Analysis of a discontinuous Galerkin method for the Biot's consolidation problem

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**A B S T R A C T**

In this paper, a fully discrete stabilized discontinuous Galerkin method is proposed to solve the Biot's consolidation problem. The existence and uniqueness of the finite element solution are obtained. The stability of the fully discrete solution is discussed. The corresponding error estimates for the approximation of displacement and pressure in a mesh dependent norm are obtained. The error estimate for the approximation of pressure in $L^2$ norm is also obtained.

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1. Introduction

Variational principles for the Biot's consolidation problem and finite element approximations based on the Galerkin method were presented in [17,30,31]. With this formulation, certain combinations of finite element interpolations (including equal order for both fields) were discarded, due to the incompressibility constraint on the displacement field in the initial state. In [22], the authors analyzed mixed Galerkin methods for the Biot's equations. Error estimates for the semidiscrete and fully discrete approximations, with a first order backward scheme in time, were presented. Asymptotic behavior of semi-discrete finite element approximations for the Biot's consolidation problem was discussed in [23]. All those methods are based on the continuous finite element spaces, and require that the discrete displacement and pressure satisfy the Babuška-Brezzi stability condition. To avoid the stability condition, some stabilized methods are proposed for the Stokes and Navier-Stokes problems in [7,9,14,32]. The aim of this paper is to provide a discontinuous Galerkin method for the Biot's consolidation problem.

The discontinuous Galerkin (DG) method was firstly introduced by Reed and Hill in [26] for hyperbolic equations in 1973, but less attention has been paid to it. Since the late 1980s, the Runge–Kutta DG (RKDG) method was developed by Cockburn and Shu in [11–13], and extended to conservation law and system of conservation laws, respectively. The mathematical analysis of its convergence behavior has been conducted. From that time on, more attention has been paid to the DG method.

In general, the DG method keeps the good properties of the finite element method (FEM) and the finite volume method (FVM). The DG method is locally conservative, stable, and high-order accurate method which can easily handle complex geometries, irregular meshes with hanging nodes, and approximations that have polynomials of different degrees in different elements. Because of these advantages, the DG method has become a very active area in research [2,4,8,10,18–21,27–29]. Unified analysis of the discontinuous Galerkin method for elliptic problems can be found in [2].

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In this paper, we analyze a fully discrete finite element method with a stabilized mixed DG method for the Biot’s consolidation problem. The starting point of the stabilized formulation is the introduction of additional terms, i.e., the jump terms of the displacement and the pressure, in the discrete problem which enhance the stability of the numerical solution. This method is stable for $P_1/P_0$ and $P_1/P_1$ ($l \geq 1$) combination of discontinuous discrete velocity and pressure spaces. The rest of paper is organized as follows. In Section 2, we introduce a fully discrete stabilized DG method for the Biot’s equations. The existence and uniqueness of the finite element solution are obtained in Section 3. Stability analysis is given in Section 4. Finally in Section 5, we establish some error estimates for the discretization formulation.

Throughout the rest of the paper, vector valued functions are written in boldface. We employ $\mathbf{0}$ to denote a generic null vector and use $C$ and $c$, with or without subscripts, to denote generic constants which may take different values at different places.

In this paper, we analyze finite element approximations of the classical two-dimensional Biot’s consolidation problem of fluid-filled homogeneous, isotropic porous materials composed of incompressible grains [22,23]. For each place.

As the initial condition we have

$$
\mathbf{u} = \mathbf{0} \quad \text{and} \quad p = 0 \quad \text{on} \quad \partial \Omega \quad \text{for} \quad t > 0.
$$

As the initial condition we have

$$
\nabla \cdot \mathbf{u}(0) = 0 \quad \text{in} \quad \Omega.
$$

2. Discontinuous finite element formulation

In this section, we introduce a discontinuous discretization for the Biot’s consolidation problem. We begin by introducing some notation. Let $\mathcal{I}$ be a quasi-uniform family of triangulations of $\Omega$, consisting of triangles $K$ with diameter $h_K$. The index $h$ denotes $h := \max_{K \subseteq \mathcal{I}} h_K$. Let $\Gamma$ be the union of the edges of the triangles $K$ of $\mathcal{I}$, $\Gamma_0 = \Gamma \setminus \partial \Omega$ and $H(\Gamma) := \Pi_{K \subseteq \mathcal{I}} L^2(\partial K)$. We denote by $\epsilon(\mathbf{u}) = \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right)$ the strain tensor of the porous skeleton. We also denote by $\mathbf{v} \otimes \mathbf{n}$ the matrix whose $ij$th component is $v_i n_j$ and write

$$
\nabla \mathbf{u} : \nabla \mathbf{v} := \sum_{ij=1}^2 (\nabla u_i v_j + \nabla v_i u_j), \quad \mathbf{v} \cdot \nabla \mathbf{u} := \sum_{ij=1}^2 v_j \nabla u_i n_j = \nabla \mathbf{u} : (\mathbf{v} \otimes \mathbf{n}).
$$

Next, we introduce notation associated with traces. Let $e$ be an interior edge shared by two neighboring elements $K^+$ and $K^-$; we write $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) to denote the unit normal vectors pointing exterior to the boundaries $\partial K^+$ (resp. $\partial K^-$). Let $\phi$ be a piecewise smooth scalar-, vector-, or matrix-valued function and let us denote by $\phi^+$ (resp. $\phi^-$) the traces of $\phi$ on $e$ taken from within the interior of $K^+$ (resp. $K^-$). Then, we define the average $\bar{\phi}$ at $x \in e$ as

$$
\{ \phi \} := \frac{1}{2} (\phi^+ + \phi^-).
$$

Further, for a generic multiplication operator $\otimes$, define the jump $[ \cdot ]$ at $x \in e$ as

$$
[ \phi \otimes \mathbf{n}] := \phi^+ \otimes \mathbf{n}^+ + \phi^- \otimes \mathbf{n}^-.
$$

On boundary edges, we set accordingly $\{ \phi \} := \phi$ and $[ \phi \otimes \mathbf{n}] := \phi \otimes \mathbf{n}$, with $\mathbf{n}$ denoting the outer unit normal vector on $\partial \Omega$. 

We omit the time variable $t$ in $\mathbf{u}(t)$, $p(t)$ and $\mathbf{f}(t)$ when there are no confusions. Here $\Omega$ denotes a convex polygonal domain of $\mathbb{R}^2$; $\partial \Omega$ denotes the time derivative; $A$ and $B$ are, respectively, the isotropic elasticity and Laplacian second-order elliptic operators

$$
\begin{align*}
&Au = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \nabla \cdot \mathbf{u}, \\
&Bp = \frac{K}{\eta_t} \Delta p,
\end{align*}
$$

where the coefficients $\lambda$, $\mu$ and $\kappa$ are the Lamé constants and the permeability of the porous medium, and $\eta_t$ denotes the viscosity of the pore fluid. The symbols $\Delta$, $\nabla$, and $\nabla$ denote the Laplacian, gradient, and divergence operators, respectively. For convenience we consider $\partial \Omega$ as a rigid and permeable boundary (with a free drainage condition) such that we can prescribe homogeneous Dirichlet boundary conditions

$$
\mathbf{u} = \mathbf{0} \quad \text{and} \quad p = 0 \quad \text{on} \quad \partial \Omega \quad \text{for} \quad t > 0.
$$

As the initial condition we have

$$
\nabla \cdot \mathbf{u}(0) = 0 \quad \text{in} \quad \Omega.
$$
Remark 1. In components, we have  \[ \|v \otimes n\|_h = (v^+_i - v^-_i) n^+_i. \] Note that  \[ \|v \cdot n\| = \sum_{k=1}^n (v^+_i - v^-_i) n^+_i. \] Let  \[ \|v \otimes n\|^2 = [v \otimes n] : [v \otimes n], \] Then  \[ \|v \otimes n\|^2 = \sum_{k=1}^n ((v^+_i - v^-_i) n^+_i)^2 \geq ||v \cdot n||^2, \] that is, the norm of the scalar-valued jump of the displacement can be controlled by the norm of the matrix-valued jump.

Define the functional spaces \( V \) and \( Q \) for the displacement and pressure respectively by  \[ V = \{ v \in L^2(\Omega)^2, \; v|_K \in H^1(K)^2, \; \forall K \in \mathcal{I} \}, \]
and  \[ Q = \{ q \in L^2(\Omega), \; q|_K \in H^1(K), \; \forall K \in \mathcal{I} \}. \]
The discrete finite element spaces \( V_h \) and \( Q_h \) are defined by  \[ V_h = \{ v \in L^2(\Omega)^2 : v|_K \in P_h(K) \times P_l(K), \; \forall K \in \mathcal{I} \}, \] and  \[ Q_h = \{ q \in L^2(\Omega) : q|_K \in P_h(K), \; \forall K \in \mathcal{I} \}, \] for approximation orders \( l, k \geq 1 \). Here, \( P_l(K) \) denotes the set of polynomials of degree at most \( l \) on \( K \). We also assume that  \[ \nabla Q_h \subset V_h. \tag{2.1} \]
Next, we will show that the DG method defines a unique approximation solution provided that for each element \( K \in \mathcal{I} \), the following condition holds  \[ q \in Q_h : \int_K v \cdot \nabla q \, dx = 0 \; \forall v \in V_h \; \text{implies} \; \nabla q \equiv 0 \; \text{on} \; K. \tag{2.2} \]
Note that (2.1) implies the assumption (2.2) on the local spaces if \( k \leq l + 1 \). For example, we can choose \( k = l \), or \( k = l - 1 \).

The discontinuous Galerkin method is often derived from the governing equations locally on each element. The use of integration by parts over each element then leads to a variational formulation. Multiplying the Eqs. (1.1) and (1.2) by test functions \( v \in V \) and \( q \in Q \), respectively, and integrating by parts on each element \( K \in \mathcal{I} \), then summing over all elements \( K \), we have  \[ 2\mu (\epsilon_h(u), \epsilon_h(v)) + \lambda (\nabla_h \cdot u, \nabla_h \cdot v) - 2\mu \sum_{K} \epsilon_h(u) : (v \otimes n) \, ds = 2\mu \sum_{K} \int \nabla_h \cdot uv \cdot nds - (p, \nabla_h \cdot v) + \sum_{K} \int p \cdot nds \]
\[ = (f, v) \; \forall v \in V, \; t > 0, \tag{2.3} \]
\[ (\nabla \cdot D, u, q) + \frac{K}{\eta} (\nabla_h p, \frac{\eta}{K} \sum_{K} \nabla_h p \cdot nq \, ds = 0 \; \forall q \in Q, \; St > 0. \tag{2.4} \]
The initial data is governed by the the following Stokes problem  \[ 2\mu (\epsilon_h(u(0)), \epsilon_h(v)) - (p(0), \nabla_h \cdot v) = (f(0), v) \; \forall v \in (H_0^1(\Omega))^2, \]
\[ (\nabla_h \cdot u(0), q) = 0 \; \forall q \in L^2(\Omega). \]

From Brezzi’s theorem, the stability of the initial problem depends on the inf-sup or LBB condition. Therefore, in this paper, we would solve the initial problem by the stabilized DG method.

The initial configuration of the saturated mixture is usually obtained by combining the equilibrium Eq. (1.1) with the constraint (1.4), which means that the mixture responds initially like an incompressible medium governed by the equations whose variational form is given as follows.

\[ 2\mu (\epsilon_h(u(0)), \epsilon_h(v)) - 2\mu \sum_{K} \epsilon_h(u(0)) : (v \otimes n) \, ds - (p(0), \nabla_h \cdot v) + \sum_{K} \int p(0)v \cdot nds = (f(0), v) \; \forall v \in V, \tag{2.5} \]
\[ (\nabla_h \cdot u(0), q) = 0 \; \forall q \in L^2_0(\Omega). \tag{2.6} \]
Here, \( (\cdot, \cdot) = \sum_K (\cdot, \cdot)_K \). \( \epsilon_h(v) \), \( \nabla_h \cdot v \) and \( \nabla_h v \) are the functions whose restriction to each element \( K \in \mathcal{I} \) are equal to \( \epsilon(v), \nabla \cdot v \) and \( \nabla v \), respectively.

To deal with the sums of the form \( \sum_K \int qv \cdot nds \), we introduce the following equality in [2]. For all \( q \in H(\Gamma) \) and for all \( \mathbf{v} \in [H(\Gamma)]^2 \), there holds  \[ \sum_{K} \int qv \cdot nds = \int_{\Gamma_0} q[v] \, ds + \int_{\Gamma} \{q\} |v| \, ds. \tag{2.7} \]
For any $T > 0$, let $(u, p) \in L^2(0, T; (H^1_0(\Omega) \cap H^2(\Omega))^2) \times L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega))$ with $l, k \geq 2$, then we have $[\epsilon_h(u)] = 0, [\nabla_h \cdot u] = 0, [p] = 0$ and $[\nabla_h p] = 0$ on $\Gamma_0$. Therefore, we can rewrite Eqs. (2.3) and (2.4) as

\[
2\mu(\epsilon_h(u), \epsilon_h(v)) + \lambda(\nabla_h \cdot u, \nabla_h \cdot v) - 2\mu \sum_{e \in I}^e \{ \epsilon_h(u) \} : [v \otimes n_e] ds - \lambda \sum_{e \in I}^e \{ \nabla_h \cdot u \} [v \cdot n_e] ds - (p, \nabla_h \cdot v) + \sum_{e \in I}^e \{ p \} [v \cdot n_e] ds = (f, v) \quad \forall v \in V, t > 0,
\]

(2.8)

and initial data is given by

\[
2\mu(\epsilon_h(u(0)), \epsilon_h(v)) - 2\mu \sum_{e \in I}^e \{ \epsilon_h(u(0)) \} : [v \otimes n_e] ds - (p(0), \nabla_h \cdot v) + \sum_{e \in I}^e \{ p(0) \} [v \cdot n_e] ds = (f(0), v) \quad \forall v \in V.
\]

(2.10)

Due to the saddle-point property of the initial state (2.10) and (2.11), the two finite element spaces must be constructed such that the inf-sup condition of Brezzi [6] and Babuška [3] is satisfied. Therefore, we shall stabilize the Eqs. (2.10) and (2.11) by using the jump terms of the displacement and the pressure.

To summarize, our variational form is given by: for each $t \in (0, T)$, seeking $(u, p) \in L^2(0, T; V) \times L^2(0, T; Q)$ such that

\[
A(u, v) + b(v, p) = (f, v) \quad \forall v \in V, t > 0,
\]

(2.12)

\[-b(D(u, q) + \frac{K}{\eta t} d(p, q) + g(p, q) = 0 \quad \forall q \in Q, t > 0,
\]

(2.13)

where

\[
a(u, v) = 2\mu(\epsilon_h(u), \epsilon_h(v)) - 2\mu \sum_{e \in I}^e \{ \epsilon_h(u) \} : [v \otimes n_e] ds + \delta 2\mu \sum_{e \in I}^e \{ \epsilon_h(v) \}
\]

\[
+ [u \otimes n_e] ds + C_{11} \sum_{e \in I} \int h_e^{-1} [u \otimes n_e] : [v \otimes n_e] ds.
\]

(2.14)

\[A(u, v) = a(u, v) + \lambda(\nabla_h \cdot u, \nabla_h \cdot v) - \lambda \sum_{e \in I} \{ \nabla_h \cdot u \} [v \cdot n_e] ds
\]

(2.15)

\[b(v, p) = -(p, \nabla_h \cdot v) + \sum_{e \in I} \{ p \} [v \cdot n_e] ds = (\nabla_h p, v) - \sum_{e \in I} \{ v \cdot n_e \} [p] ds,
\]

(2.16)

\[d(p, q) = (\nabla_h p, \nabla_h q) - \sum_{e \in I} \{ \nabla_h p \} [q] ds + \delta \sum_{e \in I} \{ \nabla_h q \} [p] ds,
\]

(2.17)

\[g(p, q) = C_{22} \sum_{e \in I} \int h_e^{-1} [p] [q] ds.
\]

(2.18)

The initial solution $(u(0), p(0))$ satisfies

\[
a(u(0), v) + b(v, p(0)) = (f(0), v) \quad \forall v \in V,
\]

(2.19)

\[-b(u(0), q) + c(p(0), q) = 0 \quad \forall q \in Q,
\]

(2.20)

where

\[
c(p, q) = C_{33} \sum_{e \in I} \int h_e[p] [q] ds.
\]

(2.21)

The coefficients $C_{11}, C_{22}, C_{33}$ are three positive constants and $\delta = \pm 1$. The parameter $C_{11} > 0$ is chosen to guarantee coercivity of the form $a(u, v)$. We assume that $C_{11}$ is bounded below by a large enough positive constant for $\delta = -1$.

**Remark 2.** More precisely, for $\delta = -1$, we obtain the classical interior penalty method [1], for $\delta = 1$ the stabilized version of the Baumann–Oden method [5]. When $\delta = -1$ and $C_{22} = 0$, that is the method proposed by X. Ye in [28] for the stationary Stokes problem.
Proposition 1. If \( u \in V \cap C^2(\Omega) \), \( p \in Q \cap C^2(\Omega) \), then the problem (1.1) and (1.2) and the weak formulation (2.12) and (2.13) are equivalent.

Proof. We use the skills in [16] to complete the proof. Let \((u, p)\) be the solution of the problem 1.1.1.2. From the regularities of \((u, p)\) and \(u|_{\partial \Omega} = 0, p|_{\partial \Omega} = 0\), we can obtain that those stabilized terms in 2.12,2.13 vanishes, so it concludes that this solution also satisfies the variational problem 2.12,2.13.

Conversely, let \((u, p)\) be the solution of variational problem 2.12,2.13. Let \( K \) belong to \( S \) and \( D(K) \) denote the space of infinitely differentiable functions with compact support on \( K \). First, choose \( v \in D(K)^2 \), extended by zero outside \( K \). Then, for each \( t \in (0, T) \), \((u, p)\) satisfies the following equations in the sense of distributions

\[
-\Delta u + \nabla p = f \quad \text{in} \ K, \tag{2.22}
\]

\[
\nabla \cdot D_t u - B p = 0 \quad \text{in} \ K. \tag{2.23}
\]

Next, we consider \( v \in C^1(\overline{K})^2 \) such that \( v = 0 \) on \( \partial K \), extended by 0 outside \( K \), \( \epsilon_h(v) \neq 0 \) on one side \( e \) and \( \epsilon_h(v) = 0 \) on \( \partial K \setminus e \), then the Eq. (2.12) reduces to

\[
2\mu(\epsilon_h(u), \epsilon_h(v)) + \lambda(\nabla_v \cdot u, \nabla_h \cdot v)_K - (p, \nabla_h \cdot v)_K + \delta 2\mu \int_e \{ \epsilon_h(v) \} : [u \otimes n_e] ds = (f, v)_K. \tag{2.24}
\]

We multiply the Eq. (2.22) by \( v \) and integrate by parts

\[
2\mu(\epsilon_h(u), \epsilon_h(v)) + \lambda(\nabla_v \cdot u, \nabla_h \cdot v)_K - (p, \nabla_h \cdot v)_K = (f, v)_K. \tag{2.25}
\]

From the above two equations, we have

\[
\int_e \{ \epsilon_h(v) \} : [u \otimes n_e] ds = 0. \tag{2.26}
\]

Since \( \epsilon_h(v) \neq 0 \) on the side \( e \), then \([u \otimes n_e] = 0\) on the side \( e \). If \( e \) belongs to the boundary \( \partial \Omega \), this implies that \( u|_{\partial \Omega} = 0 \). Similarly, we can from the equation

\[
\delta \int_e \{ \nabla_q h \} : [p] ds = 0, \tag{2.27}
\]

to conclude that \( p|_{\partial \Omega} = 0 \). Finally, let \( e = K_1 \cap K_2 \) and choose \( v \in C^1(\overline{K}_1)^2 \), with \( v = 0 \) on \( \partial K_1 \) except on the side \( e \), extended by 0 outside. From (2.12) and (2.22), we have

\[
2\mu(\epsilon_h(u), \epsilon_h(v)) + \lambda(\nabla_v \cdot u, \nabla_h \cdot v)_K - (p, \nabla_h \cdot v)_K - 2\mu \int_e (\epsilon_h(u)) : [v \otimes n_e] ds - \lambda \int_e (\nabla_h \cdot u)[v \cdot n_e] ds \\
+ \int_e (p)[v \cdot n_e] ds = 2\mu(\epsilon_h(u), \epsilon_h(v)) + \lambda(\nabla_v \cdot u, \nabla_h \cdot v)_K - (p, \nabla_h \cdot v)_K - 2\mu \int_e \epsilon_h(u) : [v \otimes n_e] ds \\
- \lambda \int_e \nabla_h \cdot uv \cdot n_e ds + \int_e pv \cdot n_e ds, \tag{2.28}
\]

which implies

\[
\int_e \{-2\mu\epsilon_h(u)n_e - \lambda \nabla_h \cdot un_e + pn_e\} \cdot v ds = \int_e \{-2\mu\epsilon_h(u)n_e - \lambda \nabla_h \cdot un_e + pn_e\} \cdot v ds. \tag{2.29}
\]

Since \( v \) is arbitrary, this means that the quantity \(-2\mu\epsilon_h(u)n_e - \lambda \nabla_h \cdot un_e + pn_e\) is continuous across \( e \). Therefore,

\[
-\mu A u - (\mu + \lambda) \nabla \cdot u + \nabla p = f \quad \text{holds in} \ \Omega.
\]

Similarly, we can obtain that \( \nabla \cdot D_t u - Bp = 0 \) holds in \( \Omega \). This completes the proof. \( \square \)

In the rest part of this section, we will derive a fully discrete formulation based on 2.12,2.13. We shall consider the discretization of the time domain by a first order backward Euler scheme. Second order approximations in time can be achieved applying the Crank-Nicolson scheme. However this is not our purpose, since we are mainly interested in the improvement of convergence rates in \( h \). Let \( N \) be a positive integer and the time step \( \Delta t = \frac{T}{N} \). And let the pair \((u^n, p^n)\) denote the approximation of \((u(t), p(t))\) at the discrete time \( t = m\Delta t \). The Euler operator \( \partial_t \) approximates the time derivative by the quotient \( \partial_t u^n = (u^n - u^{n-1})/\Delta t \). Therefore, we obtain the following fully discrete discontinuous formulation corresponding to the Biot’s consolidation problem.

For each \( 1 \leq m \leq N \), find \((u^n, p^n) \in V^h \times Q^h\) such that

\[
A(u^n, v) + b(v, p^n) = (f^n, v) \quad \forall v \in V^h, \tag{2.30}
\]

\[
-b(\partial_t u^n, q) + \frac{K}{\eta_t} d(p^n, q) + g(p^n, q) = 0 \quad \forall q \in Q^h, \tag{2.31}
\]

with \( u^n_0 \) and \( p^n_0 \) satisfying
\[ a(u_h^m, v) + b(v, p_h^0) = (f^m, v) \quad \forall v \in V^h, \]  
(2.32)

\[-b(u_h^m, q) + c(p_h^0, q) = 0 \quad \forall q \in Q^h. \]  
(2.33)

Finally, we rewrite (2.30) and (2.31) in the following equivalent form. For each \(1 \leq m \leq N\), find \((u_h^m, p_h^m) \in V^h \times Q^h\) such that

\[ \phi(u_h^m, p_h^m; v, q) = (f^m, v) \quad \forall (v, q) \in V^h \times Q^h, \]  
(2.34)

where

\[ \phi(u_h^m, p_h^m; v, q) = A(u_h^m, v) + b(v, p_h^m) - b(\delta u_h^m, q) + \frac{K}{\eta_f} d(p_h^m, q) + g(p_h^m, q). \]

### 3. Existence and uniqueness of DG solutions

Following [2], we define the following seminorms and norms

\[ ||e_h(v)||_0^2 := \sum_K ||e(v)||_{\Omega,K}^2, \quad ||\nabla_h \cdot v||_0^2 := \sum_K ||\nabla_h \cdot v||_{\partial\Omega,K}^2, \]

\[ ||v||_{1,K}^2 := \sum_{e \in \partial K} h_e^{-1} \int_e |v \otimes n_e|^2 ds + \sum_{e \in \partial K} h_e^{-1} ||v \otimes n_e||_{L^2}^2, \]

\[ ||v||_{0,K}^2 := ||e_h(v)||_0^2 + ||\nabla_h \cdot v||_0^2 + ||v||_{L^2}^2, \]

\[ ||q||_0^2 := \sum_K h_K^{-1} \int_K |q|^2 ds := |q|_{L^2}^2 + |q|_{H^{-1}}^2, \]

\[ ||(v, q)||_0^2 := ||v||_{0,K}^2 + ||q||_{0,K}^2. \]

We recall a basic trace inequality. There exists a constant \(C\) independent of \(h\) such that

\[ \forall v \in H^1(K), \quad ||v||_{0,K}^2 \leq C(h^{-1} ||v||_{L^2}^2 + h ||\nabla v||_{L^2}^2). \]  
(3.1)

**Lemma 1.** We assume that \(c_{11}\) is bounded below by a large enough positive constant for \(\delta = -1\). There exists a unique solution \((u_h^m, p_h^m) \in V^h \times Q^h\) for the weak formulation (2.32) and (2.33).

**Proof.** As (2.32) and (2.33) represent a system of linear equations, it is enough to show that the only possible solution to the system with \(f = 0\) is \((u_h^0, p_h^0) = (0, 0)\).

If \(\delta = 1\), taking \(v = u_h^0, q = p_h^0\) in (2.32) and (2.33) yields

\[ a(u_h^0, u_h^0) + c(p_h^0, p_h^0) = 2\mu (e_h(u_h^0), e_h(u_h^0)) + c_{11} \sum_{e \in \partial K} h_e^{-1} [u_h^0 \otimes n_e] : [u_h^0 \otimes n_e] ds + c(p_h^0, p_h^0) = 0. \]  
(3.2)

Since the coefficient \(c_{11}\) and \(c_{33}\) in (3.2) are positive, we can obtain that \(e_h(u_h^0) = 0\) holds on every \(K \in \mathcal{K}, |p_h^0| = 0\) and \([u_h^0 \otimes n_e] = 0\) hold on the interior edges \(\Gamma_0, p_h^0 = 0\) and \(u_h^0 \otimes n_e = 0\) hold on \(\partial\Omega\). Therefore, \(u_h^0 = 0\). Taking \(u_h^0 = 0\), Eq. (2.32) becomes

\[ \sum_{K \in \mathcal{K}} \int_K v \cdot \nabla p_h^0 dx = 0 \quad \forall v \in V^h. \]

We conclude from assumption (2.1) that \(\nabla p_h^0 = 0\) on every \(K \in \mathcal{K}\), and, since \(|p_h^0| = 0\) on \(\Gamma_0\), that \(p_h^0 = 0\) is a constant. Since we also have \(\int_{\partial\Omega} p_h^0 dx = 0\), we conclude that \(p_h^0 = 0\).

If \(\delta = -1\), the term \(a\) of the left side of the Eq. (3.2) is

\[ a(u_h^0, u_h^0) = 2\mu (e_h(u_h^0), e_h(u_h^0)) - 4\mu \sum_{e \in \partial K} \int_e (e_h(u_h^0)) : [u_h^0 \otimes n_e] ds + c_{11} \sum_{e \in \partial K} h_e^{-1} [u_h^0 \otimes n_e] : [u_h^0 \otimes n_e] ds. \]  
(3.3)

Combining the Cauchy–Schwarz inequality, (3.1) and the inverse inequality gives

\[ \sum_{e \in \partial K} \int_e (e_h(v_h)) : [w_h \otimes n_e] ds \leq \left( \sum_{e \in \partial K} \int_e (e_h(v_h))^2 ds \right)^{1/2} \left( \sum_{e \in \partial K} ||w_h \otimes n_e||_0^2 ds \right)^{1/2} \]

\[ \leq C \left( \sum_{e \in \partial K} (||e_h(v_h)||_0^2 + h_e^2 ||e_h(v_h)||_{1,K}^2) \right)^{1/2} \left( \sum_{e \in \partial K} h_e^{-1} \int_e ||w_h \otimes n_e||_0^2 ds \right)^{1/2} \leq C ||e_h(v_h)||_0 ||w_h||_0. \]  
(3.4)
Using (3.4) and Young’s inequality \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \), we have
\[
a(\mathbf{u}_h^0, \mathbf{u}_h^0) \geq 2\mu(\|\mathbf{\varepsilon}_h(\mathbf{u}_h^0)\|_0^2 + C_{11}\|\mathbf{u}_h^0\|_0^2 - \left(\frac{2\mu}{\nu}\|\mathbf{\varepsilon}_h(\mathbf{u}_h^0)\|_0^2 + 2\mu\epsilon^2\|\mathbf{u}_h^0\|_0^2\right) \geq C(\|\mathbf{\varepsilon}_h(\mathbf{u}_h^0)\|_0^2 + \|\mathbf{u}_h^0\|_0^2).
\]
We assume that \( C_{11} \) is bounded below by a large enough positive constant such that \( C_{11} - 2\mu\epsilon^2 > 0 \) hold. The last inequality holds for \( \epsilon > 1 \). Following the same way as in the proof of the case \( \delta = 1 \), (3.2) and (3.5) conclude the proof. \( \square \)

**Theorem 1.** The discontinuous finite element scheme (2.34) has one and only one solution \((\mathbf{u}_h^n, p_h^n) \in V^h \times Q^h \) for \( m \geq 1 \).

**Proof.** It is enough to show that the only possible solution to the Eq. (2.34) with \( f = 0 \) and \( \mathbf{u}_h^0 = 0 \) is \((\mathbf{u}_h^n, p_h^n) = (0,0)\). Setting \( \mathbf{v}_h = \frac{\mathbf{u}_h^n}{h}, q_h = p_h^n \) in (2.34), we have
\[
A(\mathbf{u}_h^n, \mathbf{u}_h^n) + b(\mathbf{u}_h^n, p_h^n) + \frac{\Delta t\kappa}{\eta_f} d(p_h^n, p_h^n) + \Delta t g(p_h^n, p_h^n) = 0.
\]
We complete the proof by an induction argument. For \( m \geq 1 \), let \( \mathbf{u}_h^{m-1} = 0, p_h^{m-1} = 0 \), Eq. (3.6) becomes
\[
A(\mathbf{u}_h^n, \mathbf{u}_h^n) + \frac{\Delta t\kappa}{\eta_f} d(p_h^n, p_h^n) + \Delta t g(p_h^n, p_h^n) = 0.
\]
We have
\[
A(\mathbf{u}_h^n, \mathbf{u}_h^n) = 2\mu(\|\mathbf{\varepsilon}_h(\mathbf{u}_h^n)\|_0^2 + \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0^2 + C_{11}\|\mathbf{u}_h^n\|_0^2 + \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0^2 + (\delta - 1)2\mu\sum_{e \in F} \int_e \|\mathbf{\varepsilon}_h(\mathbf{u}_h^n)\|_0^2 + \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0^2 - \lambda\sum_{e \in F} \int_e \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0^2 \mathbf{u}_h^n \cdot \mathbf{n}_e ds - \lambda\sum_{e \in F} \int_e \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0^2 \mathbf{u}_h^n \cdot \mathbf{n}_e ds,
\]
and
\[
\frac{\Delta t\kappa}{\eta_f} d(p_h^n, p_h^n) + \Delta t g(p_h^n, p_h^n) = \frac{\Delta t\kappa}{\eta_f} \left( |p_h^n|_{1,h} + (\delta - 1)\sum_{e \in F} \int_e \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0^2 \mathbf{n}_e ds + \|p_h^n\|_{1,h} \right).
\]
From Remark 1, we obtain
\[
\left( \sum_{e \in F} h_e^{-1} \int_e \|\mathbf{w}_h \cdot \mathbf{n}_e\|^2 ds \right)^{1/2} \leq \left( \sum_{e \in F} h_e^{-1} \int_e \|\mathbf{w}_h \cdot \mathbf{n}_e\|^2 ds \right)^{1/2}.
\]
Following the same way as in the proof of (3.4), from (3.10) we have
\[
\sum_{e \in F} \int_e \|\mathbf{\nabla}_h \cdot \mathbf{v}_h\|_0 \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0 \left( \sum_{e \in F} h_e^{-1} \int_e \|\mathbf{w}_h \cdot \mathbf{n}_e\|^2 ds \right)^{1/2} \leq C\|\mathbf{\nabla}_h \cdot \mathbf{v}_h\|_0 \|\mathbf{w}_h\|_0 .
\]
It follows from (3.4),(3.8) and (3.11) that
\[
A(\mathbf{u}_h^n, \mathbf{u}_h^n) \geq C\|\mathbf{u}_h^n\|_0^2 .
\]
We can also have the following results similar to (3.11)
\[
\sum_{e \in F} \int_e \|\mathbf{\nabla}_h q_h\|_0 \|\mathbf{\nabla}_h \cdot \mathbf{u}_h^n\|_0 \|\mathbf{\nabla}_h \cdot \mathbf{v}_h\|_0 \left( \sum_{e \in F} h_e^{-1} \int_e \|\mathbf{w}_h \cdot \mathbf{n}_e\|^2 ds \right)^{1/2} \leq C\|\mathbf{\nabla}_h \cdot \mathbf{v}_h\|_0 \|\mathbf{w}_h\|_0 ,
\]
\[
C_{22} \sum_{e \in \Gamma_0} h_e^{-1} |q_h|_{1,h} ds \leq C|q_h|_{1,h} .
\]
Combining (3.9), (3.13) and (3.14), we arrive at
\[
\frac{\Delta t\kappa}{\eta_f} d(p_h^n, p_h^n) + \Delta t g(p_h^n, p_h^n) \geq C( |p_h^n|_{1,h} + |p_h^n|_{1,h} ).
\]
The rest proof is similar to that of Lemma 1. Using (3.7), (3.12) and (3.15), we conclude that \( \mathbf{u}_h^n = 0, \mathbf{\nabla}_h p_h^n = 0 \) on every \( K \in \mathcal{F}. \) Since \( |p_h^n| = 0 \) on \( \Gamma_0, \) we know that \( p_h^n \) is a constant. We also have \( p_h^n = 0 \) on \( \partial \Omega, \) so we conclude that \( p_h^n = 0. \) \( \square \)
4. Stability analysis

We first give the stability of the initial solution of (2.32) and (2.33). Define
\[
\|\mathbf{v}\|_D^2 := \|\mathbf{c}_h(\mathbf{v})\|_D^2 + |\mathbf{v}|_1^2 := \sum_{e \in T_0} h_e \int_e |\mathbf{q}|^2 \, ds,
\]
\[
\|\langle \mathbf{v}, \mathbf{q} \rangle\|_S^2 := \|\mathbf{v}\|_D^2 + |\mathbf{q}|_0^2 + \|\mathbf{q}|_1^2.
\]

Let \(\Pi_h\) denote the \(L^2(\Omega)\)-orthogonal projection from \(L^2(\Omega)\) onto \(V_h\). Then, the following result holds (see [25]).
\[
\forall \mathbf{v} \in H^1(\Omega), \quad \|\nabla_h \Pi_h \mathbf{v}\|_0^2 + \sum_{e \in T_0} h_e^{-1} \|\mathbf{v} - \Pi_h \mathbf{v}\|_0^2 \leq C\|\mathbf{v}\|_1^2. \tag{4.1}
\]

The following stability result for the bilinear form \(b(\mathbf{v}_h, q_h)\) can be found in [25].

**Lemma 2.** There exists a positive constant \(\beta\) such that
\[
\forall q_h \in Q_h, \quad \beta\|q_h\|_0 \leq \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_D} + |q_h|_{1}. \tag{4.2}
\]

**Proof.** Let \(q_h \in Q_h\). Owing to a result by Nečas [24], there is \(\mathbf{v} \in \mathcal{H}_s(\Omega)^d\), s.t. \(\nabla \cdot \mathbf{v} = q_h\) and \(|\mathbf{v}|_{H^1(\Omega)^d} \leq C\|q_h\|_0\). Using the properties of the \(L^2\)-projection and the inclusion property in (2.1), we have
\[
|q_h|_{0}^2 = \int_{\Omega} q_h \nabla \cdot \mathbf{v} = -\sum_{e \in T_0} \int_{e} \nabla_h q_h \cdot \mathbf{v}_e + \sum_{e \in T_0} \int_{e} |\mathbf{v}_e| \{\mathbf{v}_e \cdot \mathbf{n}_e\} = -\sum_{e \in T_0} \int_{e} \nabla_h q_h \cdot \Pi_h \mathbf{v}_e + \sum_{e \in T_0} \int_{e} |\mathbf{v}_e| \{\mathbf{v}_e \cdot \mathbf{n}_e\} = b(\Pi_h \mathbf{v}, q_h) + \sum_{e \in T_0} \int_{e} |\mathbf{v}_e| \{\mathbf{v}_e \cdot \mathbf{n}_e\} := M_1 + M_2.
\]

Owing to (4.1), we have \(\|\Pi_h \mathbf{v}\|_D \leq C|\mathbf{v}|_1 \leq |\mathbf{v}|_0\), then it is inferred that
\[
|M_1| \leq \frac{b(\Pi_h \mathbf{v}, q_h)}{\|\Pi_h \mathbf{v}\|_D} \|\Pi_h \mathbf{v}\|_D \leq C \left( \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_D} \right) |q_h|_0.
\]

Similarly, \(|M_2| \leq C|q_h|_1 |q_h|_0\), whence the conclusion follows. \(\square\)

Let \(l_h((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h)) := a(\mathbf{v}_h, \mathbf{w}_h) + b(\mathbf{w}_h, q_h) - b(\mathbf{v}_h, r_h) + c(q_h, r_h)\). For the initial state, Lemma 2 yields the following stability result.

**Lemma 3.** There exists a positive constant \(c\) independent of \(h\) such that
\[
\forall (\mathbf{v}_h, q_h) \in V_h \times Q_h, |l_h((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h))|_1 \leq \sup_{(\mathbf{w}_h, r_h) \in V_h \times Q_h} \frac{l_h((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h))}{\|\mathbf{w}_h\|_D} \|\mathbf{w}_h\|_S^2. \tag{4.3}
\]

**Proof.** Let \((\mathbf{v}_h, q_h) \in V_h \times Q_h\). From (3.5), we have
\[
l_h((\mathbf{v}_h, q_h); (\mathbf{v}_h, q_h)) = a(\mathbf{v}_h, \mathbf{v}_h) + c(q_h, p_h) \geq C(|\mathbf{v}_h|_D^2 + |q_h|_{1}^2). \tag{4.4}
\]

Set \(S := \sup_{(\mathbf{w}_h, r_h) \in V_h \times Q_h} \frac{l_h((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h))}{\|\mathbf{w}_h\|_D^2 + |q_h|_{1}^2} \). From (4.4), we have
\[
C(|\mathbf{v}_h|_D^2 + |q_h|_{1}^2) \leq S\|l_h((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h))\|_S, \tag{4.5}
\]

and it only remains to control \(\|q_h\|_0^2\). Using Lemma 2 yields
\[
\beta\|q_h\|_0 \leq \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{w}_h, q_h)}{\|\mathbf{w}_h\|_D} + |q_h|_{1} \leq \sup_{\mathbf{v}_h \in V_h} \frac{a(\mathbf{v}_h, \mathbf{w}_h)}{\|\mathbf{w}_h\|_D} + \sup_{\mathbf{v}_h \in V_h} \frac{l_h((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h))}{\|\mathbf{w}_h\|_D} + |q_h|_{1} \leq C|\mathbf{v}_h|_D + S + |q_h|_{1}.
\]

The conclusion is straightforward. \(\square\)

Noticing that
\[
l_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = (f^0, \mathbf{v}),
\]

From the result of Lemma 3 and \(\|\nabla_h \cdot \mathbf{v}\|_0 \leq \sqrt{2}\|\mathbf{c}_h(\mathbf{v})\|_0\) in [22], we have the stability of the initial solution \((u^0_h, p^0_h)\) of (2.32) and (2.33).
\[
\|u^0_h\|_D^2 + |q^0_h|_{1}^2 \leq C|f^0|_{\ast}^2. \tag{4.6}
\]
Next, we give the stability of the solution of (2.34).

**Theorem 2.** Assume that $C_{14}$ is bounded below by a large enough positive constant. There exists a positive constant $C$ independent of $h$ and $\Delta t$ such that the solution of the discontinuous finite element scheme (2.34) holds

$$
\max_{0 \leq m \leq N} \|u_h^m\|_V^2 + \Delta t \sum_{m=1}^{N} \|p_h^m\|_Q^2 \leq C \left( \sum_{m=1}^{N} \|f^m\|_V^2 + \|u_h^0\|_V^2 \right). \tag{4.7}
$$

**Proof.** For any $0 \leq m \leq N$, setting $v = \partial_t u_h^m$, $q = p_h^m$ in (2.34), we have

$$
\phi(u_h^m, p_h^m; \partial_t u_h^m, p_h^m) = f(\partial_t u_h^m).
$$

That is

$$
A(u_h^m, \partial_t u_h^m) + \frac{K}{\eta_f} d(p_h^m, p_h^m) + g(p_h^m, p_h^m) = (f^m, \partial_t u_h^m). \tag{4.8}
$$

Let us first consider the left side of the Eq. (4.8). The term $A$ is

$$
A(u_h^m, \partial_t u_h^m) = \frac{2\mu}{\Delta t} \left( \epsilon_h(u_h^m), \epsilon_h(u_h^m - u_h^{m-1}) \right) + \frac{i}{\Delta t} \left( \nabla_h \cdot u_h^m, \nabla_h \cdot (u_h^m - u_h^{m-1}) \right) + \frac{C_{14}}{\Delta t} \sum_{i \in T} \int_{e} h_i^{-1} \| \nabla_h \cdot u_h^m \|_e \| n_e \|_e dS.
$$

Using the inequality $a(a - b) \geq \frac{1}{2} (a^2 - b^2)$, terms $T_1$ through $T_3$ are bounded

$$
T_1 \geq \frac{\mu}{\Delta t} \left( \| \epsilon_h(u_h^m) \|_0^2 - \| \epsilon_h(u_h^{m-1}) \|_0^2 \right), \tag{4.9}
$$

$$
T_2 \geq \frac{i}{\Delta t} \left( \| \nabla_h \cdot u_h^m \|_0^2 - \| \nabla_h \cdot u_h^{m-1} \|_0^2 \right), \tag{4.10}
$$

$$
T_3 \geq \frac{C_{14}}{2 \Delta t} \left( \| u_h^m \|_0^2 - \| u_h^{m-1} \|_0^2 \right). \tag{4.11}
$$

Following the same way as in the proof of (3.4) and using Young's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, the bounds for terms $T_4$ through $T_6$ are

$$
|T_4| \leq \frac{2\mu}{\Delta t} c_1 \| \epsilon_h(u_h^m) \|_0 \| u_h^m - u_h^{m-1} \|_0, \leq \frac{2\mu}{\Delta t} \left( \frac{1}{8} \| \epsilon_h(u_h^m) \|_0^2 + 2c_1^2 \| u_h^m - u_h^{m-1} \|_0^2 \right), \tag{4.12}
$$

$$
|T_5| \leq \frac{2\mu}{\Delta t} c_2 \| \epsilon_h(u_h^m - u_h^{m-1}) \|_0 \| u_h^m \|_0, \leq \frac{2\mu}{\Delta t} \left( \frac{1}{16} \| \epsilon_h(u_h^m - u_h^{m-1}) \|_0^2 + 4c_2^2 \| u_h^m \|_0^2 \right), \tag{4.13}
$$

$$
|T_6| \leq \frac{i}{\Delta t} c_3 \| \nabla_h \cdot u_h^m \|_0 \| u_h^m - u_h^{m-1} \|_0, \leq \frac{i}{\Delta t} \left( \frac{1}{8} \| \nabla_h \cdot u_h^m \|_0^2 + 2c_3^2 \| u_h^m - u_h^{m-1} \|_0^2 \right), \tag{4.14}
$$

From the definitions of the norms, we have

$$
\frac{K}{\eta_f} d(p_h^m, p_h^m) + g(p_h^m, p_h^m) = \frac{K}{\eta_f} \| \nabla_h p_h^m \|_0^2 + C_{22} \| p_h^m \|_e^2 \geq c_4 \| p_h^m \|_Q^2. \tag{4.15}
$$

Using Young's inequality and $\| a + b \|^2 \leq 2(\| a \|^2 + \| b \|^2)$, we have

$$
(f^m, \partial_t u_h^m) \leq \frac{1}{2\Delta t} \| f^m \|_V^2 + \frac{\varepsilon}{2} \| u_h^m - u_h^{m-1} \|_V^2 \leq \frac{1}{2\Delta t} \| f^m \|_V^2 + \varepsilon \| u_h^m \|_V + \varepsilon \| u_h^{m-1} \|_V^2. \tag{4.16}
$$

Substituting Eqs. (4.9)-(4.16) into (4.8), we have
To derive error estimates for the discontinuous scheme, we define the elliptic projection
\[ \text{Subtracting (5.1) and (5.2) from (2.30) and (2.31) respectively gives} \]
\[ \text{By (4.21), we have the desired result (4.2).} \]

Therefore, we have
\[ \|u_m\|_\Omega^2 + \Delta t \sum_{l=1}^{m} \|p_l\|_\Omega^2 \leq C \left( \sum_{l=1}^{m} \|f_l\|^2_{\star} + \|u_0\|^2_{\star} \right). \]

By (4.21), we have the desired result (4.2). \(\square\)

5. Error estimates

In this section, we derive some error estimates for the discontinuous finite element scheme (2.34). The polynomials of degrees \(l\) are used for each component of the displacement and polynomial of degree \(l - 1\) for the pressure.

For each \(t \in (0, T]\), let \((u, p) \in L^2(0, T; (H^{l+1}(\Omega) \cap H_0^1(\Omega))^2 \times L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))) \) be the solution of (1.1) and (1.2), then \((u, p)\) satisfies
\[ A(u, v) + b(v, p) = (f, v) \quad \forall v \in V^h, \quad (5.1) \]
\[ -b(D_t u, q) + \frac{K}{\eta_t} d(p, q) + g(p, q) = 0 \quad \forall q \in Q^h. \quad (5.2) \]

Subtracting (5.1) and (5.2) from (2.30) and (2.31) respectively gives
\[ A(u - u_0^h, v) + b(v, p - p_0^m) = 0 \quad \forall v \in V^h, \quad (5.3) \]
\[ -b(D_t u - \partial_t u_0^h, q) + \frac{K}{\eta_t} d(p - p_0^m, q) + g(p - p_0^m, q) = 0 \quad \forall q \in Q^h. \quad (5.4) \]

To derive error estimates for the discontinuous scheme, we define the elliptic projection \((u(t), p(t)) \in V^h \times Q^h\) of the exact solution \((u(t), p(t))\) as the solution of the following discrete stationary elliptic problem.

For each \(t \in (0, T]\), find \((u(t), p(t)) \in V^h \times Q^h\) such that
\[ A(u(t) - u_0^h(t), v) + b(v, p(t) - p_0^m) = 0 \quad \forall v \in V^h, \quad (5.5) \]
\[ \frac{K}{\eta_t} d(p(t) - p_0^m, q) + g(p(t) - p_0^m, q) = 0 \quad \forall q \in Q^h. \quad (5.6) \]

From the well known analysis of elliptic problems, the error of the elliptic projection can be estimated as follows.
Lemma 4. For \( t \in (0, T) \), there exists a unique solution \( (\mathbf{u}_i(t), p_i(t)) \in \mathbb{V}^h \times Q^h \) for the problem (5.5) and (5.6). And if \( \mathbf{u} \in L^2(0, T; (H^{r+1}(\Omega) \cap H_0^1(\Omega))^2), p \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \) then the estimates of the elliptic projection satisfy
\[
\|\varepsilon(x)(\mathbf{u}(t) - \mathbf{u}_i(t))\|_{0}^2 + \|v_t \cdot (\mathbf{u}(t) - \mathbf{u}_i(t))\|_{0}^2 + \|\mathbf{u}(t) - \mathbf{u}_i(t)\|_{0}^2 \leq C(h^{2l}[\mathbf{u}(t)]_{l+1} + h^{2l-2}[p(t)]_2^2),
\]
\[
\|v_t(p(t) - p_i(t))\|_{0}^2 + \|p(t) - p_i(t)\|_{0}^2 \leq ch^{2l-2}[p(t)]_2^2.
\]

Proof. For the sake of simplicity, we introduce a notation
\[
B(x, \mathbf{v}, q, \mathbf{w}, r, s) = A(x, \mathbf{w}) + b(x, q) + \frac{K}{\eta_t}d(q, r) + g(x, r) \quad \forall (x, \mathbf{v}, q, \mathbf{w}, r) \in \mathbb{V}^h \times Q^h.
\]

For \( q_h \in Q^h \) the discrete Poincaré inequality implies that (see [2])
\[
\|q_h\|_0 \leq C(q_h^2 + \|q_h\|_Q^{2l}).
\]

Combining the inequalities (3.10) and (5.10), we have
\[
\sum_{x \in T} \int_T (q_h(x) \mathbf{w}_h \cdot \mathbf{n}_x) dx \leq C \left( \sum_{x \in T} \left( \|q_h\|_{0, x}^2 + \eta_h^2 \|q_h\|_{1, x}^2 \right) \right)^{1/2} \left( \sum_{x \in T} \int_T \|\mathbf{w}_h \cdot \mathbf{n}_x\|_x^2 dx \right)^{1/2} \leq C \|q_h\|_Q^2 \|\mathbf{w}_h\|.
\]

So for any \( \mathbf{w}_h \in \mathbb{V}^h, q_h \in Q^h \), we have
\[
b(\mathbf{w}_h, q_h) = -\mathbf{v}_h \cdot (\mathbf{w}_h, q_h) + \sum_{x \in T} \int_T (q_h(x) \mathbf{w}_h \cdot \mathbf{n}_x) dx \leq C \|\mathbf{v}_h \cdot \mathbf{w}_h\|_0 + \|\mathbf{w}_h\|_0 \|q_h\|_Q.
\]

Combining (3.4), (3.11), (3.13), (3.14) and (5.12), we conclude that \( B(\cdot, \cdot, \cdot, \cdot) \) is continuous
\[
B(\mathbf{v}_h, q_h, \mathbf{w}_h, \mathbf{r}_h) \leq C(\|\mathbf{v}_h\|_0 + \|q_h\|_Q + \|\mathbf{w}_h\| + \|\mathbf{r}_h\|_Q).
\]

Setting \( (x, \mathbf{r}) = (\mathbf{v}_h, q_h) \) and using the definition of \( B \), we have
\[
B(\mathbf{v}_h, q_h, \mathbf{v}_h, q_h) \geq C(\|\mathbf{v}_h\|_0)^2.
\]

From (5.13) and (5.14), using the technique described in [15] coupled with a duality argument and the \( H^2 \)-regularity of the Stokes operator in regular domains and the trace inequality, we complete the proof. \( \Box \)

In order to estimate the error, we introduce the following notations
\[
eu = \mathbf{u} - \mathbf{u}_i = (\mathbf{u} - \mathbf{u}_i) - (\mathbf{u}_h - \mathbf{u}_i) = \zeta - \xi,
\]
\[
ep = p - p_i = (p - p_i) - (p_h - p_i) = \eta - \tau.
\]

Let \( \mathbf{u}^m = \mathbf{u}(t_m), p^m = p(t_m) \). From Eqs. (5.3)–(5.6), we have
\[
b(\partial_t \mathbf{u}^m - D_\tau \mathbf{u}^m, q) + A(\mathbf{v}^m, \mathbf{v}) + b(\mathbf{v}, \mathbf{v}^m) + \frac{K}{\eta_t}d(\mathbf{v}^m, q) + g(\mathbf{v}^m, q) = 0.
\]

Following a similar way in [10], we have the estimation of the initial data.

Lemma 5. Let \( (\mathbf{u}(0), p(0)) \) be the solution of (1.1)–(1.4) and let \( (\mathbf{u}_h^0, p_h^0) \) be the approximate solution of (2.32) and (2.33). Under the assumption (2.1), we have
\[
\|\mathbf{u}(0) - \mathbf{u}_h^0\|_0^2 + \|p(0) - p_h^0\|_0^2 \leq Ch^{2l}.
\]

Theorem 3. Assume that \( C_{11} \) is bounded below by a large enough positive constant. Let \( t_m = m\Delta t \) and suppose that \( (\mathbf{u}(t), p(t)) \in L^2(0, T; (H^{r+1}(\Omega) \cap H_0^1(\Omega))^2) \times L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), (\mathbf{u}_h^m, p_h^m) \in \mathbb{V}^h \times Q^h \) are the exact solution and discrete solution of 2.30.2.31, respectively. Then there exists a constant \( C \) independent of \( h \) and \( \Delta t \), such that
\[
\max_m \|\mathbf{u}_i^m - \mathbf{u}_h^m\|_0^2 + \Delta t \sum_{m=1}^{\frac{T}{\Delta t}} \|p^m - p_h^m\|_Q^2 \leq C \left( \Delta t^{2l} \|\mathbf{u}\|_{2l+1, 2l+1}^2 + h^{2l} \|\mathbf{u}\|_{2l+1, 2l+1}^2 + h^{2l-2} \|\mathbf{u}\|_{2l+1, 2l+1}^2 \right).
\]

Proof. Setting \( \mathbf{v} = \partial_t \mathbf{u}^m, q = \mathbf{v}^m \) in (5.15), we have
\[
A(\mathbf{v}^m, \partial_t \mathbf{u}^m) + \frac{K}{\eta_t}d(\mathbf{v}^m, \mathbf{v}^m) + g(\mathbf{v}^m, \mathbf{v}^m) = -b(\partial_t \mathbf{u}^m, \mathbf{v}^m) - b(D_\tau \mathbf{u}^m, \partial_t \mathbf{u}^m, \mathbf{v}^m).
\]
Using (5.10), Young’s inequality and approximation results of Lemma 4, the first term of the right hand side of (5.18) can be bounded
\[
| - b(\partial_t \psi^m, \tau^m)| = |(\nabla_h \cdot \partial_t \psi^m, \tau^m) - \sum_{i \in I} \int_{\tau^m} \langle \tau^m \rangle [\partial_i \psi^m \cdot n_i] ds | \leq \|\nabla_h \cdot \partial_t \psi^m\|_0 \|\tau^m\|_0 + \|\tau^m\|_0 |\partial_i \psi^m|, \\
\leq C(\|\nabla_h \cdot \partial_t \psi^m\|_0 + |\partial_i \psi^m|)\|\nabla_h \tau^m\|_Q \\
\leq C \left( \frac{h^{2i}}{\Delta t} \|\nabla \|_2(h^{-1}, n, \mu, \xi^2(j)) + \frac{h^{2j}}{\Delta t} \|\nabla \|_2(h^{-1}, n, \mu, \xi^2(j)) \right) + C |\tau^m|_Q. 
\]
(5.19)

By Taylor series expansion with the remainder in terms of integral form
\[
f(b) = f(a) + f'(a)(b - a) + \frac{1}{2} \int_a^b f''(x)(b - x) \, dx,
\]
we have
\[
[D_t u^m - \partial_t u^m] = \frac{1}{2\Delta t} \int_{t_{m-1}}^{t_m} u_d(\cdot, t)(t - t_{m-1}) \, dt.
\]
Since
\[
\left( \int_{t_{m-1}}^{t_m} u_d(\cdot, t)(t - t_{m-1}) \, dt \right)^2 \leq \int_{t_{m-1}}^{t_m} u_d(\cdot, t)^2 \, dt \int_{t_{m-1}}^{t_m} (t - t_{m-1}) \, dt = \frac{1}{3} (\Delta t)^3 \int_{t_{m-1}}^{t_m} u_d(\cdot, t)^2 \, dt,
\]
we can obtain
\[
\left| \int_{t_{m-1}}^{t_m} u_d(\cdot, t)(t - t_{m-1}) \, dt \right| \leq \frac{1}{3} (\Delta t)^{1/2} \left( \int_{t_{m-1}}^{t_m} u_d(\cdot, t)^2 \, dt \right)^{1/2}. 
\]
(5.20)

Similarly, using the above inequality (5.20), we have
\[
-b[D_t u^m - \partial_t u^m, \tau^m] \leq C \Delta t \|\partial_t u^m\|_2(h^{-1}, n, \mu, \xi^2(j)) + C |\tau^m|_Q. 
\]
(5.21)

The left side of (5.18) has the same form as that of (4.8), so (5.18) can be estimated in the same way as in Theorem 2. Similar to (4.20),(5.19) and (5.21) gives
\[
|\partial^n \|\tau^m\|_Q^2 - \partial^n \|\tau^m\|_Q^2| \leq C \left( \Delta t^2 \|\partial_t u^m\|_2(h^{-1}, n, \mu, \xi^2(j)) + h^{2n} \|\partial^m \|_2(h^{-1}, n, \mu, \xi^2(j)) + h^{2j} \|\partial^m \|_2(h^{-1}, n, \mu, \xi^2(j)) \right) 
\]
(5.22)

The constant \( \theta \) is defined in Theorem 2. The above inequality holds by choosing \( \varepsilon = \frac{\theta}{2} \) in (5.19) and (5.21). Therefore, (5.22) deduces
\[
\max_{m} |\partial^n \|\tau^m\|_Q^2 + \Delta t^m \sum_{m=1}^{N} |\tau^m\|_Q^2 | \leq C \left( \Delta t^2 \|\partial_t u^m\|_2(h^{-1}, n, \mu, \xi^2(j)) + h^{2n} \|\partial^m \|_2(h^{-1}, n, \mu, \xi^2(j)) + h^{2j} \|\partial^m \|_2(h^{-1}, n, \mu, \xi^2(j)) \right) + |\partial^n \|\tau^m\|_Q^2. 
\]
(5.23)

Finally, the desired results follows from the triangle inequality and the estimation for the initial data in the Lemma 5. \( \square \)

We shall prove a stability result which allows us to measure the error of the pressure in the \( L^2 \) norm.

**Lemma 6.** There exist positive constants \( C_1, C_2, C_3 \) independent of \( h \) and \( \Delta t \) and such that for all \( \langle v, q \rangle \in V^h \times Q^h \), there is a \( w \in V^h \) satisfying
\[
\left\langle B(v, \Delta t^2 q; \Delta t^2 w, 0) \right\rangle \geq C_1 \Delta t ||q||_0^2 - C_2 (||v||_V^2 + \Delta t ||q||_Q^2), 
\]
and
\[
|||w|||_V \leq C_3 ||q||_0. 
\]
(5.24)

(5.25)

**Proof.** Following the same way used in [10] and Lemma 2. Given \( q \in Q^h \), there is a velocity field \( u \in H^1_0(\Omega)^d \) satisfying
\[
- \int \nabla \cdot u \, dx \geq ||q||_0^2, \quad ||u||_L \leq ||q||_0 \leq ||q||_Q. 
\]
(5.26)
\[ |A(v, \Delta t^2 \Pi_h u)| \leq \Delta t^2 (|A(v, u)| + |A(v, u)|) \leq C \Delta t^2 \| u \|_1 (\| \epsilon_h(v) \|_0 + \| \nabla_h \cdot v \|_0 + \| v \|_0). \]

Using Young's inequality \( ab \leq \alpha a^2 + \frac{b^2}{\alpha} \), we have

\[ A(v, \Delta t^2 \Pi_h u) \geq (C_4 \epsilon_1 - C_4 \epsilon_2 - C_4 \epsilon_3) \Delta t |q|_0^2 - \frac{C_4}{4 \epsilon_1} |\epsilon_h(v)|_0^2 - \frac{C_4}{4 \epsilon_2} \| \nabla_h \cdot v \|_0^2 - \frac{C_4}{4 \epsilon_3} \| v \|_0^2. \quad (5.27) \]

Using (2.1), (4.1), (5.10) and (5.26), we have

\[ |b(\Delta t^2 \epsilon_h, \Delta t^2 q)| \leq C \Delta t \| u \|_1 \| q \|_0 \leq C_5 \epsilon_4 \Delta t \| q \|_0^2 + \frac{C_5 \Delta t \| q \|_0^2}{\epsilon_4}. \quad (5.28) \]

and from \( b(\Delta t^2 u, \Delta t^2 q) = -\Delta t \int_{\Omega} \nabla \cdot u \, dx \), we obtain

\[ b(\Delta t^2 \Pi_h u, \Delta t^2 q) = b(\Delta t^2 u, \Delta t^2 q) - b(\Delta t^2 \epsilon_h, \Delta t^2 q) \geq k \Delta t \| q \|_0^2 - C_5 \epsilon_4 \Delta t \| q \|_0^2 - \frac{C_5 \Delta t \| q \|_0^2}{\epsilon_4}. \quad (5.29) \]

From (5.27) and (5.29), we conclude that

\[ B(v, \Delta t^2 q; \Delta t^2 \Pi_h u, 0) = A(v, \Delta t^2 \Pi_h u) + b(\Delta t^2 \Pi_h u, \Delta t^2 q) \]

\[ \geq \left( k - C_4 \epsilon_1 - C_4 \epsilon_2 - C_4 \epsilon_3 - C_5 \epsilon_4 \right) \Delta t \| q \|_0^2 - \frac{C_4 |\epsilon_h(v)|_0^2}{4 \epsilon_1} - \frac{C_4 \| \nabla_h \cdot v \|_0^2}{4 \epsilon_2} - \frac{C_4 \| v \|_0^2}{4 \epsilon_3} - \frac{C_5 \Delta t \| q \|_0^2}{\epsilon_4}. \]

Choosing \( \epsilon_1 = \frac{k}{2C_4} \) \((1, 2, 3), \epsilon_4 = \frac{k}{4 \epsilon_4} \) such that

\[ B(v, \Delta t^2 q; \Delta t^2 \Pi_h u, 0) \geq C_1 \Delta t \| q \|_0^2 - C_2 (\| v \|_0^2 + \| \Delta t \| q \|_0^2) \]

Furthermore, we have

\[ \| \Pi_h u \|_v \leq \| \Pi_h u - u \|_v + \| u \|_v \leq C \| u \|_1 \leq C \| q \|_0. \]

Therefore \( w = \Pi_h u \) satisfy the assertion of this Lemma. This completes the proof. \( \square \)

**Theorem 4.** There exists a positive constant \( C \), independent of \( h \) and \( \Delta t \), such that

\[ \Delta t \sum_{m=0}^{N} \| p^m - p^m_h \|_0^2 \leq C (h^{2l} + \Delta t^2). \quad (5.30) \]

**Proof.** Let \( t = t^n \), using (5.13), (5.24), (5.25) and Young's inequality, we have

\[ C_1 \Delta t \| v^n \|_0^2 \leq B(s^n, \Delta t^2 x^n; \Delta t^2 w^n, 0) + C_2 (\| s^n \|_v^2 + \| \Delta t \| v^n \|_v^2) \]

\[ \leq C (\| s^n \|_v + \Delta t \| v^n \|_v + \Delta t \| v^n \|_v^2) \Delta t^2 \| v^n \|_0 + C_2 (\| s^n \|_v^2 + \| \Delta t \| v^n \|_v^2) \leq C (\| s^n \|_v^2 + \| \Delta t \| v^n \|_v^2 + \frac{C_1 \Delta t}{4} \| v^n \|_v^2). \]

Therefore, we have

\[ \Delta t \| v^n \|_0^2 \leq C (\| s^n \|_v^2 + \| \Delta t \| v^n \|_v^2 + \| v^n \|_v^2). \quad (5.31) \]

Summing from \( m = 1 \) to \( m = N \), then using Theorem 3 and the triangle inequality, we complete the proof. \( \square \)

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**References**
