§ 0. Introduction

In [8] (1920), Łukasiewicz introduced a 3-valued propositional calculus with one designated truth-value and later in [9], Łukasiewicz and Tarski generalized it to an m-valued propositional calculus (where m is a natural number or $\mathbb{N}$) with one designated truth-value. For the original 3-valued propositional calculus, an axiomatization was given by Wajsberg [16] (1931). In a case of $m \neq \mathbb{N}$, Rosser and Turquette gave an axiomatization of the m-valued propositional calculus with an arbitrary number of designated truth-values in [13] (1945). In [9], Łukasiewicz conjectured that the $\mathbb{N}$-valued propositional calculus is axiomatizable by a system with modus ponens and substitution as inference rules and the following five axioms:

- $p \supset q \supset p$,
- $(p \supset q) \supset (q \supset r) \supset p \supset r$,
- $p \lor q \supset q \lor p$,
- $(p \supset q) \lor (q \supset r) \supset q \lor r$,
- $(\neg p \supset \neg q) \supset q \supset p$.

Here we use $P \lor Q$ as the abbreviation of $(P \supset Q) \supset Q$. We associate to the right and use the convention that $\supset$ binds less strongly than $\lor$. In [15] p. 51, it is stated as follows: "This conjecture has proved to be correct; see Wajsberg [17] (1935) p. 240. As far as we know, however, Wajsberg’s proof has not appeared in print." Rose and Rosser gave the first proof of it in print in [12] (1958). Their proof was essentially due to McNaughton’s theorem [10], so it was metamathematical in nature. An algebraic proof was given by Chang [1] [2] (1959).

On the other hand, Rose [11] (1953) showed that the cardinality of the set of all super-Lukasiewicz propositional logics is $\mathbb{N}$. Surprisingly it was before Rose and Rosser’s completeness theorem [12]. The proof in Rose [11] was also due to McNaughton’s theorem. Some of our theorems in this paper have already been obtained by Rose [11]. But our proofs are completely algebraic.

In our former paper [5], we gave a complete description of super-
Łukasiewicz implicational logics (SLIL). In this paper, we will give a complete description of super-Łukasiewicz propositional logics (SLL). We need the completeness of a theory on some ordered abelian groups in [6] to give the complete description of SLL. In the first three sections, we will develop a theory without need of the result in [6]. So some of the results in §1–§3 are included in more generalized forms in the later sections.

In §1, we will give a complete description of these SLLs which are obtained by adding only $C$ formulas to the smallest SLL $L_u$. In §2, we will discuss the inclusion relations between SLLs. And we will have the theorem stated in [15] p. 48 without proof. In §3, we will give a characterization of SLLs without finite model property. §4 is the main section of this paper. A complete description of SLLs will be given in it. In §5, we will give some applications of the complete description of SLLs. In §6, we will discuss the lattice structure of all SLLs and illustrate a finite sub-structure of it.

We suppose familiarity with [4] and [5]. Only in §4, we suppose familiarity with [6]. A CN formula (or simply, formula) is an expression constructed from propositional variables and logical connectives $\supset$ and $\sim$ in the usual way. By a super-Łukasiewicz propositional logic (SLL), we mean a set of formulas which is closed with respect to substitution and modus ponens, and contains the following five formulas:

A1. $p \supset q \supset p$
A2. $(p \supset q) \supset (q \supset r) \supset p \supset r$
A3. $p \lor q \supset q \lor p$
A4. $(p \supset q) \lor (q \supset p)$
A5. $(\sim p \supset \sim q) \supset q \supset p$.

A C algebra is an algebra $\langle A; 1, \rightarrow \rangle$ which satisfies the following axioms, where $A$ is a non empty set and 1 and $\rightarrow$ are 0-ary and 2-ary functions on $A$ respectively.

B1. $1 \rightarrow x = x$
B2. $x \rightarrow y \rightarrow x = 1$
B3. $(x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z = 1$
B4. $x \cup y = y \cup x$
B5. $(x \rightarrow y) \cup (y \rightarrow x) = 1$.
We abbreviate \((x \rightarrow y) \rightarrow y\) by \(x \cup y\). We use the same convention as before. A CN algebra is an algebra \(\langle A; 1, \rightarrow, \neg \rangle\) which satisfies the following axiom, where \(\langle A; 1, \rightarrow \rangle\) is a C algebra and \(\neg\) is an 1-ary function on \(A\).

C1. \(\neg x \rightarrow \neg y \leq y \rightarrow x\).

Here we denote \(x \rightarrow y = 1\) by \(x \leq y\). We say simply that \(A\) is a CN algebra, when \(\langle A; 1, \rightarrow, \neg \rangle\) is a CN algebra. If a formula contains no connective other than \(\supset\), it is called a C formula. In [5], we denote the set of C formulas valid in a C algebra \(A\) by \(L(A)\). In this paper, we denote the set of formulas valid in a CN algebra \(A\) by \(L(A)\). The set of C formulas valid in a CN algebra \(A\) is denoted by \(L_1(A)\). \(Lu\) denotes the set of formulas derivable from A1–A5, that is, \(Lu\) is the smallest SLL. For any SLL \(L, L_i\) denotes the set of formulas contained in \(L\). Let \(H\) be any set of formulas and \(L\) be any SLL. Then we denote the smallest SLL which includes \(L \cup H\) by \(L + H\). Sometimes, \(L + \{P_1, \ldots, P_n\}\) is denoted by \(L + P_1 + \cdots + P_n\). A SLL \(L\) is called to be finitely axiomatizable if there exists a finite set \(H\) such that \(L = Lu + H\).

We denote the set \([0, 1/m, 2/m, \ldots, (m - 1)/m, 1]\) and the set of all rationals in the interval \([0, 1]\) by \(S_m\) \((m \geq 1)\) and \(S_\omega\), respectively. We define the functions \(\rightarrow\) and \(\neg\) on \(S_m\) \((1 \leq m \leq \omega)\) by \(x \rightarrow y = \min (1, 1 - x + y)\) and \(\neg x = 1 - x\), respectively. Then we can regard \(S_m\) as a CN algebra. \(S_m\) is the well-known Łukasiewicz \((m + 1)\)-valued (or \(\mathbb{R}_\omega\)-valued if \(m = \omega\)) model. We denote also the CN algebra with only one element by \(S_0\).

§1. SLLs obtained by adding only C formulas

Let \(A\) be a CN algebra. A non-empty subset \(J\) of \(A\) is a filter of \(A\) if it satisfies the following two conditions:

1) \(1 \in J\),
2) \(x \in J\) and \(x \rightarrow y \in J \Rightarrow y \in J\).

Let \(A\) be a CN algebra, \(x\) be an element of \(A\) other than 1. \(A\) is irreducible with respect to \(x\) if \(x\) is contained within any filter of \(A\) which contains at least an element other than 1. \(A\) is irreducible, if there exists an element such that \(A\) is irreducible with respect to the element or \(A\) has only one element. By Theorem 2.10 in [4], we have

**Theorem 1.1.** Any irreducible CN algebra is linearly ordered.
We can, similarly to Theorems 3.8 and 3.9 in [5], show the following theorems.

**Theorem 1.2.** If a CN algebra \( B \) is a subalgebra of a CN algebra \( A \), or \( B = A \upharpoonright \mathcal{J} \) for some filter \( \mathcal{J} \) of \( A \), then \( L(B) \supseteq L(A) \).

**Theorem 1.3.** For any SLL \( L \), there exists a set \( \{ A_i \}_{i \in \Delta} \) of irreducible CN algebras such that \( L = \bigcap_{i \in \Delta} L(A_i) \).

Next theorem gives a complete description of SLLs obtained by adding only \( C \) formulas.

**Theorem 1.4.** Let \( \{ A_t \mid i \in I \} \) be a set of \( C \) formulas. If \( L = L_u = \{ A_i \mid i \in I \} \), then \( L = \bigcap_{k \leq n} L(S_k) \) for some \( n \leq \omega \).

**Proof.** By Theorem 4.1 in [5], if \( A_t \in L_u \), then \( A_t \) is interdeducible in \( L_u \) with \( (p \supset) \uparrow q \lor p \) for some \( m \). Here we define \( (P \supset) \uparrow^m (Q) = Q \) and \( (P \supset) \uparrow^{m+1} (Q) = P \supset (P \supset) \uparrow^m (Q) \), and we denote \( (P \supset) \uparrow^m (Q) \) by \( (P \supset)^m Q \) when no confusion occurs. Because \( L_u + (p \supset) \uparrow^m q \lor p \supset (p \supset) \uparrow^m q \lor p \) for \( l \geq m \), there exists \( n \) such that \( L = L_u + (p \supset)^n q \lor p \). As \( (p \supset)^n q \lor p \) is valid in \( S_k \) for any \( k \leq n \), \( L \subseteq \bigcap_{k \leq n} L(S_k) \). We can easily shown that if \( (p \supset)^n q \lor p \in L(A) \), then \( \text{ord} (A) \leq n \). Here we give same definition of order of a CN algebra as a \( C \) algebra, that is, \( \text{ord} (A) = \sup \{ \text{ord} (x) \mid x \in A \} \) and \( \text{ord} (x) \) is the least integer \( n \) such that \( x \cup (x \rightarrow)^n y = 1 \) for any element \( y \) of \( A \) (\( \text{ord} (x) = \omega \), if no such integer \( n \) exists). Therefore, we have that if \( (p \supset)^n q \lor p \in L(A) \) and \( A \) is irreducible, then \( A \) is isomorphic to \( S_k \) for some \( k \leq n \). Then, we have \( L = \bigcap_{k \leq n} L(S_k) \). Clearly, if \( A_t \in L_u \) for any \( i \in I \), then \( L = L_u = \bigcap_{k \leq n} L(S_k) = \bigcap_{k \leq n} L(S_k) \). Q.E.D.

If \( L_t \not\subseteq L_u \), that is, \( L_t \cong L_u \), there exists a non-negative integer \( n \) such that \( (p \supset)^n q \lor p \in L \). Let \( I \) be the set of non-negative integers \( \{ i \mid L \subseteq L(S_i) \) and \( i \leq n \} \). Then, we can show that \( L = \bigcap_{i \in I} L(S_i) \). Let \( J \) be the set of non-negative integers \( \{ i \mid L \not\subseteq L(S_i) \) and \( i \leq n \} \). For each \( i \in J \), there exists a formula \( P_i \) such that \( P_i \in L \) and \( P_i \notin L(S_i) \). Let \( H \) be the set of formulas \( \{ P_i \mid i \in J \} \). Then, without being depend on the representative \( P_i \) chosen, we have that \( L = L_u + (p \supset)^n q \lor p + H \). Therefore, we have the following theorems.

**Theorem 1.5.** If \( L_t \cong L_u \), then there exists a finite set \( I \) of non-negative integers such that \( L = \bigcap_{i \in I} L(S_i) \).

**Theorem 1.6.** If \( L_t \cong L_u \), then is finitely axiomatizable.
COROLLARY 1.7. The cardinality of the set \( \{ L | L \) is a SLL such that \( L \nmid Lu \} \) is countable.

§ 2. Inclusion relations between SLLs

Though \( L_i(S_n) \subseteq L_i(S_m) \) for \( n \geq m \) in SLILs, we can easily know that \( L(S_n) \nsubseteq L(S_m) \). In [9], it is stated that Lindenbaum proved that \( L(S_n) \subseteq L(S_m) \) if and only if \( m \) is a divisor of \( n \). We will generalize Lindenbaum's theorem. We define the CN algebras \( S^*_n \) \((n = 1, 2, 3, \ldots)\) as follows.

\[
S^*_n = \{(x, y) | x \in \{1/n, 2/n, \ldots, (n-1)/n\}, y \in \mathbb{Z}\} 
\cup \{(0, y) | y \in \mathbb{N}\} \cup \{(1, -y) | y \in \mathbb{N}\},
\]

where \( \mathbb{Z} \) and \( \mathbb{N} \) are the set of all integers and the set of all non-negative integers, respectively.

\[
(x, y) \rightarrow (z, u) = \begin{cases} 
(1, 0) & \text{if } z > x, \\
(1, \min(0, u - y)) & \text{if } z = x, \\
(1 - x + z, u - y) & \text{otherwise}.
\end{cases}
\]

\[
\neg (x, y) = (1 - x, -y).
\]

When \( n = 1 \), the first term in \( S^*_1 \) is regarded as an empty set. \( S^*_n \) is essentially equivalent to the MV-algebra \( C \) defined in Chang [1]. We can check easily that \( \langle S^*_n; (1, 0), \rightarrow, \neg \rangle \) is a CN algebra.

**THEOREM 2.1.** Let \( I \) and \( J \) be finite sets of positive integers.

\[
\bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S^*_j) \subseteq L(S_m)
\]

if and only if there exists \( n \in I \cup J \) such that \( m \) is a divisor of \( n \).

**Proof.** If there exists \( n \in I \cup J \) such that \( m \) is a divisor of \( n \), \( S_m \) is isomorphic to a subalgebra of \( S_n \) (or \( S^*_n \)). Therefore, we have \( \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S^*_j) \subseteq L(S_m) \). Conversely, suppose that \( \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S^*_j) \subseteq L(S_m) \). Let \( r \) be \( \max I \cup J \) and \( P \) be the formula

\[
[(p \supset p) \supset (p \supset (p \supset p))^{m-1} \supset (p \supset p)]^{r+1} p.
\]

If \( f \) assigns the element \((m-1)/m\) of \( S_m \) for \( p \), then \( f(P) \) is also \((m-1)/m\). Hence, we have \( P \in L(S_m) \). Therefore, we have \( P \in \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S^*_j) \). Hence, there exists \( i \in I \) such that \( P \in L(S_i) \) or there exists \( j \in J \) such that \( P \in L(S^*_j) \). Suppose that \( P \in L(S_i) \). Let \( g \) be an assignment of \( S^*_i \) such that \( g(P) \neq (1, 0) \). We can show that for any \( x, y \in S^*_i \) and any \( i > j \), if
(x→)′y ≠ (1, 0) then x is of the form (1, *). Here by c = (b, *) we mean that the first component of c is b. Hence, (a→)^{m−2} a → a = (1, *) and (a→)^{m−1} a = (1, *), where a denotes g(p). Let a = (1 − k/j, *). Then we have (m − 1)k/j ≤ (j − k)j and mkj ≥ 1. Hence, we have that j = mk.

When P ∈ L(S△), the proof is similar. Q.E.D.

COROLLARY 2.2 (Lindenbaum). L(Sn) ⊆ L(Sm) if and only if m is a divisor of n (1 ≤ m < ω, 1 ≤ n < ω).

THEOREM 2.3. Let I and J be finite sets of positive integers.

\[ \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j) \subseteq L(S_m^*) \]

if and only if there exists n ∈ J such that m is a divisor of n.

Proof. If there exists n ∈ J such that m is a divisor of n, \( \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j) \subseteq L(S_m^*) \) because \( S_m^* \) is isomorphic to a subalgebra of \( S_n^* \). Conversely, suppose that \( \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j) \subseteq L(S_m^*) \). Let r be max I ∪ J and P be the formula

\[ \left[ \left( [P]^{m−2} \sim (p \supset p)]^{r+1} [(p \supset)^{m−1} \sim p \supset]^{r+1} [(q \supset)^* s \vee q] \right. \]

Let f be an assignment of \( S_m^* \) such that f(p) = ((m − 1)/m, 0), f(q) = (1, −1) and f(s) = (0, 0). Then f(P) = (1, −1). Hence, we have P ∈ L(S_m^*). Therefore, we have P ∈ \( \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j) \). Because P ∈ \( \bigcap_{i \in I} L(S_i) \), there exists j ∈ J such that P ∈ L(S_j). Similarly to the proof of Theorem 2.1, we have this theorem. Q.E.D.

COROLLARY 2.4. L(S_m^*) ⊆ L(S_n^*) if and only if m is a divisor of n (1 ≤ m < ω, 1 ≤ n < ω).

§ 3. SLLs without fmp

By the result of [5], we know that any SLIL has the finite model property (fmp). We will show that there exist SLLs without fmp.

DEFINITION 3.1. A SLL L has fmp if there exists a set of finite CN algebras \{A_i | i ∈ I\} such that \( L = \bigcap_{i \in I} L(A_i) \).

A finite irreducible CN algebra is isomorphic to \( S_n \) for some n. Therefore, by Theorem 1.3, we have

THEOREM 3.2. A SLL L has fmp if and only if there exists a set I of non-negative integers such that L = \( \bigcap_{i \in I} L(S_i) \).
**Theorem 3.3.** If $L \not\equiv \mathbf{Lu}$, then $L_t \not\equiv \mathbf{Lu}_t$ if and only if $L$ has fmp.

**Proof.** By Theorem 1.5, $L$ has fmp if $L_t \not\equiv \mathbf{Lu}_t$. Conversely, $L$ has fmp. Then there exists a set $I$ of non-negative integers such that $L = \bigcap_{k \in I} L(S_k)$. Because $L \not\equiv \mathbf{Lu}$, $I$ is a finite set. So $(p \supset q \lor p \in L$ where $n = \max I$. Hence $L_t \not\equiv \mathbf{Lu}_t$. Q.E.D.

For any positive integers $m, n$, $S_m^n$ has a subalgebra isomorphic to $S_m$ if we regard $S_m$ and $S_m^n$ as C algebras. Then we have

**Lemma 3.4.** $L_t(S_m^n) = L_u_t$ for any positive integer $k$.

**Theorem 3.5.** If both $I$ and $J$ are finite sets of positive integers, $J \not\approx \phi$ and $L = \bigcap_{i \in I} L(S_i) \cap \bigcup_{j \in J} L(S_j)$, then $L$ has not fmp.

**Proof.** $L \not\equiv \mathbf{Lu}$ because $I \cup J$ is a finite set. By $J \not\approx \phi$ and Lemma 3.4, $L_t = \mathbf{Lu}_t$. Therefore, $L$ has not fmp by Theorem 3.3. Q.E.D.

**Corollary 3.6.** $L(S_m^n)$ has not fmp for any positive integer $n$.

§ 4. A complete description of SLLs

This section is the main part of this paper.

**Definition 4.1.** Let $A$ be a linearly ordered CN algebra, and $a$ be the maximum element of $A$. An element $x$ of $A$ is called almost maximum if $(x \rightarrow)^n a \not\approx a$ for any positive integer $n$. An element of $x$ is called infinitesimal if $\neg x$ is almost maximum. If $A$ has an element other than the maximum element, the set $M_A$ of all almost maximum elements of $A$ is a filter of $A$. The CN algebra $A/M_A$ is denoted by $\bar{A}$. rank $(A)$ is defined by rank $(A) = \text{ord} (\bar{A})$.

Clearly, only one almost maximum element of $\bar{A}$ is the maximum element, that is, $\bar{A}$ is locally finite (This is Chang's terminology [1].)

**Theorem 4.2.** Let $A$ be a linearly ordered CN algebra. If rank $(A) = \omega$, then $L(A) = \mathbf{Lu}$.

**Proof.** By Theorem 1.2, $L(A) \subseteq L(\bar{A})$. Because $\bar{A}$ is locally finite, $\bar{A}$ is isomorphic to a subalgebra of the CN algebra of all real numbers between 0 and 1 (cf. [2] p. 78). By ord $(\bar{A}) = \omega$, $A$ has an infinite number of members. Therefore, $L(\bar{A}) = \mathbf{Lu}$ (cf. [12] p. 5). Hence, we have $L(A) = \mathbf{Lu}$. Q.E.D.

For a given model $G$ of $\mathbf{SS}$ (cf. [6]), let the segment $G[c]$ determined
by a positive element \( c \) of \( G \) be the set of all elements \( x \in G \) such that \( 0 \leq x \leq c \). We define the functions \( \rightarrow \) and \( \leftarrow \) on \( G[c] \) as follows:

\[
x \rightarrow y = \min(c, c - x + y), \\
\leftarrow x = c - x.
\]

Then we can easily prove the following lemma.

**Lemma 4.3.** The algebra \( \langle G[c]; c, \rightarrow, \leftarrow \rangle \) defined above is a linearly ordered \( CN \) algebra. If \( m \) satisfies \(-1 < 2(m - c) < 1\), then \( \text{rank } (G[c]) = m \).

We now wish to establish the converse to Lemma 4.3. Let \( A \) be a linearly ordered \( CN \) algebra and \( 0 \) be the minimum element of \( A \). We let \( A^* \) be the set \( \{(s, x) | s \in \{+, -\}, x \text{ is an infinitesimal element of } A\} \). We identify \((+, 0)\) with \((- , 0)\) and denote \((\pm, x)\) by \( \pm x \), respectively. On the set \( A^* \) we define the functions \(+ \) and \(- \) and the relation \( 0< \) as follows:

\[
(+, x) + (+, y) = (+, -x \rightarrow y), \\
(−, x) + (−, y) = (−, −x \rightarrow y), \\
(+, x) + (−, y) = (−, y) + (+, x) = \\
\begin{cases} 
(+, −(x \rightarrow y)) & \text{if } y \leq x, \\
(−, −(y \rightarrow x)) & \text{if } x < y,
\end{cases}
\]

\[-(+, x) = (−, x), \\
−(−, x) = (+, x), \\
0 < (s, x) \iff s = + \text{ and } x \neq 0.
\]

Then the algebra \( \langle A^*; +, −, 0< \rangle \) is a totally ordered abelian group. Generally, the group \( ZG = Z \times G \) is a model of \( SS \) is \( G \) is a totally ordered abelian group, where \( Z \times G \) is ordered as \( 0 < (x, y) \) if and only if either \( 0 < x \) or \( x = 0 \) and \( 0 < y \). Hence \( ZA^* \) is a model of \( SS \).

**Lemma 4.4.** Let \( A \) be a linearly ordered \( CN \) algebra, \( \text{ord } (A) = \omega \) and \( \text{rank } (A) = n \). Then there exists an infinitesimal element \( b \) of \( A \) such that \( b \nmid 0 \) and \( A \cong ZA^*[\{n, +b\}] \).

**Proof.** By \( \text{rank } (A) = n, \) \( \bar{A} \cong S_n \). Let \( \varphi \) be an isomorphism from \( \bar{A} \) to \( S_n \) and \( \alpha \) be an element of \( \bar{A} \) (and hence an equivalence class of \( A \)) such that \( \varphi(\alpha) = (n - 1)/n \). Since \( \text{ord } (A) = \omega \), we can take a sufficiently large element \( x \) of \( \alpha \) such that \( (x \rightarrow)^* 0 < a \) (\( a \) is the maximum element of \( A \)). We can show that for any \( y \nmid a \) there is an unique infinitesimal
element \( z \) of \( A \) such that \( y = (x \rightarrow)^m z \) or \( y = (x \rightarrow)^{m-1} \neg (\neg x \rightarrow z) \) if \( \varphi([y]) = m/n. \) Let \( b \) denote \( \neg (x \rightarrow) 0. \) Let \( f \) be a function from \( A \) to \( ZA^\ast[(n, + b)] \) such that \( f((x \rightarrow)^m z) = (m, +z), f((x \rightarrow)^{m-1} \neg (\neg x \rightarrow z)) = (m, -z) \) and \( f(a) = (n, + b). \) Then \( f \) is an isomorphism from \( A \) onto \( ZA^\ast[(n, + b)]. \)

Q.E.D.

The first order language \( \mathcal{L}' \) is the same as in [6], which consists of \( 0, 1, -, +, 0<, n\) \{for each integer \( n > 0\) and \( =. \) Let \( \mathcal{L}'' \) be the language obtained from \( \mathcal{L}' \), by adding a binary function symbol \( \text{min}. \) The language of the theory \( SS' \) is \( \mathcal{L}'' \) and the set of axioms of \( SS' \) is obtained from \( SS \) by adding the following axiom:

\[(j)\quad z = \text{min}(x, y) \iff (x < y \rightarrow z = x) \land (y < x \rightarrow z = y).\]

It is clear that each model of \( SS \) can be regarded also as a model of \( SS'. \)

In \( SS' \), for any formula \( A(x) \), the following is derivable:

\[A(\text{min}(s, t)) \iff (s < t \rightarrow A(s)) \land (t < s \rightarrow A(t)).\]

Therefore, for any formula \( F \) of \( \mathcal{L}'' \) we can construct the formula \( F^\ast \) of \( \mathcal{L}' \) such that \( F \leftrightarrow F^\ast \) is derivable in \( SS' \) and each variable of which some occurrence is bound in \( F^\ast \) is also bound in \( F. \) Especially, \( F^\ast \) is open if \( F \) is open. Hence, by Corollary 2.3 in [6], we have

**LEMMA 4.5.** For any open formula \( F \) of \( \mathcal{L}' \) and any model \( A \) of \( SS' \) \( \cup (i), F \) is valid in \( ZQ \) if and only if \( F \) is valid in \( A. \)

We now define the term \( P^\ast \) of \( \mathcal{L}'' \) corresponding to a formula \( P \) of SLL in the following manner:

\[p^\ast = h(p),\]

\[(P \supset Q)^\ast = \text{min}(c - P^\ast + Q^\ast, c),\]

\[(\neg P)^\ast = c - P^\ast.\]

Here \( h \) is an injective mapping from the set of propositional variables of SLL to the set of variables of \( \mathcal{L}'' \) such that \( h(p) \equiv c \) for any \( p. \) We assume that \( x_1, x_2, \ldots, x_n \) are the only variables occurring in \( P^\ast. \) Next, we define the formula \( P^0 \) as \( P^0 = (0 \leq x_1 \leq c \land \cdots \land 0 \leq x_n \leq c \rightarrow P^\ast = c). \)

**LEMMA 4.6.** For any formula \( P \) of SLL and any linearly ordered CN algebra \( A \) such that \( \text{ord}(A) = \omega \) and \( \text{rank}(A) = n, P \) is valid in \( A \) if \(-1 < 2(n - c) < 1 \rightarrow P^0 \) is valid in \( ZQ. \)

**Proof.** Suppose that \( P \) is not valid in \( A. \) There exists an assignment
f of A such that \( f(P) < a \) where \( a \) is the maximum element of \( A \). By Lemma 4.4, there exists an isomorphism \( \varphi \) from \( A \) to \( \mathbb{Z}A^*[(n, +b)] \). Let \( g \) be an assignment of \( \mathbb{Z}A^* \) such that \( g(x) = \varphi(f(h^{-1}(x))) \) and \( g(c) = (n, +b) \). Then \( -1 < 2(n - c) < 1 \), \( P^* \) is not true under \( g \). Since \( \mathbb{Z}A^* \) is a model of \( SS' \cup (i) \), \( -1 < 2(n - c) < 1 \), \( P^* \) is not valid in \( \mathbb{Z}Q \) by Lemma 4.5.

Q.E.D.

**Lemma 4.7.** For any linearly ordered CN algebra \( A \) such that \( \text{ord}(A) = \omega \) and \( \text{rank}(A) = n \), \( L(A) \subseteq L(\mathbb{Z}[n, (1)]) \).

*Proof.* By Lemma 4.4, \( A \cong \mathbb{Z}A^*[(n, +b)] \). A subalgebra of \( \mathbb{Z}A^*[(n, +b)] \) generated by \( (1, 0) \) is isomorphic to \( \mathbb{Z}[n, (1)] \). Q.E.D.

**Lemma 4.8.** For any integer \( k \),

\[
L(\mathbb{Z}[n, (0)]) \subseteq L(\mathbb{Z}[n, (k)]) \subseteq L(\mathbb{Z}[n, (1)])
\]

*Proof.* By Lemma 4.7, \( L(\mathbb{Z}[n, (k)]) \subseteq L(\mathbb{Z}[n, (1)]) \). Suppose that \( P \) is not valid in \( \mathbb{Z}[n, (k)] \). Let \( f \) be an assignment of \( \mathbb{Z}[n, (k)] \) such that \( f(P) = (u, v) \neq (n, k) \). Let \( g \) be an assignment of \( \mathbb{Z}[n, nk] \) such that \( g(p) = (m, nl) \) if \( f(p) = (m, l) \) for any propositional variable \( p \). Then \( g(P) = (u, nv) \neq (n, nk) \). \( \mathbb{Z}[n, nk] \) is isomorphic to \( \mathbb{Z}[n, (0)] \) (isomorphism \( \varphi \) is given by \( \varphi((m, l)) = (m, l - mk) \)). Hence, \( P \) is not valid in \( \mathbb{Z}[n, (0)] \). Q.E.D.

**Lemma 4.9.** For any integer \( k \),

\[
L(\mathbb{Z}[n, (0)]) = L(\mathbb{Z}[n, (k)]) = L(\mathbb{Z}[n, (1)])
\]

*Proof.* By Lemma 4.8, it suffices to show that \( L(\mathbb{Z}[n, (0)]) \subseteq L(\mathbb{Z}[n, (1)]) \). Let \( P \) be a formula which is not valid in \( \mathbb{Z}[n, (0)] \) and \( f \) be an assignment of \( \mathbb{Z}[n, (0)] \) such that \( f(P) \leq (n, -1) \). Let \( g_m : \mathbb{Z}[n, (0)] \rightarrow \mathbb{Z}[n, (0)] \) be a homomorphism such that \( (i, j) \mapsto (i, mj) \). Let \( f' \) be an assignment of \( \mathbb{Z}[n, (1)] \) such that \( f'(p) = g_m f(p) \) for any propositional variable \( p \). For any formula \( F \) with the degree \( d \) (that is, the number of occurrences of logical connectives in the formula \( F \) is \( d \)), we shall show by induction on \( d \) that

\[
g_m(f) - (0, d) \leq f'(F) \leq g_m(f) + (0, d)
\]

Suppose \( F \) is \( G \supset H \) and the degrees of \( G \) and \( H \) are \( e \) and \( e' \), respectively. By the inductive hypothesis,
Since
\[ g_m f(G) - (0, e) \leq f'(G) \leq g_m f(G) + (0, e), \]
\[ g_m f(H) - (0, e') \leq f'(H) \leq g_m f(H) + (0, e') . \]

Since
\[ f'(G \supset H) = \min ((n, 1) - f'(G) + f'(H), (n, 1)) , \]
\[ g_m f(G \supset H) = \min ((n, 0) - g_m f(G) + g_m f(H), (n, 0)) \]
and \( d = e + e' + 1 \), we have
\[ g_m f(G \supset H) - (0, d) \leq f'(G \supset H) \leq g_m f(G \supset H) + (0, d) . \]

The case that \( F \) is \( \sim G \) is similar. Therefore, we have that \( f'(P) \leq (n, d - m) \). If \( m \geq d \), \( P \) is not true in \([n, 1)]\) under the assignment \( f' \).

Q.E.D.

We are now in a position to prove the following key theorem.

**Theorem 4.10.** For any linearly ordered CN algebra \( A \) such that \( \text{ord}(A) = \omega \) and \( \text{rank}(A) = n \), \( L(A) = L(ZZ[(n, 0)]) \).

**Proof.** By Lemma 4.7 and Lemma 4.9, we have \( L(A) \subseteq L(ZZ[(n, 0)]) \). We shall show that \( L(A) \supseteq L(ZZ[(n, 0)]) \). Let \( P \) be a formula valid in \( ZZ[(n, 0)] \). By Lemma 4.9, \( P \) is valid in \( ZZ[(n, k)] \) for any integer \( k \). Hence \( -1 < 2(n - c) < 1 \rightarrow P^0 \) is valid in \( ZZ \). By Lemma 4.5, \( -1 < 2(n - c) < 1 \rightarrow P^0 \) is valid in \( ZQ \). By Lemma 4.6, \( P \) is valid in \( A \). Q.E.D.

\( ZZ[(n, 0)] \) is isomorphic to \( S_n^* \) defined in \( \S \ 2 \). Now, we can prove the main theorem.

**Theorem 4.11.** For any SLL, there exist sets of non-negative integers \( I, J \) such that \( L = \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j) \). If \( L \models Lu \), then both sets \( I \) and \( J \) are finite.

**Proof.** By Theorem 1.3, there exists a set \( \{A_i\}_{i \in \Delta} \) of irreducible CN algebras such that \( L = \bigcap_{i \in \Delta} L(A_i) \). By Theorem 3.13 in [5], \( L(A_i) = L(S_n) \) if \( \text{ord}(A_i) = n \). By Theorem 4.10, \( L(A_i) = L(S_n^* \) if \( \text{ord}(A_i) = \omega \) and \( \text{rank}(A_i) = n \). By Theorem 4.2, \( L(A_i) = Lu = \bigcap_{k < \omega} L(S_k) \) if \( \text{rank}(A_i) = \omega \). Therefore, \( L = \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^*) \) for some \( I \) and \( J \). If \( I \cup J \) is infinite, then \( L \subseteq \bigcap_{i \in I \cup J} L(S_i) \) because \( L(S_n^*) \subseteq L(S_n) \). By Theorem 20 in [15] p. 49, \( \bigcap_{i \in I \cup J} L(S_i) = Lu \). Hence, we have \( L = Lu \). Q.E.D.
§ 5. Applications of the main theorem

By Theorem 4.11, Theorem 3.5 gives a complete characterization of SLLs without fmp. For example, we can show as follows that $L_u + P$ has not fmp, where $P$ is the formula $(p \supset \sim p) \supset (\sim p \supset p) \supset p \lor \sim p$. Because $P \in L(S_3) \cap L(S_7)$ and $P \in L(S_n)$ for $n = 2$ or $n \geq 4$ and $P \in L(S_{\infty})$ for $n \geq 2$, we have $L_u + P = L(S_3) \cap L(S_7)$ by Theorem 4.11. Hence, $L_u + P$ has not fmp by Theorem 3.5.

The following theorem, that was proved in Rose [10], is easily obtained from Theorem 4.11.

**Theorem 5.1.** *The cardinality of the set of all SLLs is countable.*

Rose [11] also showed that any SLL is finitely axiomatizable. We will show it as follows.

**Lemma 5.2.** $L_u + A_n = \bigcap_{k \leq n} L(S_k^r)$, where

\[ A_n = [(p \supset)^n \sim p] \supset [(p \supset)^{n-1} \sim p \supset p] \supset (p \supset)^{n-1} \sim p \lor p. \]

**Proof.** By Theorem 4.11 and $L(S_k^r) \subseteq L(S_k)$ for any $k$, it suffices to show that (1) $A_n \in L(S_k^r)$ for $k \leq n$ and that (2) $A_n \in L(S_k)$ for $k > n$.

**Proof of (1).** Let $f$ be an assignment of $S_7^r$. If $f(p) \leq ((k - 1)/k, 0)$ or $f(p) = (1, 0)$, then $f((p \supset)^n \sim p \lor p) = (1, 0)$. Therefore, $f(A_n) = (1, 0)$. If $f(p) = ((k - 1)/k, *)$, then $f((p \supset)^{n-1} \sim p \supset p) \leq f((p \supset)^{n-1} \sim p)$. Therefore, $f(A_n) = (1, 0)$. If $f(p) = (1, *)$, then $f((p \supset)^n \sim p) \leq f(p)$. Hence, $f(A_n) = (1, 0)$.

**Proof of (2).** Let $f$ be an assignment of $S_k$ such that $f(p) = 1 - [k/n + 1]/k$, where $[x]$ is the integral part of $x$. Then $f((p \supset)^n \sim p) = 1$, $f((p \supset)^{n-1} \sim p \supset p) = 1$ and $f((p \supset)^{n-1} \sim p \lor p) \equiv 1$. Therefore, $f(A_n) \equiv 1$.

Q.E.D.

**Theorem 5.3.** *Any SLL is finitely axiomatizable.*

**Proof.** Let $L$ be a SLL. If $L = L_u$, then $L$ is finitely axiomatizable. Suppose that $L \not\equiv L_u$. Then there exists a positive integer $n$ such that $\bigcap_{i \leq n} L(S_i^r) \subseteq L$. Hence $A_n \in L$. Because $A_n \in L(S_k^r)$ and $A_n \in L(S_k^r)$ for any $k > n$, there exist two sets of positive integers $I'$ and $J'$ such that $L = \bigcap_{i \in I'} L(S_i) \cap \bigcap_{j \in J'} L(S_j^r)$ and $I', J' \subseteq \{i | i \leq n\}$. Let $I$ and $J$ be the sets of positive integers $\{i | L \not\subseteq L(S_i) \text{ and } i \leq n\}$ and $\{j | L \not\subseteq L(S_j^r) \text{ and } j \leq n\}$, respectively. For each $i \in I$ ($j \in J$), there exists a formula $P_i(Q_j)$ such that
$P_i \in L$ ($Q_j \in L$) and $P_i \in L(S_i)$ ($Q_j \in L(S'_j)$). Let $G$ and $H$ be the set of formulas $\{P_i | i \in I\}$ and $\{Q_j | j \in J\}$, respectively. Then, we have that $L = Lu + G + H + A_u$. Q.E.D.

We denote the set of all formulas by $W$. By Theorem 4.11, $W - L$ is recursive enumerable for any SLL $L$. By Theorem 5.3, $L$ is recursive enumerable for any SLL $L$. Hence we have

**Theorem 5.4.** Any SLL is decidable.

Krzystek and Zachorowski [7] proved that $L(S_n)$ ($2 \leq n \leq \omega$) has not Interpolation Property. Quite similarly, we can prove the following theorem.

**Theorem 5.5.** Any SLL except $W$ and $L(S_1)$ has not Interpolation Property.

**Proof.** Let $L$ be a SLL except $W$ and $L(S_1)$. Let $P$ and $Q$ be the formulas $(r \supset r \supset p) \supset r \supset p$ and $(s \supset s \supset p) \supset s \supset p$, respectively. The formula $P \supset Q$ is valid in $S_\omega$. Hence we have $P \supset Q \in Lu$. Let $A$ be a CN algebra such that $A$ is $S_\omega$ ($n \geq 2$) or $S'_\omega$ ($n \geq 1$). Let $f$ be an assignment of $A$ such that $f(r), f(s) \in \{0,1\}$ and $f(p) = 0$. It is easy to observe that $f(P), f(Q) \in \{0,1\}$ but for every formula $R$, built up from the variable $p$ only, $f(R) \in \{0,1\}$. Hence, for every such $R$, $P \supset Q \in L(A)$ or $R \supset Q \in L(A)$. By Theorem 2.1 and Theorem 4.11, $L \subseteq L(S_n)$ for some $n \geq 2$ or $L \subseteq L(S'_1)$. Therefore, $P \supset Q \in L$ but for every $R$, built up from the variable $p$ only, $P \supset R \in L$ or $R \supset Q \in L$. Q.E.D.

§ 6. Lattice structures of SLLs

Hosoi [3] showed that the set $\mathcal{L}$ of all intermediate propositional logics is a pseudo-Boolean algebra (PBA). We can similarly prove that the set $\mathcal{S}_\mathcal{L}$ of all SLLs is a PBA. Let $\{L_i\}_{i \in A}$ be a set of SLLs. Then $\bigcap_{i \in A} L_i$ is naturally a SLL but $\bigcup_{i \in A} L$ is not always a SLL. But there exists the minimum SLL including $\bigcup_{i \in A} L_i$. So, by $\bigcup_{i \in A} L_i$, we mean the minimum SLL including $\bigcup_{i \in A} L_i$. By the definition, we have

**Theorem 6.1.** $\mathcal{S}_\mathcal{L}$ forms a complete lattice with $\subseteq$ as the order relation.

Further, we have

**Theorem 6.2.** $\bigcup_{i \in A} L_i \cap L = \bigcup_{i \in A} (L_i \cap L)$.

**Proof.** It suffices to prove that $\bigcup_{i \in A} L_i \cap L \subseteq \bigcup_{i \in A} (L_i \cap L)$. Suppose
that $P \in \bigcup_{i \in \mathcal{I}} L_i \cap L$. Then there exist formulas $Q_1, Q_2, \ldots, Q_n \in \bigcup_{i \in \mathcal{I}} L_i$ such that $Q_1 \supset Q_2 \supset \cdots \supset Q_n \supset P \in L_u$. Hence, $Q_1 \lor P \supset Q_2 \lor P \supset \cdots \supset Q_n \lor P \supset P \in L_u$ because $(Q_1 \supset Q_2 \supset \cdots \supset Q_n \supset P) \supset Q_1 \lor P \supset Q_2 \lor P \supset \cdots \supset Q_n \lor P \supset P \in L_u$. On the other hand, as each $Q_i$ belongs to some $L_i$, each $Q_i \lor P$ belongs to some $L_i \cap L$. So $P$ belongs to $\bigcup_{i \in \mathcal{I}} (L_i \cap L)$.

\textbf{Remark.} $\bigcap_{i \in \mathcal{I}} L_i \cup L = \bigcap_{i \in \mathcal{I}} (L_i \cup L)$ does not always hold. For example, $\bigcap_{i \in \mathcal{N}} L(S_i) \cup L(S_i^*) = L(S_i^*) \neq L(S_i) = \bigcap_{i \in \mathcal{N}} (L(S_i) \cup L(S_i^*))$.

Theorem 6.2 is a necessary and sufficient condition for a complete lattice to be a PBA.

\textbf{Theorem 6.3.} $\mathcal{L}$ is a PBA with $W$ and $L_u$ as the maximum element and the minimum element, respectively.

We denote by $\mathcal{L}(L)$ the set of all SLLs including $L$. By Theorem 4.11, $\mathcal{L}(L)$ is a finite set if $L \neq L_u$. Hence we have

\textbf{Theorem 6.4.} If $L \neq L_u$, then $\mathcal{L}(L)$ is a finite PBA.

We illustrate the lattice structure of $\mathcal{L}(L(S_i^*))$ in the following Figure using Theorems 2.1, 2.3 and 4.11. Here we use the abbreviation such as $(2, 3, 1^*) = L(S_2) \cap L(S_3) \cap L(S_4)$.
REFERENCES


Department of Mathematics
Faculty of Science
Shizuoka University
Ohya Shizuoka
422, Japan