Generalizations of $E$-convex and B-vex functions

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ABSTRACT

A class of functions called $E$-B-vex functions is defined as a generalization of $E$-convex and B-vex functions. Similarly, a class of $E$-B-preinvex functions, which are generalizations of $E$-convex and B-preinvex functions, is introduced. In addition, the concept of B-linear functions is also generalized to $E$-B-linear functions. Some properties of these proposed classes are studied. Furthermore, the equivalence between the class of $E$-B-vex functions and that of $E$-quasiconvex functions is proved.

1. Introduction

Convexity and generalized convexity play important roles in optimization theory. Various generalizations of convexity have appeared in the literature. A significant generalization of convex functions is preinvex functions, introduced by Hanson and Mond [1] but so named by Jeyakumar [2]. Recently, another generalization of convex functions, called B-vex functions, was introduced by Bector and Singh [3]. Later, Suneja et al. [4] introduced a class of functions called B-preinvex functions which are generalizations of preinvex and B-vex functions. Li et al. [5] proved that the class of B-vex functions is equivalent to that of quasiconvex functions.

Youness [6] introduced a class of sets and a class of functions, called $E$-convex sets and $E$-convex functions, which generalize the definitions of convex sets and convex functions based on the effect of an operator $E$ on the sets and domain of definition of the functions. The initial results of Youness [6] inspired a great deal of subsequent work which has greatly expanded the role of $E$-convexity in optimization theory; see for example [7–11]. In an earlier paper [10], we introduced a class of functions, called $E$-quasiconvex functions, which are a generalization of $E$-convex functions and quasiconvex functions. Fulga and Preda [9] extended the classes of preinvex and $E$-convex functions to $E$-prequasiinvex functions.

Motivated both by earlier research works [3,9,4,10,6] and by the importance of convexity and generalized convexity, we introduce a class of functions called $E$-B-vex functions which are generalizations of $E$-convex and B-vex functions, and a class of $E$-B-preinvex functions which are generalizations of $E$-convex and B-preinvex functions. In addition, the concept of B-linear functions is also generalized to $E$-B-linear functions. Some properties of these proposed classes are studied. Furthermore, the equivalence between the class of $E$-B-vex functions and that of $E$-quasiconvex functions is proved.

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2. Preliminaries

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space; let \( X \) be a nonempty subset of \( \mathbb{R}^n \); let \( \mathbb{R}^a \) denote the set of nonnegative real numbers; let \( b : X \times X \times [0, 1] \rightarrow \mathbb{R}^n \), with \( \lambda b(x, y, \lambda) \in [0, 1] \) for all \( x, y \in X \) and \( \lambda \in [0, 1] \).

Following along the lines of Bector and Singh [3], the definitions of \( B \)-vex and \( B \)-linear functions can be given as follows.

**Definition 2.1** ([Ref. 5]). Let \( C \subseteq X \) be a nonempty convex set. A function \( f : C \rightarrow \mathbb{R}^1 \) is said to be:

1. \( B \)-vex on \( C \) with respect to (w.r.t. in short) \( b(x, y, \lambda) \) if for all \( x, y \in C \) and \( \lambda \in [0, 1] \),
   \[
   f(\lambda x + (1 - \lambda) y) \leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y);
   \]
2. \( B \)-linear on \( C \) w.r.t. \( b(x, y, \lambda) \) if for all \( x, y \in C \) and \( \lambda \in [0, 1] \),
   \[
   f(\lambda x + (1 - \lambda) y) = \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y).
   \]

For the sake of brevity, we shall omit the argument of \( b \) unless it is needed for specification.

Recall [12] that, by definition, a set \( K \subseteq \mathbb{R}^n \) is called an invex set w.r.t. a given mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) if

\[
\lambda \eta(x) + (1 - \lambda)\eta(y) \in M, \quad \forall x, y \in M, \quad \forall \lambda \in [0, 1].
\]

In what follows, let \( E : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be two fixed mappings. Now, we state the concept of \( E \)-invex sets which are generalizations of invex and \( E \)-convex sets.

**Definition 2.2** ([Ref. 9, Definition 2.2]). A set \( A \subseteq \mathbb{R}^n \) is said to be \( E \)-invex w.r.t \( \eta \) if

\[
x, y \in A, \quad \lambda \in [0, 1] \implies E(y) + \lambda \eta(E(x), E(y)) \in A.
\]

Let \( S \) be a nonempty subset of \( \mathbb{R}^n \); \( E(S) \) is defined as follows:

\[
E(S) = \{ E(x) : x \in S \}.
\]

**Lemma 2.1** ([Ref. 6, Proposition 2.2]). Let \( M \subseteq \mathbb{R}^n \) be a nonempty \( E \)-convex set; then \( E(M) \subseteq M \).

**Lemma 2.2** ([Ref. 9, Lemma 2.1]). Let \( A \subseteq \mathbb{R}^n \) be a nonempty \( E \)-invex set; then \( E(A) \subseteq A \).

Finally, we describe several generalized convex functions, viz. preinvex, \( B \)-preinvex, \( E \)-convex, \( E \)-quasiconvex, and \( E \)-preinvex.

**Definition 2.3** ([Ref. 12]). Let \( K \subseteq \mathbb{R}^n \) be a nonempty invex set w.r.t. \( \eta \). A function \( f : K \rightarrow \mathbb{R}^1 \) is said to be preinvex on \( K \) w.r.t. \( \eta \), if for all \( x, y \in K \) and \( \lambda \in [0, 1] \),

\[
f(y + \lambda \eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y).
\]

**Definition 2.4** ([Ref. 4]). Let \( K \subseteq \mathbb{R}^n \) be a nonempty invex set w.r.t. \( \eta \). A function \( f : K \rightarrow \mathbb{R}^1 \) is said to be \( B \)-preinvex on \( K \) w.r.t. \( \eta, b \), if for all \( x, y \in K \) and \( \lambda \in [0, 1] \),

\[
f(y + \lambda \eta(x, y)) \leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y).
\]

**Definition 2.5** ([Refs. 10, 6]). Let \( M \subseteq \mathbb{R}^n \) be a nonempty \( E \)-convex set. A function \( f : M \rightarrow \mathbb{R}^1 \) is said to be:

1. \( E \)-convex on \( M \) if for all \( x, y \in M \) and \( \lambda \in [0, 1] \),
   \[
f(\lambda E(x) + (1 - \lambda) E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y));
   \]
2. \( E \)-quasiconvex on \( M \) if for all \( x, y \in M \) and \( \lambda \in [0, 1] \),
   \[
f(\lambda E(x) + (1 - \lambda) E(y)) \leq \max\{f(E(x)), f(E(y))\};
   \]
3. \( E \)-quasiconcave on \( M \) if for all \( x, y \in M \) and \( \lambda \in [0, 1] \),
   \[
f(\lambda E(x) + (1 - \lambda) E(y)) \geq \min\{f(E(x)), f(E(y))\}.
   \]

**Definition 2.6** ([Ref. 9, Definition 2.3]). Let \( A \subseteq \mathbb{R}^n \) be a nonempty \( E \)-invex set w.r.t. \( \eta \). A function \( f : A \rightarrow \mathbb{R}^1 \) is said to be \( E \)-preinvex on \( A \) w.r.t. \( \eta \) if for all \( x, y \in A \) and \( \lambda \in [0, 1] \),

\[
f(E(y) + \lambda \eta(E(x), E(y))) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)).
\]
3. Basic results

First, a class of E-B-vex functions is introduced as a generalization of E-convex and B-vex functions, and the concept of B-linear functions is also generalized to E-B-linear functions.

**Definition 3.1.** Let $M \subseteq X$ be a nonempty E-convex set. A function $f : M \rightarrow R^1$ is said to be:

1. E-B-convex on $M$ w.r.t. $b$ if $x, y \in M$ and $\lambda \in [0, 1],$
   \[ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \]
2. E-B-linear on $M$ w.r.t. $b$ if $x, y \in M$ and $\lambda \in [0, 1],$
   \[ f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y) \]

**Theorem 3.1.** Let $M \subseteq X$ be a nonempty E-convex set. A function $f : M \rightarrow R^1$ is well defined.

**Remark 3.1.** Let $M \subseteq R^n$ be a nonempty E-convex set. It follows from **Lemma 2.1** that $E(M) \subseteq M$. Hence, for any $f : M \rightarrow R^1$, the restriction $\tilde{f} : E(M) \rightarrow R^1$ of $f : M \rightarrow R^1$ to $E(M)$ defined by
   \[ \tilde{f}(\tilde{x}) = f(x) \quad \text{for all } \tilde{x} \in E(M) \]
is well defined.

Let $M \subseteq X$ be a nonempty E-convex set. Direct examination of the definition of E-B-vex (resp. E-B-linear) functions shows that the set of E-B-vex (resp. E-B-linear) functions on $M$ w.r.t. the same $b$ is closed under addition and nonnegative scalar multiplication. This is formalized in the following theorem.

**Theorem 3.1.** Let $M \subseteq X$ be a nonempty E-convex set, and let $\alpha \geq 0$. If $f$ and $g$ are E-B-convex (resp. E-B-linear) functions on $M$ w.r.t. the same $b$, then $f + g$ and $\alpha f$ are E-B-convex (resp. E-B-linear) functions on $M$ w.r.t. $b$.

**Corollary 3.1.** Let $M \subseteq X$ be a nonempty E-convex set. Let $f_j, j = 1, 2, \ldots, N$ be E-B-convex (resp. E-B-linear) functions on $M$ w.r.t. the same $b$. Then the function $f : M \rightarrow R^1$ defined by
   \[ f(x) = \sum_{j=1}^{N} k_j f_j(x), \quad k_j \geq 0, \]
is E-B-convex (resp. E-B-linear) on $M$ w.r.t. $b$.

Next, we introduce a new class of functions called E-B-preinvex functions by relaxing the definitions of E-convex and B-preinvex functions.

**Definition 3.2.** Let $A \subseteq X$ be a nonempty E-invex set w.r.t. $\eta$. A function $f : A \rightarrow R^1$ is said to be E-B-preinvex on $A$ w.r.t. $\eta$, $b$, if for all $x, y \in A$ and $\lambda \in [0, 1],$
   \[ f(\lambda y + (1 - \lambda)x) \leq \lambda f(y) + (1 - \lambda)f(x) \]

**Remark 3.2.** Let $A \subseteq R^n$ be a nonempty E-invex set. It follows from **Lemma 2.2** that $E(A) \subseteq A$. Hence, for any $f : A \rightarrow R^1$, the restriction $\tilde{f} : E(A) \rightarrow R^1$ of $f : A \rightarrow R^1$ to $E(A)$ defined by
   \[ \tilde{f}(\tilde{x}) = f(x) \quad \text{for all } \tilde{x} \in E(A) \]
is well defined.

An analogous result to **Theorem 3.1** for the E-B-preinvex case is as follows.

**Theorem 3.2.** Let $A \subseteq X$ be a nonempty E-invex set w.r.t. $\eta$, and let $\alpha \geq 0$. If $f$ and $g$ are E-B-preinvex functions on $A$ w.r.t. the same $\eta, b$, then $f + g$ and $\alpha f$ are E-B-preinvex functions on $A$ w.r.t. $\eta, b$.

**Corollary 3.2.** Let $A \subseteq X$ be a nonempty E-invex set. Let $f_j, j = 1, 2, \ldots, N$ be E-B-preinvex functions on $A$ w.r.t. the same $\eta, b$. Then the function $f : A \rightarrow R^1$ defined by
   \[ f(x) = \sum_{j=1}^{N} k_j f_j(x), \quad k_j \geq 0, \]
is E-B-preinvex on $A$ w.r.t. $\eta, b$.

Finally, we derive a property of E-preinvex functions.

**Theorem 3.3.** Let $A \subseteq X$ be a nonempty E-invex set w.r.t. $\eta$. Suppose that $f : A \rightarrow R^1$ is E-preinvex on $A$ w.r.t. $\eta$, and that $\phi : R^1 \rightarrow R^1$ is nondecreasing and convex. Then $\phi \circ f : A \rightarrow R^1$ is E-preinvex on $A$ w.r.t. $\eta$. 


Proof. Since $f : A \to R^1$ is $E$-preinvex on $A$ w.r.t. $\eta$, and $\phi : R^1 \to R^1$ is nondecreasing and convex, we have, for any $x, y \in A$ and $\lambda \in [0, 1]$,
\[
\phi \circ f(E(y) + \lambda \eta(E(x), E(y))) = \phi(f(E(y) + \lambda \eta(E(x), E(y)))) \\
\leq \phi(\lambda f(E(x)) + (1 - \lambda)f(E(y))) \\
\leq \lambda \phi(f(E(x))) + (1 - \lambda)\phi(f(E(y))) \\
= \lambda \phi \circ f(E(x)) + (1 - \lambda)\phi \circ f(E(y)).
\]
That is, $\phi \circ f : A \to R^1$ is $E$-preinvex on $A$ w.r.t. $\eta$. □

4. Main results

We first study the relations between $E$-B-vex and $B$-vex (resp. $E$-quasiconvex) functions.

Theorem 4.1. Let $M \subseteq X$ be a nonempty $E$-convex set, and let $C$ be a nonempty convex subset of $E(M)$. If $f : M \to R^1$ is $E$-B-vex on $M$ w.r.t. $b$, then the restriction $\hat{f} : C \to R^1$ of $f : M \to R^1$ to $C$ defined by
\[
\hat{f}(\hat{x}) = f(\hat{x}) \text{ for all } \hat{x} \in C
\]
is a $B$-vex function on $C$ w.r.t. $b$.

Proof. Let $f : M \to R^1$ be $E$-B-vex on $M$ w.r.t. $b$, and let $C$ be a nonempty convex subset of $E(M)$. Then for $\hat{x}, \hat{y} \in C$ ($\hat{x}$ and $\hat{y}$ may not be distinct), there exist $x, y \in M$ such that $\hat{x} = E(x)$ and $\hat{y} = E(y)$. Since $\lambda \hat{x} + (1 - \lambda)\hat{y} \in C$, it follows from the $E$-B-vertex of $f$ on $M$ that
\[
\hat{f}(\lambda \hat{x} + (1 - \lambda)\hat{y}) = f(\lambda E(x) + (1 - \lambda)E(y)) \\
\leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda)b(E(x), E(y), \lambda)f(E(y)) \\
= \lambda b(\hat{x}, \hat{y}, \lambda)\hat{f}(\hat{x}) + (1 - \lambda)b(\hat{x}, \hat{y}, \lambda)\hat{f}(\hat{y})
\]
for all $\lambda \in [0, 1]$, which implies that $\hat{f} : C \to R^1$ is a $B$-vex function on $C$ w.r.t. $b$. □

Corollary 4.1. Let $M \subseteq X$ be a nonempty $E$-convex set, and let $f : M \to R^1$ be $E$-B-vex on $M$ w.r.t. $b$. If $E(M)$ is a convex set, then the restriction $\hat{f} : E(M) \to R^1$ of $f : M \to R^1$ is a $B$-vex function on $E(M)$ w.r.t. $b$.

Theorem 4.2. Let $M \subseteq X$ be a nonempty $E$-convex set such that $E(M)$ is convex. Then a function $f : M \to R^1$ is $E$-B-vex on $M$ w.r.t. $b$ if and only if its restriction $\hat{f} : E(M) \to R^1$ is $B$-vex on $E(M)$ w.r.t. $b$.

Proof. The direct implication is true due to Corollary 4.1. Conversely, suppose that $\hat{f} : E(M) \to R^1$ is a $B$-vex function on $E(M)$ w.r.t. $b$, and that $x, y \in M$. Then $E(x), E(y) \in E(M)$, and by the convexity of $E(M)$ follows $\lambda E(x) + (1 - \lambda)E(y) \in E(M)$ for all $\lambda \in [0, 1]$. Since $\hat{f} : E(M) \to R^1$ is $B$-vex on $E(M)$ w.r.t. $b$, we have
\[
f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda)b(E(x), E(y), \lambda)f(E(y))
\]
for all $\lambda \in [0, 1]$, which implies that $f : M \to R^1$ is $E$-B-vex on $M$ w.r.t. $b$. This completes the proof. □

The following result can be easily established.

Theorem 4.3. Let $M \subseteq X$ be a nonempty $E$-convex set, and let $\{f_j : j \in J\}$ be an arbitrary nonempty collection of $E$-B-vex functions on $M$ w.r.t. the same $b$ such that for each $x \in M$, $\sup_{j \in J} f_j(x)$ exists in $R^1$. Then the function $f : M \to R^1$ defined by
\[
f(x) = \sup_{j \in J} f_j(x) \text{ for each } x \in M,
\]
is $E$-B-vex on $M$.

The following theorem which can be established along the lines of Theorem 2.1 of Li et al. [5] presents the equivalence between the class of $E$-B-vex functions and that of $E$-quasiconvex functions. For the convenience of reading, the proof will be given.

Theorem 4.4. Let $M \subseteq X$ be a nonempty $E$-convex set. The following conditions are equivalent:

(1) $f : M \to R^1$ is $E$-B-vex on $M$ w.r.t. some $b$.

(2) $f : M \to R^1$ is $E$-quasiconvex on $M$. 

Proof. (1) \(\Rightarrow\) (2) Let \( f \) be E-B-vex on \( M \) w.r.t. \( b \). Noting that \( \lambda b(E(x), E(y), \lambda) \in [0, 1] \) for all \( x, y \in M \) and \( \lambda \in [0, 1] \), we have
\[
f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y)) \leq \max \{ f(E(x)), f(E(y)) \},
\]
for all \( x, y \in M \) and \( \lambda \in [0, 1] \). This shows that \( f : M \to R^1 \) is E-quasiconvex on \( M \).

(2) \(\Rightarrow\) (1) Define \( b : X \times X \times [0, 1] \to R^* \) by
\[
b(x, y, \lambda) = \begin{cases} 1/\lambda, & \text{if } \lambda \in (0, 1) \text{ and } f(x) \geq f(y); \\ 0, & \text{if } \lambda = 0 \text{ or } f(x) < f(y). \end{cases}
\]
It follows that \( \lambda b(x, y, \lambda) \in [0, 1] \) for all \( x, y \in X \) and \( \lambda \in [0, 1] \), and that
\[
\lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y)) = \max \{ f(E(x)), f(E(y)) \}
\]
for all \( x, y \in X \) and \( \lambda \in (0, 1) \). Then, by E-quasiconvexity of \( f \) on \( M \), we have
\[
f(\lambda E(x) + (1 - \lambda)E(y)) \leq \max \{ f(E(x)), f(E(y)) \}
\]
along the lines of \( \lambda \). This completes the proof. \( \square \)

Theorem 4.5. Let \( M \subseteq X \) be a nonempty E-convex set. The following conditions are equivalent:

1. \( f : M \to R^1 \) is E-B-linear on \( M \) w.r.t. some \( b \).
2. \( f : M \to R^1 \) is both E-quasiconvex and E-quasiconcave on \( M \).

The following theorem which is an analogous result to Theorem 4.1 for the E-B-preinvex case can be easily established along the lines of Theorem 4.1.

Theorem 4.6. Let \( A \subseteq X \) be a nonempty E-invex set w.r.t. \( \eta \), and let \( K \) be a nonempty invex subset of \( E(M) \) w.r.t. \( \eta \). If \( f : A \to R^1 \) is E-B-preinvex on \( A \) w.r.t. \( \eta, b \), then the restriction \( \hat{f} : K \to R^1 \) of \( f : A \to R^1 \) to \( K \) defined by
\[
\hat{f}(\hat{x}) = f(\hat{x}) \quad \text{for all } \hat{x} \in K
\]
is a B-preinvex function on \( K \) w.r.t. \( b \).

Corollary 4.2. Let \( A \subseteq X \) be a nonempty E-invex set w.r.t. \( \eta \), and let \( f : A \to R^1 \) be E-B-preinvex on \( A \) w.r.t. \( \eta, b \). If \( E(A) \) is an invex set w.r.t. \( \eta \), then the restriction \( \hat{f} : E(A) \to R^1 \) of \( f : A \to R^1 \) is a B-preinvex function on \( E(A) \) w.r.t. \( b \).

The following theorem which is an analogous result to Theorem 4.2 for the E-B-preinvex case can be easily established along the lines of Theorem 4.1.

Theorem 4.7. Let \( A \subseteq X \) be a nonempty E-invex set w.r.t. \( \eta \) such that \( E(A) \) is invex w.r.t. \( \eta \). Then a function \( f : A \to R^1 \) is E-B-preinvex on \( A \) w.r.t. \( \eta, b \) if and only if its restriction \( \hat{f} : E(A) \to R^1 \) is a B-preinvex function on \( E(A) \) w.r.t. \( b \).

The following result can be easily established.

Theorem 4.8. Let \( A \subseteq X \) be a nonempty E-invex set w.r.t. \( \eta \). If \( \{ f_j : j \in J \} \) is an arbitrary nonempty collection of E-B-preinvex functions on \( A \) w.r.t. the same \( \eta, b \) such that for each \( x \in A \), \( \sup_{j \in J} f_j(x) \) exists in \( R^1 \), then the function \( f : A \to R^1 \) defined by
\[
f(x) = \sup_{j \in J} f_j(x) \quad \text{for each } x \in A,
\]
is E-B-preinvex on \( A \) w.r.t. \( \eta, b \).

References