Discrete Optimization

The lower and upper forcing geodetic numbers of block–cactus graphs

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Abstract

A vertex set \( D \) in graph \( G \) is called a geodetic set if all vertices of \( G \) are lying on some shortest \( u–v \) path of \( G \), where \( u, v \in D \). The geodetic number of a graph \( G \) is the minimum cardinality among all geodetic sets. A subset \( S \) of a geodetic set \( D \) is called a forcing subset of \( D \) if \( D \) is the unique geodetic set containing \( S \). The forcing geodetic number of \( D \) is the minimum cardinality of a forcing subset of \( D \), and the lower and the upper forcing geodetic numbers of a graph \( G \) are the minimum and the maximum forcing geodetic numbers, respectively, among all minimum geodetic sets of \( G \). In this paper, we find out the lower and the upper forcing geodetic numbers of block–cactus graphs.

Keywords: Graph theory; Geodetic set; Block–cactus graphs; Forcing geodetic number

1. Introduction

All graphs considered in this paper are finite and simple (i.e., without loops and multiple edges).

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \), where \( |V(G)| \) and \( |E(G)| \) are called order and size of \( G \), respectively. In a connected graph \( G \), the distance \( d(u,v) \) of two vertices \( u,v \in V(G) \) is the number of edges of a shortest path between \( u \) and \( v \). A shortest path between \( u \) and \( v \) is called a \( u–v \) geodesic. Let \( I(u,v) \) denote the set of vertices such that a vertex is in \( I(u,v) \) if and only if it is in some \( u–v \) geodesic, and \( I(S) = \bigcup_{u,v \in S} I(u,v) \) for a nonempty subset \( S \) of \( V(G) \). We also call \( I(S) \) the geodetic closure of \( S \).
A vertex set $D$ in a graph $G$ is called a geodetic set if $I(D) = V(G)$. In particular, a geodetic set with the minimum cardinality is called a minimum geodetic set ($g$-set for short) and its cardinality, denoted by $g(G)$, is called the geodetic number of $G$ [16]. Notice that $2 \leq g(G) \leq |V(G)|$ for a nontrivial connected graph $G$. In [2], Buckley et al. characterized those connected graphs $G$ for which the geodetic number is equal to $|V(G)|$, $|V(G)| - 1$, or 2. Two classes of graphical games called achievement and avoidance games were examined by applying the geodetic number [1,12,14]. Lately, Chartrand et al. boost up the research related to geodetic numbers of graphs [4–8]. Although finding the geodetic number of a general graph is known to be NP-hard [11], determining the geodetic number and its bound for certain classes of graphs can be found in [5,11].

The geodetic set problem has applications in graph theory and sociology. For example, consider a friendship graph where each vertex is considered as a person and each edge represents friendship. Social scientists search for geodetic closure in graphs to help clarify the relationships about the group. Besides, geodetic concepts in graphs are closely related to convexity and Steiner concepts [4,9,10]. Convexity plays a key role in geometry, topology, functional analysis and control theory [1,12,14], and Steiner set problem often arises in network design and wiring layout problems [13].

A subset $S$ of a $g$-set $D$ is called a forcing subset of $D$ if $D$ is the unique $g$-set containing $S$. This means that $D$ can be figured out after $S$ is determined. The forcing geodetic number of $D$, denoted by $f(D)$, is the minimum cardinality of a forcing subset of $D$ [7]. In addition, if $S$ is a forcing subset of $D$ such that $|S| = f(D)$, then $S$ is called a minimum forcing subset of $D$. In such a case, every vertex $s \in S$ is called a forcing vertex of $D$ with respect to $S$. The upper forcing geodetic number of a graph $G$, denoted by $f^+(G)$, is the maximum forcing geodetic number among all $g$-sets of $G$ [16]. In contrast, we define the lower forcing geodetic number of a graph $G$, denoted by $f^-(G)$, to be the minimum forcing geodetic number among all $g$-sets of $G$.

We use Fig. 1 as an example to illustrate the concepts above. In Fig. 1, the vertex set $\{a, b, c, d, e\}$ is intuitively a geodetic set of $G$.

Fig. 1. A graph $G$ with $g(G) = 3, f^-(G) = 1$ and $f^+(G) = 2$.

There are only three $g$-sets in the graph $G$, namely $D_1 = \{a, b, e\}, \ D_2 = \{a, c, d\}$ and $D_3 = \{a, d, e\}$. Thus, $g(G) = 3$. Since $D_1$ is the only $g$-set containing $b$, it follows that $f(D_1) = 1$. Also, $D_2$ is the only $g$-set containing $c$, and thus $f(D_2) = 1$. In $D_3$, we can find that every vertex of $D_3$ is also contained in the other $g$-set, so $f(D_3) \geq 2$. Since $D_3$ is the unique $g$-set containing $\{d, e\}$, $f(D_3) = 2$. Therefore, $f^-(G) = \min\{f(D_1), f(D_2), f(D_3)\} = 1$ and $f^+(G) = \max\{f(D_1), f(D_2), f(D_3)\} = 2$.

Researches on forcing concepts related to other graph parameters have been widely studied, such as forcing domination number [3], forcing convexity number [9], forcing perfect matching [15] and forcing geodetic number [7]. Recently, Zhang determined the upper forcing geodetic numbers for trees, cycles, complete bipartite graphs and hypercubes [16]. In this paper, we shall find out the lower and the upper forcing geodetic numbers of block–cactus graphs which generalize cycles and trees. For simplicity, we may assume that the given block–cactus graph $G$ is connected. For the case where $G$ is disconnected, we can find the forcing geodetic number of each connected component of $G$, and the forcing geodetic number of $G$ is defined to be the sum of the forcing geodetic numbers of all components.

The remaining part of this paper is organized as follows. In Section 2, we introduces the termino
logies and notation of the structure of block–cactus graphs. Also, we present some basic properties related to the geodetic set and forcing geodetic number. In Section 3, we deal with the problem of finding the lower and the upper forcing geodetic numbers on block–cactus graphs. Finally, we give concluding remarks and address our future researches in the last section.

2. Preliminaries

For any subset of vertices $S \subseteq V(G)$, we denote $G[S]$ the subgraph of $G$ induced by $S$, i.e., the maximal subgraph of $G$ with vertex set $S$. Also, we use $G - S$ to denote the graph $G[V(G) \setminus S]$, where $V(G) \setminus S$ is the set $\{v \in V(G) | v \not\in S\}$. For simplicity, we write $G - v$ for $G - \{v\}$. Let $D$ be a $g$-set of a graph $G$ and $S$ be a minimum forcing subset of $D$. For a subgraph $H$ of $G$, the contribution of $H$ to $f(D)$ with respect to $S$ is defined to be $V(H) \cap S$.

A clique in a graph $G$ is a complete subgraph of $G$. A vertex $v$ is called extreme if the subgraph induced by the neighbors of $v$ is a clique. A vertex $v$ is a cut vertex if the removal of $v$ together with their incident edges increases the number of components. A block of a graph is a maximal subgraph without a cut vertex. A graph $G$ is called a block graph if and only if every block of $G$ is a clique. Clearly, every vertex in a block graph is either a cut vertex or an extreme vertex. Fig. 2 depicts a block graph, where vertices 3, 4, 5, 8 and 10 are cut vertices, and all other vertices are extreme vertices.

A block that is a cycle is called a cyclic block. A cyclic block $B$ is odd (respectively, even) if $|B|$ (i.e., the order of $B$) is odd (respectively, even). A cyclic block $B$ in a graph $G$ is said to be a cyclic end-block of $G$ (CEB for short) if there is only one cut vertex of $G$ that is contained in $B$; otherwise, we call $B$ a cyclic internal-block of $G$ (CIB for short). A cactus graph is a graph in which every block with three or more vertices is a cyclic block. For example, Fig. 3 illustrates a cactus graph $G$ with blocks $B_i$ for $i = 1, \ldots, 8$, where $B_1, B_2, B_3$ and $B_4$ are cyclic blocks. Blocks $B_1, B_2$ and $B_4$ are CIBs, while $B_3$ is a CEB. Also, it is easy to check that $B_1, B_3$ and $B_4$ are odd blocks, $B_2$ is an even block, and the cut vertices of $G$ are 2, 3, 7, 11, 19 and 20.

Furthermore, a graph whose blocks are either cycles or cliques is called a block–cactus graph [17]. This class of graphs generalizes the known classes of block graphs and cactus graphs. Fig. 4 demonstrates a block–cactus graph that is obtained from the graph of Fig. 3 by replacing the cyclic blocks $B_2$ and $B_4$ by cliques.
Let $G$ be a block–cactus graph and $B$ be a CIB in $G$. A path $P$ in $B$ is called a segment if both of its end vertices are cut vertices and all internal vertices of $P$ are not cut vertices of $G$. In particular, $P$ is a long segment if the length of $P$ is greater than $\frac{|B|}{2}$, and short segment otherwise. Note that every CIB contains at most one long segment. A CIB $B$ is called a long-segment cyclic internal-block (L-CIB for short) if the following conditions are satisfied: (i) $B$ contains a long segment $P$; and (ii) the length of $P$ is less than $|B| - 1$ if $B$ is an odd cyclic block of order $\geq 5$. We use $l(G)$ to denote the number of L-CIBs in a graph $G$.

We again use the graph $G$ in Fig. 3 as an example to illustrate the above concept. All CIBs in $G$ are $B_1$, $B_2$, and $B_4$. Block $B_1$ consists of three segments, where $\langle 3, 4, 5, 6, 7 \rangle$ is a long segment and $\langle 7, 1, 2 \rangle$ and $\langle 2, 3 \rangle$ are two short segments. So, $B_1$...
is an L-CIB. Since $B_2$ is an even cyclic block which has two segments with the same length $\frac{|B_2|}{2}$, $B_2$ is not an L-CIB. Also, the odd cyclic block $B_4$ contains a long segment $(20, 16, 17, 18, 19)$ and a short segment $(19, 20)$. Since the length of the long segment is equal to $|B_4| - 1$, $B_4$ is not an L-CIB. Therefore, $l(G) = 1$.

In the remaining part of this section, we shall introduce some basic properties which are helpful to clarify our proof for determining the upper and lower forcing geodetic numbers of block–cactus graphs. In [16], Zhang showed that for a graph $G$, the following three statements are equivalent:

1. $f^+(G) = 0$;
2. $G$ has a unique $g$-set;
3. $f^-(G) = 0$.

From the fact that every complete graph and tree has a unique $g$-set consisting of all extreme vertices [5], the following corollary is an immediate consequence of Lemma 1.

**Corollary 2.** If $G$ is a complete graph or tree, then $f^+(G) = f^-(G) = 0$.

The following two lemmas respectively state the inclusion of extreme vertices in geodetic sets and the exclusion of cut vertices in $g$-sets.

**Lemma 3** [16]. If $u$ is an extreme vertex of a graph $G$, then $u$ must be contained in every geodetic set of $G$.

**Lemma 4** [16]. If $u$ is a cut vertex of a graph $G$, then $u$ cannot be contained in any $g$-set of $G$.

By Lemma 3, if $G$ has at least two $g$-sets, then every extreme vertex cannot be contained in any forcing subset of each $g$-set. Also, if $G$ has a unique $g$-set $D$, then the minimum forcing subset of $D$ is an empty set. These arguments imply that every extreme vertex cannot be a forcing vertex of any $g$-set. Recall that every vertex of a block graph is either a cut vertex or an extreme vertex. Lemmas 3 and 4 indicate that a block graph has a unique $g$-set. Furthermore, we immediately obtain the following theorem from Lemmas 1, 3, and 4.

**Theorem 5.** Let $c$ be the number of cut vertices in a block graph $G$. Then, $g(G) = |V(G)| - c$ and $f^-(G) = f^+(G) = 0$.

An easy observation given in [5] showed that for any cycle $C_n$ of order $n \geq 3$, $g(C_n) = 2$ if $n$ is even and $g(C_n) = 3$ if $n$ is odd. The next result shows that the upper forcing geodetic numbers of cycles can be completely determined.

**Lemma 6** [16] \[ f^+(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n = 5, \\ 3 & \text{if } n \geq 7 \text{ is odd.} \end{cases} \]

In what follows, we determine the lower forcing geodetic number of a cycle.

**Lemma 7**

\[ f^-(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd and } n \neq 3. \end{cases} \]

**Proof.** It is clear that $C_3$ has a unique $g$-set containing three vertices. By Lemma 1, $f^-(C_3) = 0$. For an even cycle $C_{2k} = (v_0, v_1, \ldots, v_{2k-1}, v_0)$, every $g$-set has the form $\{v_i, v_{(i+k) \mod 2k}\}$ which is the only $g$-set containing $v_i$, where $0 \leq i \leq 2k - 1$. Thus, $f^-(C_n) = 1$ for $n$ is even.

We now show that $f^-(C_n) \geq 2$ for $n$ is odd and $n \neq 3$. In this case, it is clear that $g(C_n) = 3$ and $C_n$ has more than one $g$-set. Thus, by Lemma 1, $f^-(C_n) > 0$. To complete the proof, we have to show that $f^-(C_n) \neq 1$. Let $n = 2k + 1$ with $k \geq 2$ and $C_n = (v_0, v_1, \ldots, v_{2k}, v_0)$. Suppose to the contrary that there exists a $g$-set $D$ such that $f^-(C_n) = f(D) = 1$. From the vertex symmetry property of a cycle, we may assume that there exist two vertices $v_i$ and $v_j$ where $0 < i, j \leq 2k$ and $i \neq j$ such that $D = \{v_0, v_i, v_j\}$ is the unique $g$-set containing $v_0$ which is the forcing vertex of $D$. However, we can also find that both $\{v_0, v_i, v_{i+1}\}$...
and \( \{v_0, v_k, v_{k+1} \} \) are g-sets containing \( v_0 \). This leads to a contradiction. Moreover, it is easy to check that \( \{v_0, v_1, v_{k+1} \} \) is the unique g-set containing both \( v_0 \) and \( v_1 \). Hence, \( f^-(C_n) = 2 \).

3. The forcing geodetic numbers of block–cactus graphs

Before we computing the lower and upper forcing geodetic numbers on block–cactus graphs, we consider a quite simple situation, namely cactus graphs. Since Lemmas 6 and 7 completely characterize the forcing geodetic number of cycles, we assume that the given cactus graphs are not cycles. Two kinds of cyclic blocks CEBs and CIBs are considered as follows.

Let \( B \) be a CEB in a cactus graph \( G \) and let \( w \) be the unique cut vertex of \( B \). If \( |B| = 3 \), then it is clear that the vertices in \( V(B) \setminus \{w\} \) are extreme vertices, and hence are not forcing vertices of any \( g \)-set of \( G \) (i.e., \( V(B) \setminus \{w\} \) has no contribution to \( f(D) \) for any \( g \)-set \( D \) in \( G \)). In the following lemma we consider a CEB of order \( \geq 4 \).

Lemma 8. Let \( G \) be a cactus graph and \( B \) be a CEB with order \( \geq 4 \) in \( G \). Then, the contributions of \( B \) to \( f^-(G) \) and \( f^+(G) \) are \( f^-(B) - 1 \) and \( f^+(B) - 1 \), respectively.

Proof. Let \( v \) be the unique cut vertex of \( B \) and \( D \) be any \( g \)-set of \( G \) which contains a minimum forcing subset \( S \). By Lemma 4, \( v \notin D \) and so \( v \notin S \). Let \( D_v = (D \setminus V(B)) \cup \{v\} \) and \( S_v = (S \setminus V(B)) \cup \{v\} \). Since \( B \) is a CEB, \( D_v \) is a \( g \)-set of \( B \) and it contains \( S_v \) as a minimum forcing subset. We now consider the order of \( B \) as follows. If \( |B| \) is even, by Lemmas 6 and 7, \( f^-(B) = f^+(B) = f(D_v) = |S_v| = 1 \). Since \( B \) is a cyclic block, we may choose \( v \) to be the forcing vertex of \( D_v \) with respect to \( S_v \). Thus, \( S \cap V(B) = S_v \setminus \{v\} = \emptyset \) and there are no contribution of \( B \) to \( f(D) \). On the other hand, consider that \( B \) is an odd block. Since \( |B| > 3 \), by Lemmas 6 and 7, \( f(D_v) = |S_v| \geq 2 \). Similarly, we can choose \( v \in S_v \). Therefore, \( |S \cap V(B)| = |S_v| - 1 \) and \( B \) provides the contribution \( f(D_v) - 1 \) to \( f(D) \). Further, this implies that \( B \) has \( f^-(B) - 1 \) contribution to \( f^-(G) \) and \( f^+(B) - 1 \) contribution to \( f^+(G) \).

We now introduce some notation which will be used for proving the following lemmas. Let \( G \) be a cactus graph and \( B \) be a CIB in \( G \). If \( P \) is a segment of \( B \) with two end vertices \( a \) and \( b \), then \( a \) and \( b \) are cut vertices of \( G \). In such a case, let \( G_a \) (respectively, \( G_b \)) denote the subgraph consisting of all components of \( G - a \) (respectively, \( G - b \)) except the component containing the vertices of \( P \). Also, we use \( G_{a,b} \) to denote the graph \( G - \{V(G_a) \cup V(G_b)\} \). For instance, in Fig. 3, let \( P \) be the segment \((2,3) \) in \( B_1 \). We can see that \( G_2 \) is the subgraph induced by the vertex set \( \{8, 9, 10, 11, 12, 13, 14, 15\} \), \( G_3 \) is the subgraph induced by the vertex set \( \{16, 17, 18, 19, 20, 21\} \), and \( G_{2,3} \) contains the two blocks \( B_1 \) and \( B_5 \) of \( G \).

Lemma 9. If \( w \) is a vertex on a short segment, then \( w \) cannot be contained in any \( g \)-set.

Proof. Let \( G \) be a cactus graph and \( P \) be a short segment with end vertices \( a \) and \( b \) in a CIB of \( G \). We now assume that \( G_a \), \( G_b \), and \( G_{a,b} \) are defined as above. Let \( D \) be any \( g \)-set of \( G \). Clearly, \( V(G_a) \cap D \neq \emptyset \) and \( V(G_b) \cap D \neq \emptyset \). Let \( a' \in V(G_a) \cap D \) and \( b' \in V(G_b) \cap D \). Suppose to the contrary, we assume that there is a vertex \( w \in V(P) \cap D \). By definition, \( G \) contains at least a vertex \( u \notin I(D \setminus \{w\}) \) such that \( u \notin I(w, w') \) for some \( w' \in D \). Since \( P \) is a short segment, \( V(P) \subseteq I(a', b') \) and hence \( u \notin I(P) \). There are three cases to be considered depending on the position of \( u \).

Case 1: \( u \in V(G_a) \). Since \( u \in I(w, w') \) and \( u \notin V(P) \), we have \( w' \in V(G_a) \). Thus, \( u \) lies on an \( a-w' \) geodesic, say \( P' \). However, \( P' \) is indeed a subpath of a \( b'-w' \) geodesic. Consequently, \( u \) lies on a \( b'-w' \) geodesic. This contradicts to the fact that \( u \notin I(D \setminus \{w\}) \).

Case 2: \( u \in V(G_b) \). This proof is similar to Case 1.

Case 3: \( u \in V(G_{a,b} - P) \). Since \( u \in I(w, w') \), we have \( w' \in V(G_{a,b} - P) \). Let \( P' \) be a \( w'-w' \) geodesic which passes through \( u \). Clearly, either \( a-w' \) geodesic or \( b'-w' \) geodesic is a subpath of \( P' \). The former case indicates that \( u \in I(a', w') \), and the later case indicates that \( u \in I(b', w') \). Therefore, both the cases contradict the fact that \( u \notin I(D \setminus \{w\}) \).
Lemma 10. Let $B$ be a CIB with $n$ vertices in a cactus graph $G$, where $n \geq 3$ is an odd integer. If $P$ is a long segment of length $n - 1$ in $B$, then $P$ has no contribution to $f(D)$ for any g-set $D$ of $G$.

**Proof.** Let $a$ and $b$ be the two end vertices of $P$, and suppose that $G_a$, $G_b$, and $G_{a,b}$ are mentioned as above. Let $D$ be any g-set of $G$ which contains two vertices $a' \in V(G_a) \cap D$ and $b' \in V(G_b) \cap D$. Clearly, $a, b \notin D$. Since $P$ is a long segment, every vertex of $V(P - \{a, b\})$ cannot be contained in $I(a', b')$. Thus, we have $V(P - \{a, b\}) \cap D \neq \emptyset$.

Suppose that the length of $P$ is $n - 1$ where $n$ is odd. Let $P = (a = v_0, v_1, \ldots, v_{n-1} = b)$. Clearly, $V(P) \subset I(a', v_{n-1}) \cup I(b', v_{n-1})$. Thus, if $v_{n-1} \in D$ then $V(P - \{a, b\}) \subset I(D)$. Since $|V(P - \{a, b\}) \cap D| \geq 1$ and $D$ is a minimum geodetic set of $G$, we conclude that $v_{n-1}$ is the only vertex of $P$ that belongs to $D$. Since $v_{n-1}$ is contained in every g-set of $G$, it cannot be a forcing vertex. This implies that $P$ has no contribution to $f(D)$. □

Lemma 11. Let $D$ be any g-set of a cactus graph $G$ and $S$ be a minimum forcing subset of $D$. If $B$ is an L-CIB of $G$, then the long segment of $B$ has contribution 1 to $f(D)$ with respect to $S$.

**Proof.** Suppose that $B$ is an L-CIB with $n$ vertices in $G$. By definition, $B$ has a long segment, say $P$. In particular, if $n \geq 5$ is odd, then $|P| < n - 1$. Let $a$ and $b$ be the two end vertices of $P$, and suppose that $G_a$, $G_b$, and $G_{a,b}$ are defined as above. Let $a' \in V(G_a) \cap D$ and $b' \in V(G_b) \cap D$. Since $P$ is a long segment, we have $(V(P - B) \cup \{a, b\}) \subset I(a', b')$ and $V(P - \{a, b\}) \cap D \neq \emptyset$. This implies that if a vertex $v \in V(B)$ is a forcing vertex of $D$ with respect to $S$, then $v$ must be in $V(P - \{a, b\})$. Since $B$ is an L-CIB, either $n$ is even and $|P| > \frac{n}{2}$ or $n \geq 5$ is odd and $\frac{n}{2} < |P| < n - 1$. In both cases, it is easy to verify that $B$ has at least a vertex in any forcing subset. Thus, $|V(P - \{a, b\}) \cap S| = |V(B) \cap S| \geq 1$. Since $S$ is a minimum forcing subset of $D$, we have $|V(P - \{a, b\}) \cap S| = |V(B) \cap S| = 1$. That is, $P$ has contribution 1 to $f(D)$ with respect to $S$. □

We are now at a position to combine the results of the previous lemmas and obtain the main theorem for cactus graphs. For convenience, we use $\alpha(G)$ and $\beta(G)$ to denote the number of odd CEBs with order 5 and with order $\geq 7$ in a block–cactus graph $G$, respectively.

**Theorem 12.** If $G$ is a cactus graph, then $f^-(G) = \alpha(G) + \beta(G) + l(G)$ and $f^+(G) = \alpha(G) + 2 \cdot \beta(G) + l(G)$.

**Proof.** Let $D$ be any g-set of a cactus graph $G$ and $S$ be a minimum forcing subset of $D$. In the following proof, we individually consider the different types of subgraph of $G$ including CEB, CIB, and other vertices for their contributions to $f(D)$ with respect to $S$. Consequently, combining these results the theorem follows.

Obviously, if a vertex $v$ does not belong to any CEB or CIB of $G$, then $v$ is either an extreme vertex or a cut vertex. By Lemmas 3 and 4, $v$ cannot be a forcing vertex of $D$ and thus no contribution.

We now consider the contribution of CEBs as follows. By Lemmas 6–8, every CEB of order 5 has contribution 1 to both $f^-(G)$ and $f^+(G)$; every CEB of order $\geq 7$ has contribution 1 to $f^-(G)$ and contribution 2 to $f^+(G)$; while other CEBs (including those of order 3 and even order) have no contribution to $f(D)$ with respect to $S$. This indicates that all CEBs totally have contribution $\alpha(G) + \beta(G)$ to $f^-(G)$ and contribution $\alpha(G) + 2 \beta(G)$ to $f^+(G)$.

Finally, we consider the contribution of CIBs. Let $B$ be a CIB of $G$ with $n$ vertices and $P$ be a segment in $B$. By Lemmas 9 and 10, if $P$ is a short segment or $|P| = n - 1$ and $n$ is odd, then $P$ has no contribution to $f(D)$ with respect to $S$. On the other hand, if $P$ is a long segment and the length of $P$ is less than $n - 1$ whenever $n$ is odd (i.e., $B$ is an L-CIB), by Lemma 11 $|V(P) \cap S| = 1$. That is, every L-CIB has contribution 1 to $f(D)$. Therefore, all CIBs totally have contribution $l(G)$ to $f(D)$. □

For a block–cactus graph $G$, if a vertex $v \in V(G)$ does not belong to any cyclic block, then $v$ must be an extreme vertex or a cut vertex. By Lemmas 3 and 4, $v$ cannot be a forcing vertex of
any g-set. Therefore, a similar proof of Theorem 12 achieves the following result.

**Theorem 13.** If $G$ is a block–cactus graph, then $f^-(G) = \alpha(G) + \beta(G) + \ell(G)$ and $f^+(G) = \alpha(G) + 2 \cdot \beta(G) + \ell(G)$.

### 4. Concluding remarks

In [16], Zhang determined the upper forcing geodetic numbers for trees, cycles, complete bipartite graphs and hypercubes. In the paper, we propose a similar concept of graph parameter, namely lower forcing geodetic number. Further, we explore the lower and the upper forcing geodetic numbers of block–cactus graphs, i.e., a class of graphs generalize block graphs and cacti. An obvious continuation of this work is to investigate the problem of finding the forcing geodetic numbers of graphs on larger classes of graphs. Another line of progression will be devoted to developing efficient algorithms for finding the minimum forcing subset of any g-set.

### References