An extremal graph with given bandwidth

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Abstract

The graph bandwidth problem is a well-known NP-complete problem. The relation between size of a graph and bandwidth is very interesting. The minimum size required in \( G \) with bandwidth \( B \) is denoted as \( m(n, B) \) while the graph \( G \) of order \( n \) and bandwidth \( B \) with size \( m(n, B) \) is called an extremal graph. This paper provides the minimum size for a graph of odd order \( n \), \( n \geq 9 \), and bandwidth \( (n + 1)/2 \), and shows that \( K_{2,n-2} \) is the only extremal graph of \( m(n, (n + 1)/2) \).

Keywords: Bandwidth; Extremal graph

1. Introduction and related works

Let \( G = (V, E) \) be a simple graph of order \( n \). A proper numbering \( f \) of \( G \) is a bijective function \( f : V(G) \to \{1, 2, \ldots, n\} \). The bandwidth \( B_f(G) \) of a proper numbering \( f \) of \( G \) is \( \max\{|f(u) - f(v)| : uv \in E(G)\} \) and the bandwidth \( B(G) \) of \( G \) is \( \min\{B_f(G) : f \) is a proper numbering of \( G\} \). A proper numbering \( f \) is called a bandwidth numbering of \( G \) if \( B_f(G) = B(G) \).

The graph bandwidth problem is the problem of finding the bandwidth of a graph. The graph bandwidth problem has been studied since the 1950s. The decision version of this problem was shown to be NP-complete in 1976 [10]. It is shown that the problem is NP-complete even for trees of maximum degree 3 [6]. Graph bandwidth has many applications in lots of areas. There are several survey papers on the results and applications for graph bandwidth; see [2,3,8].

People are interested in finding characteristics of a graph which are related to its bandwidth. The relations between the bandwidth and the order of the graph, the maximum degree of vertices in the graph, the diameter of the graph, and the size of the graph have all been investigated in the past. The relation between the graph bandwidth and the size of the graph was first studied in 1980 by Chvátalová [4]. In 1989, Dutton and Brigham [5] proved Chinn’s conjecture that the graph size must be at least twice the bandwidth minus one.

Proposition 1 ([5]). A graph \( G \) has the minimum number of edges over all graphs having bandwidth \( B \) if and only if \( G \) is \( K_{1,2B-1} \) or \( K_3 \), along with any number of isolated vertices.

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According to Proposition 1, we have the following corollaries.

**Corollary 1** ([5]). If graph $G$ has bandwidth $B$, then $|E(G)| \geq 2B - 1$.

**Corollary 2** ([5]). If $B \leq n/2$, then $m(n, B) = 2B - 1$ where $m(n, B)$ denotes the minimum number of edges required in $G$ of order $n$ and bandwidth $B$.

From Corollary 2 we can see that when $B(G) \leq \lfloor n/2 \rfloor$, adding 1 to the bandwidth only increases the minimum size by 2. For $B > \lfloor n/2 \rfloor$, they also gave a lower bound as follows:

**Proposition 2** ([5]). If graph $G$ of order $n$ has bandwidth $B$ and $B \geq n/2$, then $m(n, B) \geq n(n - 1)/(2n - 2B)$.

According to Proposition 2, when the bandwidth is changed from $\lfloor n/2 \rfloor$ to $\lfloor n/2 \rfloor + 1$,

$$m(n, \lfloor n/2 \rfloor + 1) \geq \begin{cases} n & \text{if } n \text{ is odd,} \\ n + 2 & \text{if } n \text{ is even.} \end{cases}$$

So the same property (adding 1 to the bandwidth only increases the minimum size by 2) does not hold when $n$ is even and $B > \lfloor n/2 \rfloor$. In fact, there is a big jump from $m(n, \lfloor n/2 \rfloor)$ to $m(n, \lfloor n/2 \rfloor + 1)$; [1] gave a tighter lower bound in this case:

**Proposition 3** ([1]). If $n \geq 8$, then $m(n, \lfloor n/2 \rfloor + 1) \geq 3\lfloor n/2 \rfloor + 2$.

By Proposition 3, the difference between $m(n, \lfloor n/2 \rfloor)$ and $m(n, \lfloor n/2 \rfloor + 1)$ is at least $\lfloor n/2 \rfloor + 3$. The authors of [1] believe that the bound given in Proposition 3 is not sharp when $n \geq 10$. Knowing that $m(2k + 1, k + 1) \leq m(2k, k + 1)$, for all positive integers $k$, the case of odd $n$ will provide a lower bound for the case of even $n$. Hence, the case of odd $n$ is more interesting as regards finding the gap between $m(n, \lfloor n/2 \rfloor)$ and $m(n, \lfloor n/2 \rfloor + 1)$.

Follow the definition of [1], the graph $G$ of order $n$ and bandwidth $B$ with size $m(n, B)$ is called an extremal graph. For odd $n$, $n \leq 9$, [1] presented the facts that:

1. $m(5, 3) = 6$ and the only extremal graphs are $K_{2,3}$ and $K_1 \cup K_4$;
2. $m(7, 4) = 9$ and the only extremal graph is $K_1 \cup K_{3,3}$;
3. $m(9, 5) = 14$.

They established the facts by checking all graphs on $n$ vertices with $m(n, \lfloor n/2 \rfloor) − 1$ edges for $n = 5, 7$ and considering the maximum degree $\Delta(G) \geq 3$. For $n = 9$, they proved the result by considering two cases of the minimum degree $\delta(G) = 0$ and $\delta(G) \geq 1$. For general odd $n$, $n \geq 9$, they have the following conjecture:

**Conjecture.** If $k \geq 4$, then $m(2k + 1, k + 1) = 4k - 2$ and the only extremal graph for $m(2k + 1, k + 1)$ is $K_{2,2k-1}$.

Some researchers worked on the case $B = \lfloor n/2 \rfloor + 2$ and considered only connected graphs; the results are as follows:

**Proposition 4** ([7]). Let $n$ be a positive integer and $B = \lfloor n/2 \rfloor + 2$. Then:

1. $m^*(n, B) = 4(n - 4)$ for odd $n$, $n \geq 15$, and $K_{4,n-4}$ is an extremal graph;
2. $m^*(n, B) = 5(n - 5)$ for even $n$, $n \geq 18$, and $K_{5,n-5}$ is an extremal graph;

where $m^*(n, B)$ denotes the minimum of edges required for a connected graph $G$ of order $n$ and bandwidth $B$.

For bandwidth closed to the order of the graph, [11] considered a special class of graphs with maximum degree no more than the bandwidth, and presented some results as follows:

**Proposition 5** ([11]). Suppose that $n = 3k + r$, ($k \geq 1$, $1 \leq r \leq 3$). Then:

1. For $r = 1$, $f(n, n - 2) = n(n - 1)/2 - n + 2$ and the complete $(k + 1)$-partite graph $K_{3,3,\ldots,2,2}$ is an extremal graph;
2. For $r = 2$ or 3, $f(n, n - 2) = n(n - 1)/2 - n + 3 - r$, and the complete $(k + 1)$-partite graph $K_{3,3,\ldots,3,r}$ is an extremal graph;

where $f(n, B)$ is the minimum of edges required for a graph $G$ of order $n$ and bandwidth $B$ with maximum degree $\Delta(G) < B$.

This paper provides a proof of the conjecture given by [1] which is the minimum size required for a graph with odd order $n$ and $n \geq 9$ to have bandwidth about half of its order.
2. Main result

Let $G(U)$ denote the subgraph of $G$ induced by $U$ for some $U \subseteq V(G)$. Define $H_k$ for $k \geq 1$ to be a bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and edge set $E(H_k) = \{x_i y_j | i + j > k\}$. Fig. 1 illustrates $H_k$ for $1 \leq k \leq 3$.

If we consider the bandwidth numbering of $G$ and look at the complement graph $\overline{G}$, [9] provided a proof of the following lemma which comes directly from the definitions of the graph bandwidth and $H_k$.

**Lemma 1** ([9]). Let $G$ be a graph of order $n$ and $k \leq n/2$. Then $B(G) \geq n - k$ if and only if $H_k \not\subseteq \overline{G}$. In other words, $B(G) < n - k$ if and only if $H_k \subseteq \overline{G}$.

**Lemma 2** ([2]). Let $G = K_{m,n}$ for $m \leq n$ be a complete bipartite graph. Then $B(K_{m,n}) = m + \lfloor (n - 1)/2 \rfloor$.

We restate the conjecture of [1] as the following theorem.

**Theorem.** Let $n$ be odd, $n \geq 9$. Then $m(n, (n + 1)/2) = 2(n - 2)$ and $K_{2,n-2}$ is the only extremal graph for $m(n, (n + 1)/2)$.

**Proof.** Let $G$ be such a graph with minimum size. According to **Lemma 1**, let $k = (n - 1)/2$; then $\overline{G}$ satisfies the following conditions: (i) $H_{k-1} \subseteq \overline{G}$ and $H_k \not\subseteq \overline{G}$ (by **Lemma 1**); (ii) $\overline{G}$ has maximum size under condition (i). Since $n = 2k + 1$ and $H_{k-1} \subseteq \overline{G}$, we may suppose that $V_1 = V(H_{k-1}), |V_1| = 2k - 2$ and $V_2 = V - V_1 = \{w_1, w_2, w_3\}$. Let $N(V_1) = \{w|w \in V_2, \exists v \in V_1 \text{ such that } v w \in E(\overline{G})\}$ be the neighboring set of $V_1$ in $\overline{G}$.

**Case 1:** $|N(V_1)| = 0$, that is $\overline{G}(V_1)$ and $\overline{G}(V_2)$ are disconnected. Since $\overline{G}$ has maximum possible number of edges, $\overline{G}(V_1)$ and $\overline{G}(V_2)$, the subgraphs of $\overline{G}$ induced by $V_1$ and $V_2$ respectively must be complete subgraphs. Hence, $G = K_{3,n-3}$. By **Lemma 2** we have $B(K_{3,n-3}) = (n + 1)/2$ and $B(K_{2,n-2}) = (n + 1)/2$. Since $|E(K_{3,n-3})| = 3(n - 3) \geq 2(n - 2) = |E(K_{2,n-2})|$ when $n \geq 9$, so $K_{3,n-3}$ is not an extremal graph of $m(n, (n + 1)/2)$ which leads us to conclude that case 1 is not possible.

**Case 2:** $|N(V_1)| = 1$, that is, only one of $w_1$, $w_2$, $w_3$ (say $w_1$) is adjacent to some vertices of $V_1$ in $\overline{G}$.

**Subcase 2.1:** Assume $\overline{G}(V_1)$ is a complete subgraph of $\overline{G}$.

**Subcase 2.1.1:** If $w_1$ is adjacent to all vertices in $V_1$ and none of the vertices in $V_2$, then by maximality of $\overline{G}$ (condition (ii)), $w_2 w_3 \in E(\overline{G})$ which implies $G = K_{2,n-2}$.

**Subcase 2.1.2:** If $w_1$ is adjacent to all vertices in $V_1$ and some of the vertices in $V_2$ (say $w_2$), then $H_k \subseteq \overline{G}(V_1 \cup \{w_1, w_2\}) \subseteq \overline{G}$ which contradicts condition (i).

**Subcase 2.1.3:** If $w_1$ is adjacent to some (but not all) vertices in $V_1$ and none of the vertices in $V_2$, then $|E(\overline{G})| < |E(K_{2,n-2})|$ which contradicts condition (ii) so $G$ is not extremal.

**Subcase 2.1.4:** Let $w_1$ be adjacent to some (but not all) vertices in $V_1$ and some of the vertices in $V_2$ (say $w_2$). If $w_1$ is adjacent to at least $k - 1$ vertices in $V_1$ then $H_k \subseteq \overline{G}(V_1 \cup \{w_1, w_2\}) \subseteq \overline{G}$ which contradicts condition (i). If $w_1$ is adjacent to less than $k - 1$ vertices in $V_1$, then $|E(\overline{G})| < |E(K_{2,n-2})|$ for $k \geq 4$, which contradicts condition (ii) so $G$ is not extremal.

**Subcase 2.2:** Assume that $\overline{G}(V_1)$ is not a complete subgraph of $\overline{G}$, say $v_1 v_2 \notin E(\overline{G}(V_1))$. Similar to what we just discussed, we have following subcases.
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