Feedback vertex sets in star graphs

Fu-Hsing Wang, Yue-Li Wang, Jou-Ming Chang

Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan, Republic of China

Department of Information Management, National Taipei College of Business, Taipei, Taiwan, Republic of China

Received 11 December 2002; received in revised form 15 October 2003

Communicated by Wen-Lian Hsu

Abstract

In a graph \( G = (V, E) \), a subset \( F \subseteq V(G) \) is a feedback vertex set of \( G \) if the subgraph induced by \( V(G) \setminus F \) is acyclic. In this paper, we propose an algorithm for finding a small feedback vertex set of a star graph. Indeed, our algorithm can derive an upper bound to the size of the feedback vertex set for star graphs. Also by applying the properties of regular graphs, a lower bound can easily be achieved for star graphs.

Keywords: Feedback vertex sets; Interconnection networks; Star graphs; Algorithms; Analysis of algorithms

1. Introduction

Let \( G = (V, E) \) be a simple graph, i.e., loopless and without multiple edges, with vertex set \( V(G) \) and edge set \( E(G) \). A set of vertices \( F \subseteq V(G) \) is called a feedback vertex set if the subgraph induced by \( V(G) \setminus F \) is acyclic [7,14], where \( V(G) \setminus F = \{ x \mid x \in V(G) \text{ and } x \notin F \} \). In addition, if the cardinality \( |F| \) is minimum among all possible feedback vertex sets, then \( F \) is called a minimum feedback vertex set. In particular, we use \( F_{\text{min}}(G) \) to denote a certain minimum feedback vertex set of the graph \( G \).

The problem of finding a minimum feedback vertex set is NP-hard for general graphs [7]. The best known approximation algorithm for this problem has approximation ratio 2 [4]. Furthermore, most recent research have been devoted to solving the problem for certain special graphs in polynomial time, e.g., reducible graphs [14], cocomparability graphs [10], convex bipartite graphs [10], cyclically reducible graphs [15], and interval graphs [11]. Recently, the lower and upper bounds to the size of the feedback vertex sets have been established and improved on meshes, tori, butterflies, cube connected cycles, and hypercubes [5,6,12].

The feedback vertex set problem has important applications to several fields, for example, deadlock prevention in operating systems. Once a deadlock has been detected, a strategy is needed to break up the deadlock. Usually, a deadlock in a system can be described by using a wait-for graph [13]. In a wait-for graph, each vertex represents a process, and the existence of an edge \( (i, j) \) indicates that process \( i \) is waiting for process \( j \) to release a resource requested...
by process $i$. A deadlock exists in a system if and only if the corresponding wait-for graph contains a cycle. One of the best-known approaches for solving the deadlock problem can be carried out to abort as less deadlocked processes as possible in the wait-for graph. Using graph-theoretic terminology, the strategy is equivalent to finding a feedback vertex set for such a system.

In this paper, we consider the problem of a particular interconnection network, namely, star graph. Star graphs were proposed as an attractive alternative to hypercubes with many nice topological properties [1,2]. Both star graphs and hypercubes provide attractive interconnection schemes for massively parallel systems. Hence characterizations of these architectures have been widely investigated. Star graphs are vertex and edge symmetric, highly regular, strongly hierarchical, and maximally fault-tolerant for connectivity. Due to their strongly hierarchical structure, star graphs can be defined recursively. Moreover, star graphs have many superior advantages over hypercubes, such as smaller degree and diameter. In this paper, we present a simple algorithm for finding an upper bound of the minimum feedback vertex set on star graphs. In contrast, we also give a lower bound to the problem on star graphs.

2. Main results

A permutation is a sequence of elements in which no element appears more than once. Let $N = \{1, 2, \ldots, n\}$ and $p = [p_1, p_2, \ldots, p_n]$ be a permutation, where $p_i \in N$ for all $1 \leq i \leq n$. The $n$-dimensional star graph (n-star for short), denoted by $S_n$, is an undirected graph consisting of $n!$ vertices labeled with distinct permutations $[p_1, p_2, \ldots, p_n]$ from $N$. For each vertex $v = [p_1, p_2, \ldots, p_n]$, $p_i$ is called the $i$th number of $v$. Two vertices are connected by an edge if and only if the label of one vertex can be obtained from the other by swapping the first number and the $i$th number, where $2 \leq i \leq n$ [1,2]. Note that an $n$-star is a regular graph of degree $n - 1$. Fig. 1 depicts $S_4$. Two vertices are connected by an edge indicated by the same symbol. For instance, $[1, 2, 3, 4]$ and $[4, 2, 3, 1]$ are neighbors since their labels differ only in the first and the last positions.

![Fig. 1. A 4-dimensional star graph $S_4$. The set $F_m(S_4)$ is represented by the circled vertices.](image)

For a permutation $[p_1, p_2, \ldots, p_n]$ the pair $p_i$ and $p_j$ constitute an inversion, if $p_i > p_j$ and $i < j$. A vertex is odd (respectively, even) if the number of inversions in its permutation is odd (respectively, even). Throughout the rest, we use $I_e$ and $I_o$ to denote the sets containing all even vertices and odd vertices of $S_n$, respectively.

**Proposition 1.** Permutation $[n, n - 1, \ldots, 1]$ is even if and only if $n \equiv 0, 1 \pmod{4}$.

**Proof.** The number of inversions for the permutation $[n, n - 1, \ldots, 1]$ is $1 + 2 + \cdots + (n - 1) = \frac{n(n - 1)}{2}$. The inversions-sum formula makes the condition necessary. To prove the converse, suppose $n \equiv 0, 1 \pmod{4}$. For an integer $k$, if $n = 4k$, then $\frac{n(n - 1)}{2} = 2k(4k - 1)$ is even. For the case where $n = 4k + 1$, the proposition follows from the fact that $\frac{n(n - 1)}{2} = 2k(4k + 1)$ is also even. \qed

An independent set $S$ of a graph $G$ is a set of vertices in which no two vertices of $S$ are adjacent in $G$. If the cardinality of $S$ is maximum among all possible independent sets, then $S$ is called a maximum independent set of $G$. Note that a star graph is a bipartite graph with equal partite size and $I_e$ and $I_o$ are the two partite sets [9]. Thus, $I_e$ is a maximum independent set of $S_n$ and $|I_e| = \frac{n}{2}$. Furthermore, the
sets in $S_n$. Therefore, $V(S_n) \setminus I_e$ is a trivial feedback vertex set.

For $i, j \in N$, let $N_i = N \setminus \{i\}$ and $N_{i,j} = N \setminus \{i, i + 1, \ldots, j\}$, where $i < j$. Define classes of vertex sets in $S_n$ as follows.

$$
\Phi_1 = \{[1, p_2, p_3, \ldots, p_n] | \ \\
\quad \text{if } j, k \geq 2 \text{ and } j \neq k \} \cap I_o,
$$

$$
\Phi_i = \{[i, p_2, p_3, \ldots, p_{n-1}, i-1, i-2, \ldots, 2, 1] | \ \\
\quad \text{for } i = 2, 3, \ldots, n-2, \text{ and } \ \\
\quad \text{if } 2 \leq j, k \leq n-i+1 \text{ and } j \neq k \} \cap I_o,
$$

$$
\Phi_n-1 = \{ \ \\
\quad \text{if } n \equiv 0, 1 \pmod{4}, \ \\
\quad \{[n-1, n-2, n-3, \ldots, 1] \}, \ \\
\quad \text{otherwise}. \}
$$

From the above definition and Proposition 1, $\Phi_{n-1}$ is a set containing only one vertex. It is obvious that all $\Phi_i$, $1 \leq i \leq n-1$, are independent sets of $S_n$ since they contain only odd vertices. Consider $G_0 = I_e$ and let $G_i, 1 \leq i \leq n-1$, be the subgraph of $S_n$ induced by $I_e \cup \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_i$. We shall show that, for $i = 1, 2, \ldots, n-1$, $V(S_n) \setminus V(G_i)$ is a feedback vertex set and its size is smaller than that of $V(S_n) \setminus V(G_{i-1})$.

The neighborhood $N(v)$ of a vertex $v$ is the set of vertices which are adjacent to $v$. A vertex $v \in V(G_i)$, $0 \leq i \leq n-2$, is called a port vertex of $G_i$ if there exists a vertex $u \in N(v)$ in $\Phi_j$ with $j > i$. We use Fig. 2 as an example to illustrate the above notation. In $S_4$, $\Phi_1 = \{[1, 3, 2, 4], [1, 2, 4, 3], [1, 4, 3, 2] \}$, $\Phi_2 = \{[2, 3, 4, 1] \}$ and $\Phi_3 = \{[3, 4, 2, 1] \}$. Fig. 2(a) depicts the induced subgraph $G_0$ of $S_4$. Consider the induced subgraph $G_1$ of $S_4$ in Fig. 2(b). Vertices $[4, 3, 2, 1], [3, 2, 4, 1], [2, 4, 3, 1], [1, 3, 4, 2]$ and $[1, 4, 2, 3]$ are port vertices of $G_1$ since vertices $[4, 3, 2, 1], [3, 2, 4, 1]$ and $[1, 3, 4, 2]$ are neighbors of vertex $[2, 3, 4, 1]$ in $\Phi_2$ in $S_4$, and vertices $[2, 4, 3, 1]$ and $[1, 4, 2, 3]$ are neighbors of vertex $[3, 4, 2, 1]$. $G_1 \in \Phi_3$ in $S_4$. Fig. 2(c) illustrates $G_2$ of $S_4$ where vertices $[4, 3, 2, 1], [2, 4, 3, 1]$ and $[1, 4, 2, 3]$ are port vertices of $G_2$. However, vertices $[1, 3, 4, 2]$ and $[3, 4, 2, 1]$ are not port vertices of $G_2$ since they are not adjacent to any vertex in $\Phi_3$. Finally, Fig. 2(d) is a maximum acyclic induced subgraph of $S_4$. Thus, $V(S_4) \setminus V(G_2) = \{[2, 1, 3, 4], [2, 4, 1, 3], [3, 1, 4, 2], [3, 2, 1, 4], [4, 1, 2, 3], [4, 2, 3, 1], [4, 3, 1, 2] \}$ is a feedback vertex set (see Fig. 1). We will show later on that this set is a minimum feedback vertex set.

![Fig. 2](image-url)
A set \( D \subseteq V(G) \) is a dominating set of \( G \) if for every vertex \( u \in V(G) \setminus D \) there exists a vertex \( v \in D \) such that \( u \) is adjacent to \( v \). We also say that \( v \) dominates \( u \) and \( u \) is dominated by \( v \). In particular, we call \( D \) a perfect dominating set if every vertex in \( V(G) \setminus D \) is dominated by exactly one vertex in \( D \) [8]. We call \( D \) an independent dominating set if \( D \) is also an independent set of \( G \). A dominating set \( D \) is independent perfect if it is both independent and perfect. In [3], Arumugam and Kala showed that all vertices in \( S_n \) having the same first number in their labels form a minimum independent perfect dominating set. Note that every \( \Phi_i, i = 1, 2, \ldots, n - 1 \), is a subset of some minimum independent perfect dominating set.

Lemma 2. \( G_1 \) is acyclic and each component of \( G_1 \) has at most one port vertex.

Proof. Let \( u \) and \( v \) be two vertices of \( \Phi_1 \). Since all vertices having 1 as the first number of their labels form a minimum independent perfect dominating set, no vertex is dominated by both \( u \) and \( v \) in \( G_1 \). This means that the component of \( G_1 \) is either an isolated vertex or a nontrivial tree since both \( I_r \) and \( \Phi_1 \) are independent sets. Thus, \( G_1 \) is acyclic. (See Fig. 2(b) for \( S_4 \).) To complete the proof, let \( T \) be a component of \( G_1 \). If \( T \) contains only one isolated vertex, then the lemma holds immediately. For the case where \( T \) is a nontrivial tree, there is only one vertex having 1 as the first number and another vertex having 1 as the \( n \)th number in their labels. However, only one of these two vertices can be a port vertex in \( T \) since, by definition, a port vertex of \( G_1 \) must be adjacent to a vertex of \( \Phi_1 \) for \( i \geq 2 \) and every vertex of \( \Phi_1, i \geq 2 \), has 1 as the last number in its label. Specifically, if \( w \) is the vertex with 1 as the first number in \( T \), then \( w \in \Phi_1 \) and \( w \) is an odd vertex. Thus, \( w \) cannot be adjacent to any vertex of \( \Phi_1, i \geq 2 \). Therefore, \( w \) is not a port vertex and we conclude that the component \( T \) has at most one port vertex in \( G_1 \). \( \square \)

Before proving Lemma 3, we introduce some notation used in the proof. For each vertex \( u = [i, p_2, p_3, \ldots, p_{n-i+1}, i-1, i-2, \ldots, 2, 1] \in \Phi_i \), \( 2 \leq i \leq n-2 \), let \( NB_j(u) = [j, p_{j+1}, p_{j+2}, \ldots, p_{n-i+1}, i-1, i-2, \ldots, 2, 1] \) be the \( j \)th neighbor of \( u \). Notice that, by definition, \( NB_j(u), 2 \leq j \leq n \), are port vertices in \( G_{i-1} \) as \( u \in N(NB_j(u)) \) in \( \Phi_i \). Since \( u \) is an odd vertex, all \( NB_j(u), 2 \leq j \leq n \), are even vertices and are contained in \( G_i \). Therefore, the degree of \( u \) in \( G_i \) is \( n-1 \). Let \( T_u \) be the component of \( G_1, 1 \leq i \leq n-2 \), containing vertex \( u \in \Phi_i \). For example, Fig. 2(c) depicts \( G_2 \) of \( S_4 \) where vertex \( u = [2, 3, 4, 1] \) is the only vertex of \( G_2 \) in \( \Phi_2 \). Besides, vertices \([3, 2, 4, 1], [4, 3, 2, 1], \) and \([1, 3, 4, 2] \) are \( NB_2(u), NB_3(u), \) and \( NB_4(u), \) respectively, of vertex \( u \). Thus, the degree of \( u \) is 3. The leftmost tree in Fig. 2(c) is the component \( T_u \) of \( G_2 \). Let us now prove Lemma 3.

Lemma 3. Each component of \( G_k, k = 1, 2, \ldots, n-2 \), has at most one port vertex.

Proof. We prove this lemma by induction on \( k \). The basis \((k = 1) \) follows directly from Lemma 2. For the inductive hypothesis, we assume that the lemma is true for all \( 1 \leq k < i \), where \( i < n-2 \). Now, consider \( k = i \). In \( G_i \), the component containing no vertex in \( \Phi_i \) is also a component of \( G_{i-1} \) and, by hypothesis, has at most one port vertex. By contrast, \( T_u, u \in \Phi_i \), intuitively has at most \( n-1 \) port vertices namely \( NB_j(u), j = 2, 3, \ldots, n \). To complete the proof, we now show that there are at least \( n-2 \) neighbors of \( u \) which are not port vertices in \( G_i \). In fact, we want to show that \( NB_j(u), 2 \leq j \leq n \) and \( j \neq n-i+1 \), are not port vertices in \( G_i \). Suppose to the contrary that \( NB_j(u), 2 \leq j \leq n \) and \( j \neq n-i+1 \), is a port vertex in \( G_i \). In such a case, a vertex \( w = [m, p_2, \ldots, p_{n-i}, i, i-1, \ldots, 1] \) in \( G_m, m > i \), exists such that \( NB_j(u) \subseteq N(w) \). Consider the possible positions of the number \( i \) for those vertices of \( N(u) \) and \( N(w) \). Since the first number of \( u \) is \( i \), the first number of \( NB_j(u) \) cannot be \( i \). However, the number \( i \) is either in the first position or the \((n-i+1)\)th position for every neighbor of \( u \). Therefore, \( NB_j(u) \) must have \( i \) as the \((n-i+1)\)th number, i.e., \( j = n-i+1 \). This contradicts the assumption that \( j \neq n-i+1 \). Thus, the lemma follows. \( \square \)

Note that \( G_1 \) is acyclic. We next show that \( G_2, G_3, \ldots, G_{n-1} \) are acyclic.

Lemma 4. \( G_k \) is acyclic, for \( k = 1, 2, \ldots, n-1 \).
Proof. The proof is also by induction on $k$. The basis ($k = 1$) follows directly from Lemma 2. Assume the lemma is true for all $G_k$, $1 \leq k < i$, where $i < n - 1$. Now, consider the case $k = i$. We first show that each component of $G_i$ contains at most one vertex of $\Phi_i$. Let $u$ and $v$ be any two distinct vertices of $\Phi_i$. Since $\Phi_i$ is a subset of some independent perfect dominating set, $N(u) \cap N(v) = \emptyset$. From the inductive hypothesis and Lemma 3, $u$ and $v$ are in different components of $G_i$. Further, we show that all components of $G_i$ are acyclic to complete the proof. In $G_i$, the component containing no vertex in $\Phi_i$ is also a component of $G_{i-1}$ and, by hypothesis, is acyclic. It suffices to show that the component $T_u$ containing $u \in \Phi_i$ is acyclic. The number of vertices in $T_u$ is

$$|V(T_u)| = |V(C_2)| + |V(C_3)| + \cdots + |V(C_n)| + 1,$$

where $C_j$, $j = 2, 3, \ldots, n$, is the component containing $NB_j(u)$ in $G_{i-1}$. Since the degree of $u$ is $n - 1$ and each component $C_i$, $2 \leq i \leq n$, is acyclic, the number of edges in $T_u$ is

$$|E(T_u)| = |E(C_2)| + |E(C_3)| + \cdots + |E(C_n)| + (n - 1)$$

$$= (|V(C_2)| - 1) + (|V(C_3)| - 1) + \cdots + (|V(C_n)| - 1) + (n - 1)$$

$$= |V(C_2)| + |V(C_3)| + \cdots + |V(C_n)|.$$

Since $T_u$ is connected and $|E(T_u)| = |V(T_u)| - 1$, $T_u$ is acyclic. This completes the proof. \hfill \Box

Let us now compute an upper bound to the size of the minimum feedback vertex set $F_m(S_n)$. Theorem 5. $|F_m(S_n)| \leq \frac{13}{2}[n! - (n - 1)! - (n - 2)! - \cdots - 2!] - 1$ for $n \geq 3$.

Proof. By Lemma 4, $G_{n-1}$ is acyclic. Thus, $V(G) \setminus V(G_{n-1})$ is a feedback vertex set. Therefore, for a minimum feedback vertex set $F_m(S_n)$ we have the following bound:

$$|F_m(S_n)| \leq |V(G) \setminus V(G_{n-1})|$$

$$= n! - |I_e \cup \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_{n-1}|$$

$$= n! - (|I_e| + |\Phi_1| + |\Phi_2| + \cdots + |\Phi_{n-1}|)$$

$$= n! - \left[\frac{n!}{2} + \frac{(n - 1)!}{2} + \frac{(n - 2)!}{2} + \cdots + \frac{2!}{2} + 1\right]$$

$$= \frac{1}{2}[n! - (n - 1)! - (n - 2)! - \cdots - 2!] - 1. \quad \Box$$

The following lemma given in [5] shows a lower bound to the size of a minimum feedback vertex set of $G$ with maximum degree $r$.

Lemma 6. $|F_m(G)| \geq \frac{|E(G)| - |V(G)| + 1}{r - 1}$.

Note that the $n$-star has $n!$ vertices and $\frac{n(n - 1)}{2}$ edges and each vertex has degree $n - 1$. The next result directly follows from Lemma 6.

Corollary 7. $|F_m(S_n)| \geq \frac{(n - 3)n + 2}{2(n - 2)}$ for $n \geq 3$.

3. Concluding remarks

By Theorem 5 and Corollary 7, we have found that $\frac{13}{2} \leq |F_m(S_4)| \leq 7$. This implies that $|F_m(S_4)| = 7$. Therefore, our algorithm finds an optimal feedback vertex set of $S_4$. The equality of Lemma 6 holds only when the subgraph $G'$ induced by $V(G) \setminus F_m(G)$ is a forest that contains the least number of components. Moreover, the lower bound will increase to adapt to the least number of components of $G'$. Therefore, we are now trying to determine the cardinality of components in $G'$. It will be helpful to explore the exact value for this problem.

References