Fuzzy Convexity and Multiobjective Convex Optimization Problems

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Abstract—In this paper, based on the more restrictive definition of fuzzy convexity due to Ammar and Metz [1], several useful composition rules are developed. The advantages in using the more restrictive definition of fuzzy convexity are that local optimality implies global optimality, and that any convex combination of such convex fuzzy sets is also a convex fuzzy set. As shown in this paper, these properties are lacking in the usual convex fuzzy sets. In addition, to illustrate the applications in fuzzy convex optimization, two examples in multiple objective programming are considered. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Since most of practical decision problems are fuzzy and approximate, fuzzy decision making becomes one of the most important practical approaches. However, the resulting problems are frequently complicated and difficult to solve. One effective way to overcome these difficulties is to explore the fuzzy convexity properties of the resulting problems. However, in order to formulate these desirable resulting problems, we must have a complete, or, at least, reasonable understanding about the basic convexity properties of fuzzy sets. For example, in an earlier paper [2], we formulated several fuzzy nonlinear programming problems based on the concept of fuzzy convexity. Different types of convexity and generalized convexity of fuzzy sets were studied by several authors, including Ammar and Metz [1], Ramik and Vlach [3], Sarkar [4], Syau and coworkers [2,5,6], and Yang [7-9], aiming at applications to fuzzy nonlinear programming.

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We shall restrict attention here to fuzzy sets on the n-dimensional Euclidean space $R^n$. The concept of convex fuzzy sets was introduced by Zadeh [10], in which a fuzzy set with membership function $\mu: R^n \rightarrow [0, 1]$ was called convex if

$$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\} \quad (1.1)$$

for all $x, y \in \text{supp}(\mu) = \{t \in R^n : \mu(t) > 0\}$ and $\lambda \in [0, 1]$. Consider the following fuzzy set with membership function,

$$\mu(x) = \begin{cases} 
\frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\
1, & \text{if } \frac{1}{2} < x \leq 1,
\end{cases}$$

satisfying (1.1). Any point in $[0, 1/2)$ is a local maximizer of $\mu$ but not a global maximizer. This indicates that, in maximizing a fuzzy decision (for details, see [11]), such a convexity does not ensure that a local maximizer is also a global maximizer. Therefore, we will study a more restrictive definition of fuzzy convexity due to Ammar and Metz [1], which ensures that a local maximizer is also a global maximizer as will be shown later. Another important property of the more restrictive convex fuzzy sets is that any convex combination of such convex fuzzy sets is also a convex fuzzy set. It will be shown that this property is also lacking in the usual convex fuzzy sets which will be called quasiconvex fuzzy sets in this paper.

In this paper a fuzzy set with membership function $\mu: R^n \rightarrow [0, 1]$ satisfying (1.1) will be called a quasiconvex fuzzy set; and a strictly quasiconvex fuzzy set if strict inequality holds for all $x, y \in \text{supp}(\mu), x \neq y$ and $\lambda \in (0, 1)$.

We shall say, a fuzzy set with membership function $\mu: R^n \rightarrow [0, 1]$ is a convex fuzzy set if

$$\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y)$$

for all $x, y \in \text{supp}(\mu)$ and $\lambda \in [0, 1]$; and a strictly convex fuzzy set if strict inequality holds for all $x, y \in \text{supp}(\mu), x \neq y$ and $\lambda \in (0, 1)$.

From the above definitions, it can be seen that any convex fuzzy set is a quasiconvex fuzzy set (but not vice versa) and that any strictly convex fuzzy set is a convex fuzzy set (but not vice versa). It can also be easily checked that any strictly convex fuzzy set is a strictly quasiconvex fuzzy set, and that any strictly quasiconvex fuzzy set is a quasiconvex fuzzy set. From the above arguments, we can, therefore, conclude that strictly convex fuzzy sets, convex fuzzy sets and strictly quasiconvex fuzzy sets are all quasiconvex fuzzy sets.

A fuzzy set with membership function $\mu: R^n \rightarrow [0, 1]$ is said to be nonempty if $\text{supp}(\mu) \neq \emptyset$. In what follows, let $\mathcal{F}(R^n)$ denote the set of all nonempty fuzzy sets in $R^n$. The intersection of two fuzzy sets $\mu_1, \mu_2 \in \mathcal{F}(R^n)$, denoted by $\mu_1 \wedge \mu_2$, is defined for all $x \in R^n$ by

$$(\mu_1 \wedge \mu_2)(x) = \min \{\mu_1(x), \mu_2(x)\}, \quad (1.2)$$

where the right-hand side of (1.2) denotes the minimum of $\mu_1(x)$ and $\mu_2(x)$. Let $\mu_1, \mu_2, \ldots, \mu_k \in \mathcal{F}(R^n)$, then it can be easily checked that

$$\text{supp} \left( \bigwedge_{j=1}^{k} \mu_j \right) = \bigcap_{j=1}^{k} \text{supp} (\mu_j). \quad (1.3)$$

### 2. PRELIMINARIES

For the convenience, several definitions and results without proof from [1,2,5,6] will be listed below.
For $0 \leq \alpha \leq 1$, the $\alpha$-level set of a fuzzy set $\mu : \mathbb{R}^n \to [0, 1]$ is defined as
\[
[\mu]_\alpha = \begin{cases}
\{ x \in \mathbb{R}^n \mid \mu(x) \geq \alpha \}, & \text{if } 0 < \alpha \leq 1; \\
\text{cl}(%(\text{supp}(\mu)), & \text{if } \alpha = 0,
\end{cases}
\]
where cl(supp($\mu$)) denotes the closure of supp($\mu$).

A fuzzy set $\mu : \mathbb{R}^n \to [0, 1]$ is normal if there exists a point $x \in \mathbb{R}^n$ such that $\mu(x) = 1$, i.e., $[\mu]_1 \neq \emptyset$. A fuzzy number we treat in this study is a quasiconvex fuzzy set $\mu : \mathbb{R}^1 \to [0, 1]$ which is normal and upper semicontinuous.

It is easily verified that the $\alpha$-level set of a fuzzy number $\mu$ is a closed and bounded interval, which can be represented as: $[\mu]_\alpha = [a(\alpha), b(\alpha)]$, where the limits $a(\alpha) = -\infty$ and $b(\alpha) = \infty$ are admissible, and that a fuzzy set $\mu : \mathbb{R}^1 \to [0, 1]$ is a fuzzy number if and only if

(i) $[\mu]_\alpha$ is a closed and bounded interval for each $\alpha \in [0, 1]$, and

(ii) $[\mu]_1 \neq \emptyset$.

The most widely used fuzzy numbers are the so-called trapezoidal or triangular fuzzy numbers. This is because of the fuzzy or approximate nature of the problem and a straight line instead of nonlinear curve is good enough approximation. Trapezoidal fuzzy numbers are especially important in applications in fuzzy optimization problem.

**Definition 2.1.** A trapezoidal fuzzy number $\mu : \mathbb{R}^1 \to [0, 1]$ is specified by four parameters \{a, b, c, d\} as following
\[
\mu(x) = \begin{cases}
\frac{x - a}{b - a}, & a \leq x < b, \\
1, & b \leq x < c, \\
\frac{x - d}{c - d}, & c \leq x \leq d, \\
0, & \text{otherwise},
\end{cases}
\]
where $a, b, c, d \in \mathbb{R}^1$ and $(a \leq b \leq c \leq d)$. A triangular fuzzy number can be considered as a special case of the trapezoidal fuzzy number with $b = c$. Due to their simple formulas and the ease of computation, both trapezoidal and triangular fuzzy numbers are most commonly used in practice.

A fuzzy set $\mu : \mathbb{R}^1 \to [0, 1]$ is said to be open left if
\[
\lim_{x \to -\infty} \mu(x) = 1 \text{ and } \lim_{x \to \infty} \mu(x) = 0;
\]
Likewise, a fuzzy set $\mu : \mathbb{R}^1 \to [0, 1]$ is said to be open right if
\[
\lim_{x \to -\infty} \mu(x) = 0 \text{ and } \lim_{x \to \infty} \mu(x) = 1.
\]

**Definition 2.2.** An open left trapezoidal fuzzy number $\mu : \mathbb{R}^1 \to [0, 1]$ is specified by two parameters \{c, d\} as following
\[
\mu(x) = \begin{cases}
1, & x \leq c, \\
\frac{x - d}{c - d}, & c \leq x \leq d, \\
0, & \text{otherwise},
\end{cases}
\]
where $c, d \in \mathbb{R}^1$ and $(c < d)$. Likewise, an open right trapezoidal fuzzy number $\mu : \mathbb{R}^1 \to [0, 1]$ is specified by two parameters \{a, b\} as following
\[
\mu(x) = \begin{cases}
\frac{x - a}{b - a}, & a \leq x \leq b, \\
1, & x \geq b, \\
0, & \text{otherwise},
\end{cases}
\]
where $a, b \in \mathbb{R}^1$ and $(a < b)$.
 Remark 2.1. It can be easily checked that open left (resp., right) trapezoidal fuzzy numbers are nonincreasing (resp., nondecreasing), and that triangular, trapezoidal, open left trapezoidal and open right trapezoidal fuzzy numbers are all not only quasiconvex fuzzy sets, but also convex fuzzy sets.

It can be easily seen from Remark 2.1 that the maximum of a convex fuzzy set can be attained at more than one point.

3. MAIN RESULTS

We have seen that strictly convex fuzzy sets, convex fuzzy sets and strictly quasiconvex fuzzy sets are all quasiconvex fuzzy sets. It can be easily checked that supp(μ) of any quasiconvex fuzzy set μ : R^n → [0, 1] is a (crisp) convex subset of R^n. It follows that any of the aforementioned fuzzy sets has convex support. Thus, we conclude that a fuzzy set μ : R^n → [0, 1] is a convex (resp., quasiconvex) fuzzy set if it is a concave (resp., quasiconcave) function in the common sense on its support, and that a fuzzy set μ : R^n → [0, 1] is a strictly convex (resp., strict quasiconvex) fuzzy set if it is a strictly concave (resp., strictly quasiconcave) function in the common sense on its support. Due to this observation, the following extremum properties can be established immediately from well known extremum properties of concave and generalized concave functions (for details, see [12,13]).

Theorem 3.1. Let μ ∈ F(R^n) be a convex fuzzy set, and let x* ∈ supp(μ) be a local maximizer of μ. Then x* is also a global maximizer of μ over supp(μ). If μ is a strictly convex fuzzy set, then x* is the unique global maximizer.

As mentioned in the introduction, not every local maximizer of a quasiconvex fuzzy set is a global maximizer. However, that are not global cannot be strict local maximizers, as we state below.

Theorem 3.2. Let μ ∈ F(R^n) be a quasiconvex fuzzy set. If x* ∈ supp(μ) is a strict local maximizer of μ, then x* is also a strict global maximizer of μ over supp(μ). The set of points at which μ attains its global maximum over its support is a (crisp) convex set.

Theorem 3.3. Let μ ∈ F(R^n) be a strictly quasiconvex fuzzy set.

1) If x* ∈ supp(μ) is a local maximizer of μ, then it is the unique global maximizer.
2) μ attains its maximum over its support at no more than one point.

Definition 3.1. Let x, y ∈ R^n. The line segment [x, y] (with endpoints x and y) is the segment \{γx + (1 − γ)y : 0 ≤ γ ≤ 1\}. If x ≠ y, the interior (x, y) of [x, y] is the segment \{γx + (1 − γ)y : 0 < γ < 1\}. In a similar way, we can define [x, y) and (x, y].

We have already seen that convexity and strict quasiconvexity of fuzzy sets both ensure that every local maximizer is also a global maximizer, and that the maximum of a convex fuzzy set can be attained at more than one point. However, a strictly quasiconvex fuzzy set μ : R^n → [0, 1] attains its maximum at no more than one point. We can, therefore, conclude that strict quasiconvexity for fuzzy sets is not a proper generalization of convexity, but only of strict convexity. The following theorem which is motivated by Thompson [14] gives sufficient conditions for strict quasiconvexity of fuzzy sets.

Theorem 3.4. Let μ : R^n → [0, 1] be a fuzzy set with nonempty convex support. If μ assumes its least upper bound and has a unique local maximizer on every closed interval in its support, then μ is a strictly quasiconvex fuzzy set.

Proof. The proof is by contradiction. Suppose that μ is not a strictly quasiconvex fuzzy set. Then there exist distinct points ̄x, ̄y ∈ supp(μ) and ̄x ∈ (̄x, ̄y) ⊆ supp(μ) such that

μ(̄x) ≤ min \{μ(̄x), μ(̄y)\}.
Let

\[ u = \sup_{x \in [\bar{x}, \bar{y}]} \mu(x) \]

and

\[ v = \sup_{y \in [\bar{z}, \bar{g}]} \mu(y). \]

Since \( \mu(\bar{z}) \leq \min \{\mu(\bar{x}), \mu(\bar{y})\} \), by the hypothesis of the theorem, we may take \( u \neq \bar{z} \) and \( v \neq \bar{z} \). It follows that \( u \) and \( v \) are two local maximizers of \( \mu \) on \([\bar{x}, \bar{y}]\), which contradicts the assumption that \( \mu \) has a unique local maximizer on every closed interval in its support.

In view of the definitions of convex fuzzy sets, quasiconvex fuzzy sets, convex functions in common sense, and quasiconvex functions in common sense, the following composition rules can be easily established.

**Theorem 3.5.** Let \( h : \mathbb{R}^n \to \mathbb{R}^1 \) be a convex (resp., concave) function in common sense and let \( \nu : \mathbb{R}^1 \to [0, 1] \) be a nonincreasing (resp., nondecreasing) convex fuzzy set. Then the composite function \( \mu : \mathbb{R}^n \to [0, 1] \), defined by

\[
\mu(x) = \begin{cases} 
\nu(h(x)), & \text{if } h(x) \in \text{supp}(\nu); \\
0, & \text{if } h(x) \notin \text{supp}(\nu),
\end{cases}
\]

is a convex fuzzy set.

**Proof.** Assume that \( x, y \in \text{supp}(\mu) \). Then \( h(x), h(y) \in \text{supp}(\nu) \). Since \( h : \mathbb{R}^n \to \mathbb{R}^1 \) is a convex function, and \( \nu : \mathbb{R}^1 \to [0, 1] \) is nonincreasing, we have for every \( \lambda \in [0, 1] \)

\[
\mu(\lambda x + (1 - \lambda)y) = \nu(h(\lambda x + (1 - \lambda)y)) \geq \nu(\lambda h(x) + (1 - \lambda)h(y)).
\]

Since \( h(x), h(y) \in \text{supp}(\nu) \), by the fuzzy convexity of \( \nu \), it follows that

\[
\nu(\lambda h(x) + (1 - \lambda)h(y)) \geq \lambda \nu(h(x)) + (1 - \lambda)\nu(h(y))
\]

\[
= \lambda \mu(x) + (1 - \lambda)\mu(y).
\]

Hence, we obtain

\[
\mu(\lambda x + (1 - \lambda)y) \geq \lambda \mu(x) + (1 - \lambda)\mu(y)
\]

for every \( \lambda \in [0, 1] \), which completes the proof.

From Remark 2.1 and Theorem 3.5, we obtain the following corollary.

**Corollary 3.1.** Let \( h : \mathbb{R}^n \to \mathbb{R}^1 \) be a convex (resp., concave) function in common sense and let \( \nu : \mathbb{R}^1 \to [0, 1] \) be an open left (open right) trapezoidal fuzzy number. Then the composite function \( \mu : \mathbb{R}^n \to [0, 1] \), defined by

\[
\mu(x) = \begin{cases} 
\nu(h(x)), & \text{if } h(x) \in \text{supp}(\nu); \\
0, & \text{if } h(x) \notin \text{supp}(\nu),
\end{cases}
\]

is a convex fuzzy set.
THEOREM 3.6. Let \( h : \mathbb{R}^n \rightarrow \mathbb{R}^1 \) be a quasiconvex (resp., quasiconcave) function in common sense and let \( \nu : \mathbb{R}^1 \rightarrow [0, 1] \) be a nonincreasing (resp., nondecreasing) fuzzy set. Then the composite function \( \mu : \mathbb{R}^n \rightarrow [0, 1] \), defined by

\[
\mu(x) = \begin{cases} 
\nu(h(x)), & \text{if } h(x) \in \text{supp}(\nu); \\
0, & \text{if } h(x) \notin \text{supp}(\nu),
\end{cases}
\]

is a quasiconvex fuzzy set.

PROOF. Assume that \( x, y \in \text{supp}(\mu) \). Then \( h(x), h(y) \in \text{supp}(\nu) \). Since \( h : \mathbb{R}^n \rightarrow \mathbb{R}^1 \) is a quasiconvex function, and \( \nu : \mathbb{R}^1 \rightarrow [0, 1] \) is nonincreasing, we have for every \( \lambda \in [0, 1] \)

\[
\mu(\lambda x + (1 - \lambda)y) = \nu(h(\lambda x + (1 - \lambda)y)) \\
\geq \nu(\max\{h(x), h(y)\}) \\
= \min\{\nu(h(x)), \nu(h(y))\} \\
= \min\{\mu(x), \mu(y)\},
\]

which completes the proof.

From Remark 2.1 and Theorem 3.6, we obtain the following corollary.

COROLLARY 3.2. Let \( h : \mathbb{R}^n \rightarrow \mathbb{R}^1 \) be a quasiconvex (resp., quasiconcave) function in common sense and let \( \nu : [0, 1] \rightarrow \mathbb{R}^1 \) be an open left (resp., open right) trapezoidal fuzzy number. Then the composite function \( \mu : \mathbb{R}^n \rightarrow [0, 1] \), defined by

\[
\mu(x) = \begin{cases} 
\nu(h(x)), & \text{if } h(x) \in \text{supp}(\nu); \\
0, & \text{if } h(x) \notin \text{supp}(\nu),
\end{cases}
\]

is a quasiconvex fuzzy set.

In view of definitions of convex fuzzy sets and strictly convex fuzzy sets, the following result can be easily established.

THEOREM 3.7. Let \( \mu_j : \mathbb{R}^n \rightarrow [0, 1], j = 1, 2, \ldots, k, \) be convex fuzzy sets with \( \bigcap_{j=1}^k \text{supp}(\mu_j) \neq \emptyset \). For \( \gamma_1, \gamma_2, \ldots, \gamma_k > 0 \) with \( \sum_{j=1}^k \gamma_j = 1 \), the fuzzy set \( \mu : \mathbb{R}^n \rightarrow [0, 1] \), defined by

\[
\mu(x) = \begin{cases} 
\sum_{j=1}^k \gamma_j \mu_j(x), & \text{if } x \in \bigcap_{j=1}^k \text{supp}(\mu_j); \\
0, & \text{elsewhere},
\end{cases}
\]

is a convex fuzzy set with \( \text{supp}(\mu) = \bigcap_{j=1}^k \text{supp}(\mu_j) \). If at least one \( \mu_j, j = 1, 2, \ldots, k, \) is a strictly convex fuzzy set, then \( \mu \) is strictly convex.

REMARK 3.1. Since all t-norm operators are bounded above by minimum (for details, see [15]), we restrict ourselves to the aggregated fuzzy set \( \mu \) of \( \mu_1, \mu_2, \ldots, \mu_k \) on \( \bigcap_{j=1}^k \text{supp}(\mu_j) \).

REMARK 3.2. Theorem 3.8 states that a strict convex combination of convex fuzzy sets is also a convex fuzzy set. As will be shown later, this property is very important in fuzzy decision making.

Furthermore as shown below, besides the local-global maximizer property, this is another property that is lacking in quasiconvex fuzzy sets.

EXAMPLE 3.1. Let \( \mu_1 \) and \( \mu_2 \) be defined below:

\[
\mu_1(x) = \begin{cases} 
\frac{2}{3}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\
1, & \text{if } \frac{1}{2} < x \leq 1,
\end{cases}
\]
\[
\mu_2(x) = \begin{cases} 
1, & \text{if } 0 \leq x < \frac{1}{2}; \\
\frac{1}{2}, & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

By definition, \(\mu_1\) and \(\mu_2\) are quasiconvex fuzzy sets with \(\text{supp}(\mu_1) = \text{supp}(\mu_2) = [0, 1]\).

Define
\[
\mu(x) = \begin{cases} 
\frac{1}{2}\mu_1(x) + \frac{1}{2}\mu_2(x), & \text{if } x \in [0, 1]; \\
0, & \text{elsewhere}.
\end{cases}
\]

Then it can be easily checked that
\[
\mu(x) = \begin{cases} 
\frac{5}{6}, & 0 \leq x < \frac{1}{2}, \\
\frac{7}{12}, & x = \frac{1}{2}, \\
\frac{3}{4}, & \frac{1}{2} < x \leq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Recall [1] that a fuzzy set is a quasiconvex fuzzy set if and only if its \(\alpha\)-level set is a convex set for each \(\alpha \in (0, 1]\). It follows that the fuzzy set \(\mu\) is not a quasiconvex fuzzy set since its \(\alpha\)-level set, \([0, 1/2) \cup (1/2, 1]\), for \(\alpha = 9/12\) is not a convex set.

4. APPLICATIONS TO FUZZY OPTIMIZATION PROBLEMS

Obviously, the above fuzzy convexity and aggregation results can be used to optimize fuzzy problems under different conditions. In the following, we shall present several ideas of applications.

First, let us briefly summarize the essence of Bellman and Zadeh's general approach to decision making under fuzziness [11]. A collection of \(l\) fuzzy objective functions \(G_1, G_2, \ldots, G_l\), and \(m\) fuzzy constraints \(C_1, C_2, \ldots, C_m\), defined on the decision space \(X \subseteq \mathbb{R}^3\), are assumed to be given. A fuzzy decision \(D\) in \(X\) is defined by its membership function,
\[
\mu_D(x) = \mu_{G_1}(x) \ast \mu_{G_2}(x) \ast \ldots \ast \mu_{G_l}(x)
\ast \mu_{C_1}(x) \ast \mu_{C_2}(x) \ast \ldots \ast \mu_{C_m}(x),
\]
where \(x \in X\) and \(\ast\) denotes an appropriate aggregation operator. Many different aggregation operators have been proposed. In general, \(t\)-norm aggregation operators are preferred. Due to computational tractability and simplicity, the most commonly used aggregation operator is the minimum operator. The biggest disadvantage of this operator is that it is completely non-compensatory. It is desirable to use compensatory operators. The following arithmetical average aggregation operator, \(\mu_D(x)\), is fully compensatory (for details, see [16]).
\[
\mu_D(x) = \frac{1}{l + m} \left( \sum_{i=1}^{l} \mu_{G_i}(x) + \sum_{j=1}^{m} \mu_{C_j}(x) \right).
\]

In addition, Bellman and Zadeh [11] pointed out that \(D\) might be expressed as a convex combination of the goals and constraints, with weighting coefficients reflecting the relative importance of the various terms.
If there exists a subset $M \subseteq X$ for which $\mu_D(x)$ reaches its maximum, then $M$ is called the set of maximizing decisions.

Let $\mu_j \in \mathcal{F}(\mathbb{R}^n)$, $j = 1, 2, \ldots, k$, be given fuzzy criteria and

$$X = \bigcap_{j=1}^{k} \text{supp}(\mu_j) \neq \emptyset$$

be the set of feasible points. Recall that, by definition, a point $\tilde{x} \in X$ is said to be an efficient solution to the multiple objectives, $\mu_1, \ldots, \mu_k$, if no one objective can be improved without a simultaneous detriment to at least one of the other objectives. An efficient solution is also known as nondominated solution or a Pareto-optimal solution.

The concept of proper efficiency given by Geoffrion [17] is a slightly restricted definition of efficiency that eliminates efficient points of a certain anomalous type: a point $\tilde{x} \in X$ is said to be a properly efficient solution to the multiple objectives, $\mu_1, \ldots, \mu_k$, if it is efficient and there exists a scalar $K > 0$ such that, for each $i = 1, 2, \ldots, k$, we have

$$\frac{\mu_i(x) - \mu_i(\tilde{x})}{\mu_j(\tilde{x}) - \mu_j(x)} \leq K,$$

for some $j$ such that $\mu_j(x) < \mu_j(\tilde{x})$ whenever $x \in X$ and $\mu_i(x) > \mu_i(\tilde{x})$. Direct examination from the definition of proper efficiency shows that: An efficient solution $\tilde{x} \in X$ that is not properly efficient means that to every scalar $K > 0$ (no matter how large) there is a point $x \in X$ and an $i$ such that $\mu_i(x) > \mu_i(\tilde{x})$ and

$$\frac{\mu_i(x) - \mu_i(\tilde{x})}{\mu_j(\tilde{x}) - \mu_j(x)} > K,$$

for all $j$ such that $\mu_j(x) < \mu_j(\tilde{x})$.

Motivated by Geoffrion [17], we obtain the following result.

**Theorem 4.1.** Let $\mu_j \in \mathcal{F}(\mathbb{R}^n)$, $j = 1, 2, \ldots, k$, be given fuzzy criteria and

$$X = \bigcap_{j=1}^{k} \text{supp}(\mu_j) \neq \emptyset$$

be the set of feasible points. For $\gamma_1, \gamma_2, \ldots, \gamma_k > 0$ with

$$\sum_{j=1}^{k} \gamma_j = 1,$$

if $\tilde{x} \in X$ is a global maximizer of the aggregated fuzzy decision,

$$\mu(x) = \begin{cases} 
\sum_{j=1}^{k} \gamma_j \mu_j(x), & \text{if } x \in \bigcap_{j=1}^{k} \text{supp}(\mu_j); \\
0, & \text{elsewhere},
\end{cases}$$

then $\tilde{x}$ is properly efficient to the multiple objectives, $\mu_1, \ldots, \mu_k$.

**Proof.** Let $\tilde{x} \in X$ be a global maximizer of the aggregated fuzzy decision

$$\mu(x) = \begin{cases} 
\sum_{j=1}^{k} \gamma_j \mu_j(x), & \text{if } x \in \bigcap_{j=1}^{k} \text{supp}(\mu_j); \\
0, & \text{elsewhere},
\end{cases}$$
of $\mu_1, \mu_2, \ldots, \mu_k$, then it can be easily checked that $x$ is an efficient solution to the multiple objectives, $\mu_1, \ldots, \mu_k$. Suppose, on the contrary, there exists $x \in X$ such that $\mu_j(x) \geq \mu_j(x)$ for all $j$ and $\mu_j(x) > \mu_j(x)$ for at least one $j$. Hence, it follows that

$$\mu(x) = \sum_{j=1}^{k} \gamma_j \mu_j(x) > \sum_{j=1}^{k} \gamma_j \mu_j(x) = \mu(x),$$

which contradicts the assumption that $x$ is a global maximizer of $\mu$.

We now show that $x$ is properly efficient to the multiple objectives, $\mu_1, \ldots, \mu_k$. Suppose, by contradiction, that $x$ is an efficient solution but not properly efficient to the multiple objectives, $\mu_1, \ldots, \mu_k$. Then for every scalar $K > 0$, there is a point $x \in X$ and an $i$ such that $\mu_i(x) > \mu_i(x)$ and

$$\frac{\mu_i(x) - \mu_i(x)}{\mu_j(x) - \mu_j(x)} > K$$

for all $j$ such that $\mu_j(x) < \mu_j(x)$. By taking

$$K = (k - 1) \max_{i,j} \left\{ \frac{\gamma_j}{\gamma_i} \right\},$$

it follows that

$$\mu_i(x) - \mu_i(x) > \frac{(k - 1)}{\gamma_i} \gamma_j (\mu_j(x) - \mu_j(x)), \quad \text{for all } j \neq i.$$

Multiplying the above inequality by $\gamma_i/(k - 1)$ and summing over $j \neq i$ yields

$$\gamma_i (\mu_i(x) - \mu_i(x)) > \sum_{j \neq i} \gamma_j (\mu_j(x) - \mu_j(x)),$$

which implies that $\mu(x) < \mu(x)$, which contradicts the assumption that $x$ is a global maximizer of the aggregated fuzzy decision $\mu$.

We now discuss some applications of convex and quasiconvex fuzzy sets to fuzzy decision making. Assume that we are given $l$ fuzzy goals $G_1, G_2, \ldots, G_l$, and $m$ fuzzy constraints $C_1, C_2, \ldots, C_m$ in $R^n$ such that

$$X = \left( \bigcap_{i=1}^{l} \text{supp}(\mu_{G_i}) \right) \cap \left( \bigcap_{j=1}^{m} \text{supp}(\mu_{C_j}) \right) \neq \emptyset.$$

Let $D_1$ be the resulting fuzzy decision by using the minimum aggregation operator of the goals and constraints, and let $D_2$ be the fuzzy decision defined by

$$\mu_{D_2}(x) = \begin{cases} \sum_{i=1}^{l} \gamma_i \mu_{G_i}(x) + \sum_{j=1}^{m} \gamma_{l+j} \mu_{C_j}(x), & \text{if } x \in X; \\ 0, & \text{if } x \notin X, \end{cases}$$

for some $\gamma_1, \ldots, \gamma_{l+m} > 0$ with $\gamma_1 + \cdots + \gamma_l + \cdots + \gamma_{l+m} = 1$. Let $M_1$ and $M_2$ be subsets of $X$ for which $\mu_{D_1}(x)$ and $\mu_{D_2}(x)$ reach their maximum, respectively.

It is known [1,6] that the intersection of a finite number of convex (resp., strictly convex) fuzzy sets is also a convex (resp., strictly convex) fuzzy set, and that the intersection of a finite number of quasiconvex (resp., strictly quasiconvex) fuzzy sets is also a quasiconvex (resp., strictly quasiconvex) fuzzy set. From Theorems 3.1–3.3, we obtain the following theorem.
THEOREM 4.2. Let $\tilde{x} \in M_1$.

1. If the fuzzy goals $G_1, G_2, \ldots, G_l$, and fuzzy constraints $C_1, C_2, \ldots, C_m$, are convex (resp., quasiconvex) fuzzy sets, then the fuzzy decision $D_1$ is a convex (resp., quasiconvex) fuzzy set with $\text{supp}(\mu_{D_1}) = X$ and $M_1$ is a (crisp) convex set.

2. If the fuzzy goals $G_1, G_2, \ldots, G_l$, and fuzzy constraints $C_1, C_2, \ldots, C_m$, are strictly convex (resp., strictly quasiconvex) fuzzy sets, then the fuzzy decision $D_1$ is a strictly convex (resp., strictly quasiconvex) fuzzy set with $\text{supp}(\mu_{D_1}) = X$, and $M_1 = \{\tilde{x}\}$.

REMARK 4.1. As pointed out in Lee and Li [16], the resulting fuzzy decision $D_1$ by using the minimum aggregation operator does not guarantee nondominated solutions. In contrast to the minimum operator, it follows from Theorem 4.1 that the maximizing decisions of $D_2$ are properly efficient solution to the fuzzy objectives and fuzzy constraints.

THEOREM 4.3. Let the fuzzy goals $G_1, G_2, \ldots, G_l$, and fuzzy constraints $C_1, C_2, \ldots, C_m$, be convex fuzzy sets, and let $\tilde{x} \in M_2$. Then $M_2$ is a (crisp) convex set and $\tilde{x}$ is properly efficient to the fuzzy objectives and fuzzy constraints. If at least one objective or constraint is a strictly convex fuzzy set, then $\tilde{x}$ is properly efficient and $M_2 = \{\tilde{x}\}$.

5. FUZZY NONLINEAR MULTIOBJECTIVE PROGRAMMING

Consider the following optimization problems (P1) (resp., P2) with convex (resp., quasiconvex) objective functions and convex (resp., quasiconvex) constraints:

$$\begin{align*}
\text{minimize} & \quad [f_1(x), f_2(x), \ldots, f_l(x)] \\
\text{subject to} & \quad g_j(x) \leq 0, \quad j = 1, \ldots, m,
\end{align*}$$

where $x \in \mathbb{R}^n$, and $f_1, \ldots, f_l, g_1, \ldots, g_m$ are convex (resp., quasiconvex) functions in the common sense on $\mathbb{R}^n$. In order to transform problems (P1) and (P2) in a more tractable form, we transform $f_i(x), i = 1, \ldots, l$, to fuzzy goals, and $g_j(x), j = 1, \ldots, m$, to fuzzy constraints. More precisely, by the use of suitable nonincreasing transformations

$$
\begin{align*}
\eta_i : \mathbb{R}^1 & \rightarrow [0, 1], \quad i = 1, \ldots, l, \\
\theta_j : \mathbb{R}^1 & \rightarrow [0, 1], \quad j = 1, \ldots, m,
\end{align*}
$$

we get the corresponding fuzzy goals and fuzzy constraints,

$$
\begin{align*}
\mu_{G_i}(x) & = \begin{cases} 
\eta_i(f_i(x)), & \text{if } f_i(x) \in \text{supp}(\eta_i); \\
0, & \text{if } f_i(x) \notin \text{supp}(\eta_i),
\end{cases} \\
\mu_{C_j}(x) & = \begin{cases} 
\theta_j(g_j(x)), & \text{if } g_j(x) \in \text{supp}(\theta_j); \\
0, & \text{if } g_j(x) \notin \text{supp}(\theta_j).
\end{cases}
\end{align*}
$$

From Theorem 3.5, we obtain the following theorem.

THEOREM 5.1. For solving problem (P1) by fuzzy optimization, suppose that

$$\begin{align*}
\eta_i : \mathbb{R}^1 & \rightarrow [0, 1] \\
\theta_j : \mathbb{R}^1 & \rightarrow [0, 1]
\end{align*}$$

and
are chosen to be suitable nonincreasing convex fuzzy sets, say suitable open left trapezoidal fuzzy numbers, then the corresponding fuzzy goals and fuzzy constraints

\[ \mu_{G_i}(x) = \begin{cases} \eta_i(f_i(x)), & \text{if } f_i(x) \in \text{supp}(\eta_i); \\ 0, & \text{if } f_i(x) \notin \text{supp}(\eta_i), \end{cases} \]

and

\[ \mu_{C_j}(x) = \begin{cases} \theta_j(g_j(x)), & \text{if } g_j(x) \in \text{supp}(\theta_j); \\ 0, & \text{if } g_j(x) \notin \text{supp}(\theta_j), \end{cases} \]

are convex fuzzy sets.

**Remark 5.1.** In Theorem 5.1, the min operator, the arithmetic average, or strictly convex combination can be used to aggregate the resulting convex fuzzy goals \( \mu_{G_i}(x) \) and convex fuzzy constraints \( \mu_{C_j}(x) \), and the resulting fuzzy decision is also a convex fuzzy set.

From Theorem 3.6, we obtain the following theorem.

**Theorem 5.2.** For solving problem (P2) by fuzzy optimization, suppose that

\[ \eta_i : R^1 \rightarrow [0, 1] \]

and

\[ \theta_j : R^1 \rightarrow [0, 1] \]

are chosen to be suitable nonincreasing fuzzy sets, say suitable open left trapezoidal fuzzy numbers, then the corresponding fuzzy goals and fuzzy constraints

\[ \mu_{G_i}(x) = \begin{cases} \eta_i(f_i(x)), & \text{if } f_i(x) \in \text{supp}(\eta_i); \\ 0, & \text{if } f_i(x) \notin \text{supp}(\eta_i), \end{cases} \]

and

\[ \mu_{C_j}(x) = \begin{cases} \theta_j(g_j(x)), & \text{if } g_j(x) \in \text{supp}(\theta_j); \\ 0, & \text{if } g_j(x) \notin \text{supp}(\theta_j), \end{cases} \]

are quasiconvex fuzzy sets.

**Remark 5.2.** In Theorem 5.2, the min operator can be used to aggregate the resulting quasiconvex fuzzy goals \( \mu_{G_i}(x) \) and quasiconvex fuzzy constraints \( \mu_{C_j}(x) \), and the resulting fuzzy decision is also a quasiconvex fuzzy set.

**References**