Abstract—The convexity and continuity of fuzzy mappings are defined through a linear ordering and a metric on the set of fuzzy numbers. The local-global minimum property of real-valued convex functions is extended to convex fuzzy mappings. It is proved that a strict local minimizer of a quasiconvex fuzzy mapping is also a strict global minimizer. Characterizations for convex fuzzy mappings and quasiconvex fuzzy mappings are given. In addition, the Weierstrass theorem is extended from real-valued functions to fuzzy mappings. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Fuzzy numbers, Convexity, Continuity, Convex fuzzy mappings, Linear ordering, Fuzzy Weierstrass theorem, Fuzzy optimization.

1. INTRODUCTION

Let $R^n$ denote the $n$-dimensional Euclidean space. The support, $\text{supp}(\mu)$, of a fuzzy set $\mu : R^n \rightarrow I = [0, 1]$ is defined as

$$\text{supp}(\mu) = \{x \in R^n \mid \mu(x) > 0\}.$$ 

A fuzzy set $\mu : R^n \rightarrow I$ is called fuzzy convex if

$$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in \text{supp}(\mu)$, and $\lambda \in [0, 1]$. A fuzzy set $\mu : R^n \rightarrow I$ is said to be normal if there exists a point $x \in R^n$ such that $\mu(x) = 1$. A fuzzy number we treat in this study is a fuzzy set $\mu : R^1 \rightarrow I$ which is normal, fuzzy convex, upper semicontinuous and with bounded support.

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Let $\mathcal{F}$ denote the set of all fuzzy numbers. A mapping from any nonempty set into $\mathcal{F}$ will be called a fuzzy mapping. It is clear that each $r \in \mathbb{R}^1$ can be considered as a fuzzy number. Hence, each real-valued function can be considered as a fuzzy mapping. The concept of convexity for fuzzy mappings has been considered by many authors in fuzzy optimization. For example, in [1–5], the concept of convex fuzzy mappings defined through the “fuzzy-max” order was investigated. However, the “fuzzy-max” order is a partial ordering on the set of fuzzy numbers.

In [6], Goetschel and Voxman proposed a linear ordering $\preceq$ on $\mathcal{F}$. For each fuzzy mapping $f : \mathbb{R}^1 \rightarrow \mathcal{F}$, based on the linear ordering $\preceq$, they introduced a real-valued function $T_f$ on the domain of the fuzzy mapping $f$. In [7], two concepts of convexity and quasiconvexity for a fuzzy mapping $f$ are defined through the real-valued function $T_f$ introduced in [6].

In this paper, we introduce the concept of convex fuzzy mappings directly through the linear ordering proposed in [6]. We define a ranking value function $\tau$ on $\mathcal{F}$ and the concept of monotonicity for a fuzzy mapping $g : \mathcal{F} \rightarrow \mathcal{F}$. Based on the ranking value function $\tau$, the concept of quasiconvex fuzzy mappings is also introduced. The continuity of fuzzy mappings through a metric on $\mathcal{F}$ is studied, and the Weirstrass theorem is extended from real-valued functions to fuzzy mappings. The local-global minimum property of real-valued convex functions is extended to convex fuzzy mappings. As for real-valued convex functions, nonnegative linear combinations of convex fuzzy mappings are convex. Characterizations for convex fuzzy mappings and quasiconvex fuzzy mappings are also given. In addition, it is proved that every strict local minimizer of a quasiconvex fuzzy mapping is a global minimizer.

2. PRELIMINARIES

In this section, for convenience, several definitions and results without proof from [3,6,8] are summarized below.

It can be easily checked that the $\alpha$-level set of a fuzzy number $\mu \in \mathcal{F}$ is a closed and bounded interval

$$[a(\alpha), b(\alpha)] = [\mu]_\alpha = \begin{cases} \{x \in \mathbb{R}^1 \mid \mu(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1; \\ \overline{\text{cl}(\text{supp}(\mu))}, & \text{if } \alpha = 0, \end{cases}$$

where $\overline{\text{cl}(\text{supp}(\mu))}$ denotes the closure of $\text{supp}(\mu)$. It was shown in [3] that a fuzzy set $\mu : \mathbb{R}^1 \rightarrow I$ is a fuzzy number if and only if (i) $[\mu]_\alpha$ is a closed and bounded interval for each $\alpha \in [0, 1]$, and (ii) $[\mu]_1 \neq \emptyset$. Thus, we can identify a fuzzy number $\mu$ with the parameterized triples,

$$\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\},$$

where $a(\alpha)$ and $b(\alpha)$ denote the left- and right-hand endpoints of $[\mu]_\alpha$, respectively.

For fuzzy numbers $\mu, \nu \in \mathcal{F}$ represented parametrically by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$, respectively, and each nonnegative real number $r$, we define the addition $\mu + \nu$ and nonnegative scalar multiplication $r\mu$ as follows,

$$\mu + \nu = \{(a(\alpha) + c(\alpha), b(\alpha) + d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\},$$

$$r\mu = \{(ra(\alpha), rb(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}.$$ (2.1)

(2.2)

It is known that the addition and nonnegative scalar multiplication on $\mathcal{F}$ defined by (2.1) and (2.2) are equivalent to those derived from the usual extension principle, and that $\mathcal{F}$ is closed under the addition and nonnegative scalar multiplication.

According to Goetschel and Voxman [6], we metricize $\mathcal{F}$ by the metric,

$$D(\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}, \{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}) = \sup \{\max \{|a(\alpha) - c(\alpha)|, \, |b(\alpha) - d(\alpha)|\} \mid 0 \leq \alpha \leq 1\},$$

and define the following ordering, $\preceq$, for $\mathcal{F}$.
DEFINITION 2.1. (See [6, Definition 2.5].) Suppose that $\mu = \{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ and $\nu = \{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ are fuzzy numbers. Then $\mu$ precedes $\nu$ ($\mu \preceq \nu$) if

$$
\int_0^1 \alpha [a(\alpha) + b(\alpha)] \, d\alpha \leq \int_0^1 \alpha [c(\alpha) + d(\alpha)] \, d\alpha.
$$

(2.3)

REMARK 2.1. (See [6].) The ordering $\preceq$ is reflexive and transitive; moreover, any two elements of $\mathcal{F}$ are comparable under the ordering $\preceq$, i.e., $\preceq$ is a linear ordering for $\mathcal{F}$.

3. DEFINITIONS AND BASIC RESULTS

In this section, based on the linear ordering $\preceq$ on $\mathcal{F}$, we define a ranking value function and a strict ordering $\prec$ of $\preceq$ on $\mathcal{F}$. Then, the concept of monotonicity for a fuzzy mapping $g : \mathcal{F} \rightarrow \mathcal{F}$ is proposed. We also introduce the concepts of convexity and continuity for fuzzy mappings based on the ordering $\preceq$ and the metric $D$, respectively, introduced in the preceding section.

First, motivated by the notion of the linear ordering $\preceq$ on $\mathcal{F}$, we define a ranking value function $T : \mathcal{F} \rightarrow \mathbb{R}$ as follows.

DEFINITION 3.1. Let $T : \mathcal{F} \rightarrow \mathbb{R}$ be defined by

$$
T(\mu) = \int_0^1 \alpha [a(\alpha) + b(\alpha)] \, d\alpha,
$$

for each $\mu = \{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\} \in \mathcal{F}$.

LEMMA 3.1. For $\mu, \nu \in \mathcal{F}$, and $k > 0$,

1. $T(\mu + \nu) = T(\mu) + T(\nu)$,
2. $T(k\mu) = kT(\mu)$.

PROOF. Let $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ be the parametric representations of $\mu$ and $\nu$, respectively. Then, for each $\alpha \in [0, 1]$,

$$
[\mu + \nu]_\alpha = [a(\alpha) + c(\alpha), b(\alpha) + d(\alpha)]
$$

(3.2)

and

$$
[k\mu]_\alpha = [ka(\alpha), kb(\alpha)].
$$

(3.3)

From (3.1) and (3.2), we obtain $T(\mu + \nu) = T(\mu) + T(\nu)$, which proves Part (1).

From (3.1) and (3.3), we obtain $T(k\mu) = kT(\mu)$, which proves Part (2).

Combine Part (1) and Part (2) of Lemma 3.1, we obtain the following.

COROLLARY 3.1. For $\mu, \nu \in \mathcal{F}$, and $k_1, k_2 > 0$,

$$
T(k_1\mu + k_2\nu) = k_1T(\mu) + k_2T(\nu).
$$

Based on Definitions 2.1 and 3.1, we give the following.

DEFINITION 3.2. For $\mu, \nu \in \mathcal{F}$, we say that $\mu \prec \nu$ if

$$
\mu \leq \nu \quad \text{and} \quad T(\mu) \neq T(\nu).
$$

It is often convenient to write $\nu \succeq \mu$ (resp. $\nu \succ \mu$) in place of $\mu \leq \nu$ (resp. $\mu \prec \nu$).

By Definition 3.2, the following result can be easily established.
Lemma 3.2. For \( \mu, \nu \in \mathcal{F} \), if \( \mu < \nu \), then
\[
\mu < \lambda \mu + (1 - \lambda) \nu < \nu, \quad \text{for} \ \lambda \in (0, 1).
\]

From Definitions 2.1, 3.1, and 3.2, we obtain the following.

Lemma 3.3. For \( \mu, \nu \in \mathcal{F} \),
\[
\mu < \nu \iff \tau(\mu) < \tau(\nu), \quad \tau(\mu) = \tau(\nu) \iff g(\mu) = g(\nu).
\]

From (3.4), we have the following.

Corollary 3.2. For \( \mu, \nu \in \mathcal{F} \), if \( \mu \leq \nu \), and \( \nu \leq \mu \), then \( \tau(\mu) = \tau(\mu) \).

As mentioned earlier, a mapping from any nonempty set into \( \mathcal{F} \) is a fuzzy mapping. By using the notions of \( \leq \) and \( < \), we define the monotonicity for a fuzzy mapping \( g : \mathcal{F} \to \mathcal{F} \) as follows.

Definition 3.3. A fuzzy mapping \( g : \mathcal{F} \to \mathcal{F} \) is said to be
1. nondecreasing if for \( \mu, \nu \in \mathcal{F} \),
   (i) \( \mu < \nu \Rightarrow g(\mu) \leq g(\nu) \); and
   (ii) \( \tau(\mu) = \tau(\nu) \Rightarrow g(\mu) = g(\nu) \);
2. nonincreasing if for \( \mu, \nu \in \mathcal{F} \),
   (i) \( \mu < \nu \Rightarrow g(\mu) \geq g(\nu) \); and
   (ii) \( \tau(\mu) = \tau(\nu) \Rightarrow g(\mu) = g(\nu) \).

In what follows, let \( S \) be a nonempty subset of \( \mathbb{R}^n \). For any \( x \in \mathbb{R}^n \) and \( \delta > 0 \), let
\[
B_\delta(x) = \{ y \in \mathbb{R}^n \mid \| y - x \| < \delta \},
\]
where \( \| \cdot \| \) being the 2-norm of \( \mathbb{R}^n \).

Definition 3.4. For a fuzzy mapping \( f : S \to \mathcal{F} \),
1. an element \( \hat{x} \in S \) is called a local minimizer of \( f : S \to \mathcal{F} \) if there exists a \( \delta > 0 \), such that
   \[
f(\hat{x}) \leq f(x), \quad \text{for all} \ x \in S \cap B_\delta(\hat{x});
\]
2. an element \( \hat{x} \in S \) is called a strict local minimizer of \( f : S \to \mathcal{F} \) if there exists a \( \delta > 0 \) such that
   \[
f(\hat{x}) < f(x), \quad \text{for all} \ x \neq \hat{x}, \ \text{and} \ x \in S \cap B_\delta(\hat{x});
\]
3. an element \( x_\ast \in S \) is called a global minimizer of \( f : S \to \mathcal{F} \) if
   \[
f(x_\ast) \leq f(x), \quad \text{for all} \ x \in S.
\]

Definition 3.5. Let \( x_0 \in S \). A fuzzy mapping \( f : S \to \mathcal{F} \) is said to be continuous at \( x_0 \) if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
D(f(x), f(x_0)) < \varepsilon, \quad \text{whenever} \ x \in S \cap B_\delta(x_0).
\]

If \( f : S \to \mathcal{F} \) is said to be continuous if it is continuous at each \( x \in S \).

Motivated by Goetschel and Voxman [6], we give the following.
DEFINITION 3.6. For each fuzzy mapping \( f : \mathbb{R}^n \to \mathcal{F} \), define \( T_f : \mathbb{R}^n \to \mathbb{R}^1 \) by

\[
T_f(x) = \int_0^1 \alpha [a(\alpha, x) + b(\alpha, x)] \, d\alpha,
\]

(3.6)

where for each \( x \in \mathbb{R}^n \), \( f(x) \) is represented parametrically by \( \{(a(\alpha, x), b(\alpha, x), \alpha) \mid 0 \leq \alpha \leq 1\} \).

From (2.3) and (3.6), for \( x, y \in \mathbb{R}^n \), we have

\[
f(x) \leq f(y) \iff T_f(x) \leq T_f(y).
\]

(3.7)

Motivated by Goetschel and Voxman [6, Lemma 2.9], we establish the following.

LEMMA 3.4. If \( f : \mathbb{R}^n \to \mathcal{F} \) is continuous, then \( T_f : \mathbb{R}^n \to \mathbb{R}^1 \) is also continuous.

PROOF. Let \( x_0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \) be given. Choose \( \delta > 0 \) such that

\[
\|x - x_0\| < \delta \implies D(f(x), f(x_0)) < \varepsilon,
\]

which implies that for all \( \alpha \in [0, 1] \),

\[
|a(\alpha, x) - a(\alpha, x_0)| < \varepsilon \quad \text{and} \quad |b(\alpha, x) - b(\alpha, x_0)| < \varepsilon,
\]

whenever \( x \in B_\delta(x_0) \). Then, it follows that

\[
\|T_f(x) - T_f(x_0)\| = \left| \int_0^1 \alpha [a(\alpha, x) - a(\alpha, x_0) + b(\alpha, x) - b(\alpha, x_0)] \, d\alpha \right| \leq \int_0^1 2\varepsilon \alpha \, d\alpha = \varepsilon,
\]

which implies that \( T_f \) is continuous at \( x_0 \).

Finally, we introduce the concept of convexity and concavity for fuzzy mappings. We have seen that \( \mathcal{F} \) is closed under addition and nonnegative scalar multiplication. It follows that \( \mathcal{F} \) is a convex subset of \( \mathcal{V} \). Hence, it does make sense to speak of convexity for fuzzy mappings on any convex subset of \( \mathcal{F} \).

In what follows, let \( C \) be a nonempty convex subset of \( \mathcal{F} \), and let \( K \) be a nonempty convex subset of \( \mathbb{R}^n \).

DEFINITION 3.7. Let \( C \) be a nonempty convex subset of \( \mathcal{F} \). A fuzzy mapping \( g : \mathcal{F} \to \mathcal{F} \) is said to be

1. convex if for \( \mu, \nu \in C \) and \( \lambda \in (0, 1) \),

\[
g(\lambda\mu + (1 - \lambda)\nu) \leq \lambda g(\mu) + (1 - \lambda)g(\nu);
\]

2. concave if for \( \mu, \nu \in C \) and \( \lambda \in (0, 1) \),

\[
g(\lambda\mu + (1 - \lambda)\nu) \geq \lambda g(\mu) + (1 - \lambda)g(\nu).
\]

DEFINITION 3.8. A fuzzy mapping \( f : K \to \mathcal{F} \) is said to be

1. convex if for \( x, y \in K \) and \( \lambda \in (0, 1) \),

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);
\]

2. concave if for \( x, y \in K \) and \( \lambda \in (0, 1) \),

\[
f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).
\]
**Theorem 3.1.** Let $g : \mathcal{F} \to \mathcal{F}$ and $f : K \to \mathcal{F}$ be convex fuzzy mappings. If $g$ is nondecreasing, then the fuzzy mapping $g \circ f : K \to \mathcal{F}$ defined by

$$(g \circ f)(x) = g(f(x)), \quad \text{for each } x \in K$$

is convex on $K$.

**Proof.** Let $x, y \in K$ and $\lambda \in (0, 1)$. Since $f : K \to \mathcal{F}$ is convex, we have

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

(i) $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$: Since $g : \mathcal{F} \to \mathcal{F}$ is nondecreasing and convex, it follows that

$$g(f(\lambda x + (1 - \lambda) y)) \leq g(\lambda f(x) + (1 - \lambda) f(y)) \leq \lambda g(f(x)) + (1 - \lambda) g(f(y)).$$

(ii) $\tau(f(\lambda x + (1 - \lambda) y)) = \tau(\lambda f(x) + (1 - \lambda) f(y))$: Since $g : \mathcal{F} \to \mathcal{F}$ is nondecreasing and convex, it follows that

$$g(f(\lambda x + (1 - \lambda) y)) = g(\lambda f(x) + (1 - \lambda) f(y)) \leq \lambda g(f(x)) + (1 - \lambda) g(f(y)).$$

From the above arguments, we conclude that $g \circ f : K \to \mathcal{F}$ is convex on $K$.

**4. MAIN RESULTS**

In this section, we establish a theorem which extends the Weierstrass theorem from real-valued functions to fuzzy mappings, and give characterizations for convex fuzzy mappings. Motivated by a characterization of convex fuzzy mappings in terms of the ranking value function, we propose the concept of quasiconvexity for fuzzy mappings by using the ranking value function $\tau : \mathcal{F} \to \mathbb{R}^1$. In addition, the local-global minimum property of real-valued convex functions is extended to convex fuzzy mappings. Furthermore, it is proved that every strict local minimizer of a quasiconvex fuzzy mapping is a global minimizer.

**Theorem 4.1.** Let $M$ be a nonempty closed and bounded subset of $\mathbb{R}^n$. If $f : M \to \mathcal{F}$ is continuous, then $f$ attains a maximum and a minimum value on $M$.

**Proof.** Since $f : M \to \mathcal{F}$ is continuous, it follows from Lemma 3.4 that $T_f : M \to \mathbb{R}^1$ is also continuous. Since $M$ is a nonempty closed and bounded subset of $\mathbb{R}^n$, it follows from the Weierstrass theorem that $T_f : M \to \mathbb{R}^1$ attains its maximum value, say at $x^* \in M$, and its minimum value, say at $x_* \in M$, on $M$. From (3.7), we conclude that $f : M \to \mathcal{F}$ attains its maximum value at $x^* \in M$, and its minimum value at $x_* \in M$, which completes the proof.

**Theorem 4.2.** Let $K$ be a nonempty convex subset of $\mathbb{R}^n$, and let $f$ be a fuzzy mapping on $K$. The following conditions are equivalent.

1. $f : K \to \mathcal{F}$ is convex;
2. for $x, y \in K$ and $\lambda \in (0, 1)$,

$$\tau(f(\lambda x + (1 - \lambda) y)) \leq \lambda \tau(f(x)) + (1 - \lambda) \tau(f(y));$$

3. the epigraph $\text{epi}(f) = \{(x, \mu) \mid x \in K, \mu \in \mathcal{F}, f(x) \leq \mu\}$ of $f : K \to \mathcal{F}$ is a convex subset of $\mathbb{R}^n \times \mathcal{F}$. 


PROOF.

1. \(\implies\) 2. Let \(f : K \to \mathcal{F}\) be convex, and let \(x, y \in K\). Then, for \(\lambda \in (0, 1)\), we have
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).
\]
From (3.4) and Corollary 3.1, it follows that
\[
\tau(f(\lambda x + (1 - \lambda)y)) \leq \lambda \tau(f(x)) + (1 - \lambda)\tau(f(y))
\]
which proves (4.1).

2. \(\implies\) 3. Assume that \(\tau : \mathcal{F} \to \mathbb{R}^1\) satisfies (4.1). If \(\text{epi}(f)\) is the empty set or a singleton, then it is obviously a convex set. Let \((x, \mu), (y, \nu) \in \text{epi}(f)\), where \(x, y \in K\), and \(\mu, \nu \in \mathcal{F}\). It follows from the definition of \(\text{epi}(f)\) that \(f(x) \leq \mu\) and \(f(y) \leq \nu\). Then, from (4.1), (3.4) and Corollary 3.1, we have
\[
\tau(f(\lambda x + (1 - \lambda)y)) \leq \lambda \tau(f(x)) + (1 - \lambda)\tau(f(y))
\]
for each \(\lambda \in (0, 1)\). It follows from (3.4) that for all \(\lambda \in (0, 1)\),
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda \mu + (1 - \lambda)\nu,
\]
which implies that for each \(\lambda \in [0, 1]\),
\[
(\lambda x + (1 - \lambda)y, \lambda \mu + (1 - \lambda)\nu) = (x, \mu) + (1 - \lambda)(y, \nu) \in \text{epi}(f).
\]
This proves that \(\text{epi}(f)\) is a convex subset of \(\mathbb{R}^n \times \mathcal{V}\).

3. \(\implies\) 1. Assume that \(\text{epi}(f)\) is a convex subset of \(\mathbb{R}^n \times \mathcal{V}\). Let \(x, y \in K\). Then, by the definition of \(\text{epi}(f)\) and the reflexivity of \(\preceq\), we have \((x, f(x)), (y, f(y)) \in \text{epi}(f)\). By the convexity of \(\text{epi}(f)\), it follows that for all \(\lambda \in [0, 1]\),
\[
\lambda(x, f(x)) + (1 - \lambda)(y, f(y)) = (\lambda x + (1 - \lambda)y, \lambda f(x) + (1 - \lambda)f(y)) \in \text{epi}(f),
\]
which implies that for all \(\lambda \in (0, 1)\),
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).
\]
This completes the proof.

COROLLARY 4.1. Let \(C\) be a nonempty convex subset of \(\mathcal{F}\), and let \(g\) be a fuzzy mapping on \(C\). The following conditions are equivalent.

1. \(g : C \to \mathcal{F}\) is convex;
2. for \(\mu, \nu \in C\) and \(\lambda \in (0, 1)\),
\[
\tau(g(\lambda \mu + (1 - \lambda)\nu)) \leq \lambda \tau(g(\mu)) + (1 - \lambda)\tau(g(\nu));
\]
3. the epigraph \(\text{epi}(g) = \{(\mu, \omega) \mid \mu \in C, \omega \in \mathcal{F}, g(\mu) \preceq \omega\}\) of \(g : C \to \mathcal{F}\) is a convex subset of \(\mathcal{V} \times \mathcal{V}\).

PROOF. The idea of the proof is similar to that of Theorem 4.2.
THEOREM 4.3. Let $K$ be a nonempty convex subset of $R^n$, and let $f_j : K \rightarrow \mathcal{F}$, $j = 1, \ldots, l$, be convex fuzzy mappings. For $k_1, k_2, \ldots, k_l > 0$, the fuzzy mapping $f : K \rightarrow \mathcal{F}$ defined by

$$f(x) = \sum_{j=1}^{l} k_j f_j(x),$$

for each $x \in K$, (4.2) is a convex fuzzy mapping.

PROOF. Since $f_j : K \rightarrow \mathcal{F}$ is convex for each $j = 1, \ldots, l$, it follows from Theorem 4.2 that for $x, y \in K$ and $\lambda \in (0, 1)$,

$$\tau(f_j(\lambda x + (1 - \lambda)y)) \leq \lambda \tau(f_j(x)) + (1 - \lambda)\tau(f_j(y)), \quad j = 1, \ldots, l.$$

Then, by Lemma 3.1, it follows that for $x, y \in K$ and $\lambda \in (0, 1)$,

$$\tau(k_j f_j(\lambda x + (1 - \lambda)y)) \leq \lambda \tau(k_j f_j(x)) + (1 - \lambda)\tau(k_j f_j(y)), \quad j = 1, \ldots, l.$$

From (4.2) and Lemma 3.1, it follows that for $x, y \in K$ and $\lambda \in (0, 1)$,

$$\tau(f(\lambda x + (1 - \lambda) y)) = \tau \left( \sum_{j=1}^{l} k_j f_j(\lambda x + (1 - \lambda) y) \right)$$

$$= \sum_{j=1}^{l} \tau(k_j f_j(\lambda x + (1 - \lambda) y))$$

$$\leq \sum_{j=1}^{l} [\lambda \tau(k_j f_j(x)) + (1 - \lambda)\tau(k_j f_j(y))]$$

$$= \lambda \tau \left( \sum_{j=1}^{l} k_j f_j(x) \right) + (1 - \lambda)\tau \left( \sum_{j=1}^{l} k_j f_j(y) \right)$$

$$= \lambda \tau(f(x)) + (1 - \lambda)\tau(f(y)).$$

Then, by Theorem 4.2, $f : K \rightarrow \mathcal{F}$ is a convex fuzzy mapping.

COROLLARY 4.2. Let $C$ be a nonempty convex subset of $\mathcal{F}$, and let $g_j : C \rightarrow \mathcal{F}$, $j = 1, \ldots, l$, be convex fuzzy mappings. For $k_1, k_2, \ldots, k_l > 0$, the fuzzy mapping $g : C \rightarrow \mathcal{F}$ defined by

$$g(\mu) = \sum_{j=1}^{l} k_j g_j(\mu),$$

for each $\mu \in C$, is a convex fuzzy mapping.

PROOF. The idea of the proof is similar to that of Theorem 4.3.

THEOREM 4.4. Let $f : K \rightarrow \mathcal{F}$ be convex, and $\bar{x} \in K$ is a local minimizer of $f$, then $\bar{x}$ is also a global minimizer of $f$ over $K$.

PROOF. The proof is by contradiction. Let $\bar{x} \in K$ be a local minimizer of $f : K \rightarrow \mathcal{F}$ and suppose, by contradiction, that it is not a global minimizer. Then, there exists some point $x \in K$ satisfying $f(x) < f(\bar{x})$. Since $f : K \rightarrow \mathcal{F}$ is convex, it follows from Lemma 3.2 that

$$f(\lambda x + (1 - \lambda)\bar{x}) \leq \lambda f(x) + (1 - \lambda)f(\bar{x}) < f(\bar{x}),$$

for all $\lambda \in (0, 1)$. So, $f(\lambda x + (1 - \lambda)\bar{x}) < f(\bar{x})$ for arbitrary small positive number $\lambda$, and this contradiction proves the result.

Motivated by (4.1), we define the concept of quasiconvexity and quasiconcavity for fuzzy mappings as follows.
DEFINITION 4.1. Let $K$ be a nonempty convex subset of $\mathbb{R}^n$. A fuzzy mapping $f : K \rightarrow \mathcal{F}$ is said to be

1. quasiconvex if for $x, y \in K$ and $\lambda \in (0, 1)$,
   \[ \tau(f(\lambda x + (1 - \lambda)y)) \leq \max\{\tau(f(x)), \tau(f(y))\}; \]

2. quasiconcave if for $x, y \in K$ and $\lambda \in (0, 1)$,
   \[ \tau(f(\lambda x + (1 - \lambda)y)) \geq \min\{\tau(f(x)), \tau(f(y))\}. \]

DEFINITION 4.2. Let $C$ be a nonempty convex subset of $\mathbb{R}$. A fuzzy mapping $g : C \rightarrow \mathcal{F}$ is said to be

1. quasiconvex if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,
   \[ \tau(g(\lambda\mu + (1 - \lambda)\nu)) \leq \max\{\tau(g(\mu)), \tau(g(\nu))\}; \]

2. quasiconcave if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,
   \[ \tau(g(\lambda\mu + (1 - \lambda)\nu)) \geq \min\{\tau(g(\mu)), \tau(g(\nu))\}. \]

From (4.1) and Definition 4.1, we obtain the following.

THEOREM 4.5. Let $K$ be a nonempty convex subset of $\mathbb{R}^n$. If $f : K \rightarrow \mathcal{F}$ is a convex (resp., concave) fuzzy mapping, then it is also quasiconvex (resp., quasiconcave).

COROLLARY 4.3. Let $C$ be a nonempty convex subset of $\mathbb{R}$. If $g : C \rightarrow \mathcal{F}$ is a convex (resp., concave) fuzzy mapping, then it is also quasiconvex (resp., quasiconcave).

THEOREM 4.6. Let $K$ be a nonempty convex subset of $\mathbb{R}^n$. A fuzzy mapping $f : K \rightarrow \mathcal{F}$ is quasiconvex if and only if for each $\mu \in \mathcal{F}$, the lower $\mu$-level set,

\[ L_{\mu}(f) = \{x \in K \mid f(x) \leq \mu\}, \]

of $f : K \rightarrow \mathcal{F}$ is a convex subset of $\mathbb{R}^n$.

PROOF. Assume that $f : K \rightarrow \mathcal{F}$ is quasiconvex, and let $\mu \in \mathcal{F}$. If $L_{\mu}(f)$ is the empty set or a singleton, then it is obvious a convex set. Let $x, y \in L_{\mu}(f)$. It follows from the definition of $L_{\mu}(f)$ that $f(x) \leq \mu$ and $f(y) \leq \mu$. Since $f : K \rightarrow \mathcal{F}$ is quasiconvex, from (3.4), we have

\[ \tau(f(\lambda x + (1 - \lambda)y)) \leq \max\{\tau(f(x)), \tau(f(y))\} \leq \tau(\mu) \]

for each $\lambda \in (0, 1)$. It follows from (3.4) that for each $\lambda \in (0, 1)$,

\[ f(\lambda x + (1 - \lambda)y) \leq \mu, \]

which implies that $\lambda x + (1 - \lambda)y \in L_{\mu}(f)$ for each $\lambda \in [0, 1]$. Hence, $L_{\mu}(f)$ is a convex subset of $\mathbb{R}^n$.

Conversely, assume that $L_{\mu}(f)$ is a convex subset of $\mathbb{R}^n$ for each $\mu \in \mathcal{F}$. Let $x, y \in K$. Without loss of generality, we may assume that $f(x) \leq f(y)$. Let

\[ \mu = f(y). \]

(4.3)

Since $\preceq$ is reflexive and transitive, we have

\[ f(x) \preceq \mu \quad \text{and} \quad f(y) \preceq \mu, \]

which implies that $x, y \in L_{\mu}(f)$. Then, by the convexity of $L_{\mu}(f)$, we have $\lambda x + (1 - \lambda)y \in L_{\mu}(f)$ for each $\lambda \in [0, 1]$, which implies that $f(\lambda x + (1 - \lambda)y) \preceq \mu$ for all $\lambda \in [0, 1]$. This completes the proof.
COROLLARY 4.4. Let $C$ be a nonempty convex subset of $\mathcal{F}$. A fuzzy mapping $g : C \to \mathcal{F}$ is quasiconvex if and only if for each $\mu \in \mathcal{F}$, $\omega \in \mathcal{F}$, the lower $\omega$-level set,

$$L_\omega(g) = \{\mu \in C \mid f(\mu) \leq \omega\},$$

of $g : C \to \mathcal{F}$ is a convex subset of $\mathcal{F}$.

PROOF. The idea of the proof is similar to that of Theorem 4.6.

THEOREM 4.7. Let $f : K \to \mathcal{F}$ be a quasiconvex fuzzy mapping, and let $x_\ast \in K$ be a global minimizer of $f$ over $K$. Then, the set,

$$\Omega = \{x \in K \mid \tau(f(x)) = \tau(f(x_\ast))\},$$

is a convex set.

PROOF. Let $\mu = f(x_\ast)$. Since $x_\ast \in K$ is a global minimizer of $f : K \to \mathcal{F}$. From Corollary 3.2, we see that if $x \in K$ and $x \in L_\mu(f)$, then $\tau(f(x)) = \tau(f(x_\ast))$. On the other hand, if $x \in \Omega$, it is easily seen that $x \in L_\mu(f)$. From the above arguments, we conclude that $\Omega = L_\mu(f)$. Since $f : K \to \mathcal{F}$ is quasiconvex, it follows from Theorem 4.6 that $\Omega = L_\mu(f)$ is a convex set. This completes the proof.

From Theorems 4.5 and 4.7, we obtain the following.

THEOREM 4.8. Let $f : K \to \mathcal{F}$ be a convex fuzzy mapping, and let $x_\ast \in K$ be a global minimizer of $f$ over $K$. Then, the set,

$$\Omega = \{x \in K \mid \tau(f(x)) = \tau(f(x_\ast))\},$$

is a convex set.

THEOREM 4.9. Let $f : K \to \mathcal{F}$ be quasiconvex, and $\hat{x} \in K$ is a strict local minimizer of $f$, and $\hat{x}$ is also a strict global minimizer of $f$ over $K$.

PROOF. The proof is by contradiction. Let $\hat{x} \in K$ be a strict local minimizer of $f : K \to \mathcal{F}$ and suppose, by contradiction, that it is not a strict global minimizer. Then there exists some point $x \in K$ satisfying $f(x) \leq f(\hat{x})$. Since $f : K \to \mathcal{F}$ is quasiconvex, it follows from (3.5) that for all $\lambda \in (0, 1),

$$\tau(f(\lambda x + (1 - \lambda)\hat{x})) \leq \max\{\tau(f(x)), \tau(f(\hat{x}))\} = \tau(f(\hat{x})).$$

From (3.4), it follows that $f(\lambda x + (1 - \lambda)\hat{x}) \leq f(\hat{x})$ for arbitrary small positive number $\lambda$, and this contradiction proves the result.

REFERENCES