + \sum_{E \in \mathcal{E}} h_E^{-1} \|\nabla w_h\|_{L^2(E)}^2 + \text{osc}(g)^2,$$

with the energy norm $\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h)$ defined in Section 5.

### 7.3. The Arnold-Brezzi-Marini elements

This subsection uses the notation of the previous subsection. In this family of class triangular elements, the displacement space $W_h$ and the shear force space $\Gamma_h$ are the same as those defined in the previous subsection. The rotation space reads [6]

$$(7.26) \quad \Theta_h := W_h \times W_h,$$

with the reduction integration operator is defined as before by Equation (5.35).

Similar to the previous subsection, one has

$$(7.27) \quad \sup_{(v, \psi) \in W_h \times \Theta_h} \frac{(\nabla_h v - \psi, \sigma)}{\|
abla \psi\|_{\Theta} + \|v\|_{W}} \geq \|\sigma\|_{H^{-1}(\text{div}, \Omega)}, \text{ for any } \psi \in \Gamma_h.$$ 

Then, a similar argument proves that

$$\mathcal{E}(\phi - \phi_h, w - w_h, \gamma - \gamma_h)^2 + \sum_{E \in \mathcal{E}} h_E^{-1} \|w_h\|_{L^2(E)}^2 \approx \eta_h^2 + \sum_{E \in \mathcal{E}} h_E^{-1} \|\nabla w_h\|_{L^2(E)}^2 + \text{osc}(g)^2.$$

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