A generalized Nash equilibrium for a bioeconomic problem of fishing

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Résumé. With the overexploitation of many conventional fish stocks, and growing interest in harvesting new kinds of food from the sea, there is an increasing need for managers of fisheries to take account of interactions among species. In this work we define a bioeconomic equilibrium model for ‘n’ fishermen who catch three species; these species compete with each other for space or food. The natural growth of each species is modeled using a logistic law. The objective of the work is to find the fishing effort that maximizes the profit of each fisherman constrained by the conservation of the biodiversity. The existence of the steady states and its stability are studied using eigenvalue analysis. The problem of determining the equilibrium point that maximizes the profit of each fisherman is then solved by using the generalized Nash equilibrium problem. Finally, some numerical simulations are given to illustrate the results.

Mots-Clés : Bioeconomic model; Species in competition; Fishing effort; Maximizing profits for each fisherman; Generalized Nash Equilibrium GNE; Linear Complementarity Problem LCP; Biodiversity of renewable resources.
1. Introduction

Harvesting of multispecies fisheries is an important area of study in fishery modelling. The basic ideas related to this field of study were provided by Clark[7]. Clark also examined the effects of harvesting one species in the Gause’s model[11] of two competing species. Chaudhuri ([3],[4]) has studied the combined harvesting of two competing species from the standpoint of bioeconomic harvesting and has discussed dynamic optimization of the harvest policy. Chaudhuri and SahaRay[5] have studied combined harvesting of a prey-predator community with some prey hiding in refuges. Auger[1] has presented a specific stock-effort dynamic model; the stock corresponds to two fish populations growing and moving between two fishing zones, on which they are harvested by two different fleets; the effort represents the number of fishing vessels of the two fleets which operate on the two fishing zones; the bioeconomic model is a set of four ordinary differential equations governing the stocks and the fishing efforts in the two fishing areas; fish migration, as well as vessels displacements, between the two zones are assumed to take place at a faster time scale than the variation of the stocks and the changes of fleets sizes, respectively; the vessels movements between the two fishing areas are assumed to be stock dependent, i.e. the larger the stock density is in a zone the more vessels tend to remain in it. Mchich in his work[16] has presented a stock-effort dynamical model of a fishery subdivided on several fishing zones; the stock corresponds to a fish population moving between different zones, on which they are harvested by fishing fleets. Auger[2] has given a mathematical model of artificial pelagic multisite fisheries. The model is a stock–effort dynamical model of a fishery subdivided into artificial fishing sites such as fish-aggregating devices (FADs) or artificial habitats (AHs). The objective of its work is to investigate the effects of the number of sites on the global activity of the fishery.

In the present paper, we propose to define a model for ’n’ fishermen acting in an area containing three species of fish. The evolution of the population of fishes is described by a density dependent model taking into account the competition between species which compete with each other for space or food (see the model of Verhulst[21]). More specifically, the bioeconomic model includes three parts: A biological part
that connects the catch to the biomass stock, an exploitation part that connects the catch to fishing effort at equilibrium, and an economic part that connects the fishing effort to profit.

The objective of each fisherman is to maximize his income without any consultation of the others, but all of them have to respect two constraints, the first one is the sustainable management of the resources, the second one is the preservation of the biodiversity. With all these considerations, our problem leads to a generalized Nash equilibrium problem, to solve this problem we transform it into a linear complementarity problem.

The paper is organized as follows. In section 2 we define a bioeconomic equilibrium model of three species that compete with each other for space or food. In section 3 we compute the generalized Nash equilibrium point. In section 4 we give a numerical simulation of the mathematical model and discussion of the results. Finally we give conclusions in section 5.

2. Mathematical model

In this section we propose to define a bioeconomic equilibrium model of three marine species that compete with each other for space or food, and whose natural growth of each is obtained by means of a logistics law (Law of Verhulst[21]), more specifically, the bioeconomic model includes three parts: A biological part that connects the catches to the biomass stock, exploiting part that connects the catches to fishing effort at equilibrium, and an economic part that connects the fishing effort to profit.

The evolution of the biomass of fishes is modelled by the following equations

\[
\begin{align*}
\dot{B}_1 &= r_1 B_1 (1 - \frac{B_1}{K_1}) - c_{12} B_1 B_2 - c_{13} B_1 B_3 \\
\dot{B}_2 &= r_2 B_2 (1 - \frac{B_2}{K_2}) - c_{21} B_1 B_2 - c_{23} B_2 B_3 \\
\dot{B}_3 &= r_3 B_3 (1 - \frac{B_3}{K_3}) - c_{31} B_1 B_3 - c_{32} B_2 B_3
\end{align*}
\]

(1)

where \(B_1, B_2\) and \(B_3\) are the densities of populations 1, 2 and 3 respectively; \((r_j)_{j=1,2,3}\) are the intrinsic growth rates; \((K_j)_{j=1,2,3}\) are the
carrying capacities for the respective species; and \((c_{jk})_{1\leq j\neq k\leq 3}\) are the coefficients of competition between species \(k\) and species \(j\).

The steady states of the system of equations (1) are obtained by solving the equations

\[
\begin{align*}
    r_1 B_1^* (1 - \frac{B_1^*}{K_1}) - c_{12} B_1^* B_2^* - c_{13} B_1^* B_3^* &= 0 \\
    r_2 B_2^* (1 - \frac{B_2^*}{K_2}) - c_{21} B_1^* B_2^* - c_{23} B_2^* B_3^* &= 0 \\
    r_3 B_3^* (1 - \frac{B_3^*}{K_3}) - c_{31} B_1^* B_3^* - c_{32} B_2^* B_3^* &= 0
\end{align*}
\]

We have 8 solutions of this system, only one of them can give coexistence of the three species, in this case the biomasses of the three marine species are strictly positive; this solution is the point \(P(B_1^*, B_2^*, B_3^*)\) where

\[
\begin{align*}
    B_1^* &= K_1(r_1 r_2 r_3 - c_{23} K_2 r_1 K_3 c_{32} + K_2 c_{12} c_{23} K_3 r_3 - K_2 c_{12} r_2 r_3 - K_3 c_{13} r_2 r_3 + K_3 c_{13} r_2 K_2 c_{32})/\Delta \\
    B_2^* &= K_2(r_1 r_2 r_3 - r_2 c_{31} K_3 K_1 c_{13} + r_1 c_{31} K_1 c_{23} K_3 - r_1 c_{23} K_3 r_3 - r_1 r_3 c_{21} K_1 + r_3 c_{21} K_1 c_{13} K_3)/\Delta \\
    B_3^* &= K_3(r_1 r_2 r_3 - c_{21} K_2 K_1 c_{12} r_3 - r_2 r_1 c_{31} K_1 + c_{21} K_2 K_1 r_1 c_{32} + r_2 K_2 c_{31} K_1 c_{12} - r_2 K_2 r_1 c_{32})/\Delta.
\end{align*}
\]

\[
\Delta = r_1 r_2 r_3 - c_{23} K_2 r_1 K_3 c_{32} - r_2 c_{31} K_3 K_1 c_{13} - c_{21} K_2 K_1 c_{12} r_3 + c_{23} K_3 K_3 c_{31} K_1 c_{12} + c_{21} K_2 K_1 c_{13} K_3 c_{32}
\]

The variational matrix of the system at the steady state \(P(B_1^*, B_2^*, B_3^*)\) is

\[
J = \begin{bmatrix}
    J_{11} & -c_{12} B_1^* & -c_{13} B_1^* \\
    -c_{21} B_1^* & J_{22} & -c_{23} B_2^* \\
    -c_{31} B_3^* & -c_{32} B_3^* & J_{33}
\end{bmatrix}
\]

where

\[
\begin{align*}
    J_{11} &= r_1 (1 - \frac{2B_1^*}{K_1}) - c_{12} B_2^* - c_{13} B_3^* \\
    J_{22} &= r_2 (1 - \frac{2B_2^*}{K_2}) - c_{21} B_1^* - c_{23} B_3^* \\
    J_{33} &= r_3 (1 - \frac{2B_3^*}{K_3}) - c_{31} B_1^* - c_{32} B_2^*
\end{align*}
\]

Using the fact that by (2) we have

\[
\begin{align*}
    r_1 (1 - \frac{2B_1^*}{K_1}) - c_{12} B_2^* - c_{13} B_3^* &= -r_1 \frac{B_1^*}{K_1} \\
    r_2 (1 - \frac{2B_2^*}{K_2}) - c_{21} B_1^* - c_{23} B_3^* &= -r_2 \frac{B_2^*}{K_2} \\
    r_3 (1 - \frac{2B_3^*}{K_3}) - c_{31} B_1^* - c_{32} B_2^* &= -r_3 \frac{B_3^*}{K_3}
\end{align*}
\]
then

$$J = \begin{bmatrix}
-r_1 \frac{B_1^*}{K_1} & -c_{12}B_1^* & -c_{13}B_1^* \\
-c_{21}B_2^* & -r_2 \frac{B_2^*}{K_2} & -c_{23}B_2^* \\
-c_{31}B_3^* & -c_{32}B_3^* & -r_3 \frac{B_3^*}{K_3}
\end{bmatrix}$$

Note that the biological model is meaningful only insofar as the biomass of the species are strictly positive, then we must have $B_i^* > 0$.

The characteristic polynomial is given by

$$P(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

where

$$\begin{cases}
  a_0 &= 1 > 0 \\
  a_1 &= (r_3K_1K_2B_3^* + r_2K_1K_3B_2^* + r_1K_2K_3B_1^*)/(K_1K_2K_3) > 0 \\
  a_2 &= [K_1B_2^*B_3^*(r_2r_3 - c_{23}c_{32}K_2K_3) + K_2B_1^*B_3^*(r_3r_1 - c_{13}c_{31}K_1K_3) \\
  &+ K_3B_1^*B_2^*(r_1r_2 - c_{12}c_{21}K_1K_2)]/(K_1K_2K_3) > 0 \\
  a_3 &= \Delta/(K_1K_2K_3) > 0
\end{cases}$$

Using the fact that by (3) we have $a_1a_2 - a_0a_3 > 0$ and therefore by Routh-Hurwitz Stability Criterion we have $P(B_1^*, B_2^*, B_3^*)$ is locally asymptotically stable.

So, for existence and stability of (1) we need : $\Delta > 0$ and $(B_j^*)_{j=1,2,3} > 0$. We assume that it is the case in what follows.

Now, we introduce the fishing by reducing the rate of fish population growth by the amount (see [20])

$$H_j = q_j E_j B_j$$

where $(q_j)_{j=1,2,3}$ are the catchability coefficients of species $j$; and $(E_j)_{j=1,2,3}$ are the fishing efforts to exploit a species $j$.

The model for the evolution of fish population becomes :

$$\begin{cases}
  \dot{B}_1 = r_1B_1(1 - \frac{B_1}{K_1}) - c_{12}B_1B_2 - c_{13}B_1B_3 - q_1E_1B_1 \\
  \dot{B}_2 = r_2B_2(1 - \frac{B_2}{K_2}) - c_{21}B_1B_2 - c_{23}B_2B_3 - q_2E_2B_2 \\
  \dot{B}_3 = r_3B_3(1 - \frac{B_3}{K_3}) - c_{31}B_1B_3 - c_{32}B_2B_3 - q_3E_3B_3
\end{cases}$$ (5)
The catchability coefficient $q$ is a key parameter in the validation process of fishing simulation model (see [14]). In this paper this parameter is assumed to be constant.

The fishing effort is defined as the product of a fishing activity and a fishing power. The fishing effort deployed by a fleet is the sum of these products over all fishing units in the fleet. The fishing activity is in units of time. The fishing power is the ability of a fishing unit to catch fish and is a complex function depending on vessel, gear and crew. However, since measures of fishing power may not be available, activity (such as hours or days fished) has often been used as a substitute for effort.

It is interesting to note that according to the literature, the effort depends on several variables, namely for example: Number of hours spent fishing; search time; number of hours since the last fishing; number of days spent fishing; number of operations; number of sorties flown; ship, technology, fishing gear, crew, etc. However, in this paper, the effort is treated as a unidimensional variable which includes a combination of all these factors. Now we give the expression of biomass as a function of fishing effort.

The biomasses at biological equilibrium (i.e., the variation of the biomass of each species is zero), are the solutions of the system

$$\begin{cases} r_1 (1 - \frac{B_1}{K_1}) = c_{12} B_2 + c_{13} B_3 + q_1 E_1 \\ r_2 (1 - \frac{B_2}{K_2}) = c_{21} B_1 + c_{23} B_3 + q_2 E_2 \\ r_3 (1 - \frac{B_3}{K_3}) = c_{31} B_1 + c_{32} B_2 + q_3 E_3 \end{cases} \quad (6)$$

The solutions of this system are given by:

$$\begin{cases} B_1 = a_{11} E_1 + a_{12} E_2 + a_{13} E_3 + B_1^* \\ B_2 = a_{21} E_1 + a_{22} E_2 + a_{23} E_3 + B_2^* \\ B_3 = a_{31} E_1 + a_{32} E_2 + a_{33} E_3 + B_3^* \end{cases} \quad (7)$$

where

$$\begin{align*} a_{11} &= K_1 (c_{32} K_2 K_3 c_{23} q_1 - r_3 r_2 q_1) / \Delta \\ a_{12} &= K_1 (-c_{32} K_2 q_2 c_{13} K_3 + K_2 q_2 c_{12} r_3) / \Delta \\ a_{13} &= K_1 (-K_2 K_3 c_{23} c_{12} q_3 + q_3 r_2 c_{13} K_3) / \Delta \\ a_{21} &= K_2 (-K_3 c_{23} q_1 K_1 c_{31} + K_1 c_{21} r_3 q_1) / \Delta \end{align*}$$
\[ \begin{align*}
a_{22} &= K_2(q_2c_1K_3c_{31} - q_2r_1r_3)/\Delta \\
a_{23} &= K_2(+K_3c_{23}r_1q_3 - K_1c_{21}q_3c_{13}K_3)/\Delta \\
a_{31} &= K_3(-q_1K_1c_{32}K_2c_{21} + q_1K_1r_2c_{31})/\Delta \\
a_{32} &= K_3(+r_1c_{32}K_2q_2 - c_{12}K_1K_2q_2c_{31})/\Delta \\
a_{33} &= K_3(c_{12}K_1K_2c_{21}q_3 - r_1r_2q_3)/\Delta
\end{align*} \]

Or in matrix form $B = -AE + B^*$ where $A = (-a_{ij})_{1 \leq i, j \leq 3}$ and $E = (E_1, E_2, E_3)^T$.

It is natural to assume that $r_jr_k > c_{ij}c_{ji}K_iK_j$ for all $j, k = 1, 2, 3$ which implies that $a_{ii} < 0$ for all $i = 1, 2, 3$.

**The net economic revenue**

The profit for each fisherman $\pi_i(E)$ is equal to total revenue $(TR)_i$ minus total cost $(TC)_i$, in other words, the profit for each fisherman is represented by the following function

$$\pi_i(E) = (TR)_i - (TC)_i$$

Now we give the expressions of total revenue and total costs of each fisherman.

**Expression of the total revenue** : We use, as usual in the bioeconomic models, the fact that the total revenue $(TR)_i$ depends linearly on the catch, that is

\[ Total \ revenue = Price \times Catches \]

As mentioned previously (see (4)), we note that $H_{ij} = q_jE_{ij}B_j$ Catches of species $j$ by the fisherman $i$, where $E_{ij}$ is the effort of the fisherman $i$ to exploit the species $j$. It is clear that $H_j = \sum_{i=1}^3 H_{ij}$ is the total catches of species $j$ by all fisherman.

On the other hand, we denote by $E_j = \sum_{i=1}^3 E_{ij}$ the total fishing effort dedicated to species $j$ by all fisherman and by $E^i = (E_{i1}, E_{i2}, E_{i3})^T$ the vector fishing effort must provide by the fisherman $i$ to catch the three species.
With these notations we have

\[(TR)_i = p_1 H_{i1} + p_2 H_{i2} + p_3 H_{i3}\]

\[= p_1 q_1 E_{i1} B_1 + p_2 q_2 E_{i2} B_2 + p_3 q_3 E_{i3} B_3\]

\[= p_1 q_1 E_{i1} (a_{11} E_1 + a_{12} E_2 + a_{13} E_3 + B_1^*)\]

\[+ p_2 q_2 E_{i2} (a_{21} E_1 + a_{22} E_2 + a_{23} E_3 + B_2^*)\]

\[+ p_3 q_3 E_{i3} (a_{31} E_1 + a_{32} E_2 + a_{33} E_3 + B_3^*)\]

\[= p_1 q_1 E_{i1} \left( \sum_{i=1}^{n} E_{i1} + a_{12} \sum_{i=1}^{n} E_{i2} + a_{13} \sum_{i=1}^{n} E_{i3} + B_1^* \right)\]

\[+ p_2 q_2 E_{i2} \left( \sum_{i=1}^{n} E_{i1} + a_{22} \sum_{i=1}^{n} E_{i2} + a_{23} \sum_{i=1}^{n} E_{i3} + B_2^* \right)\]

\[+ p_3 q_3 E_{i3} \left( \sum_{i=1}^{n} E_{i1} + a_{32} \sum_{i=1}^{n} E_{i2} + a_{33} \sum_{i=1}^{n} E_{i3} + B_3^* \right)\]

so

\[(TR)_i = <E^i, -pqAE^i> + <E^i, pqB^* - \sum_{j=1, j \neq i}^{n} pqAE^j> \quad (8)\]

where \((p_j)_{j=1,2,3}\) is the price per unit biomass of the species \(j\). In this work, we take \(p_1, p_2\) and \(p_3\) to be constants.

**Expression of the total effort cost**: We shall assume, in keeping with many standard fisheries models (e.g., the model of Clark[6] and Gordon[12]), that

\[(TC)_i = <c, E^i> \quad (9)\]

where \((TC)_i\) is the total effort cost of the fisherman \(i\), and \((H_j)_{j=1,2,3}\) is the constant cost per unit of harvesting effort of species \(j\).

**Expression of the profit**: As mentioned previously, the net economic revenue of each fisherman is represented by the following function

\[\pi_i(E) = (TR)_i - (TC)_i.\]

It follows from (8) and (9) that
\[
\pi_i(E) = <E^i, -pqAE^i> + <E^i, pqB^* - c - \sum_{j=1, j \neq i}^n pqAE^j>
\] (10)

**Constraints of the model**: As we have mentioned previously, the biological model is meaningful only insofar as the biomass of all the marine species are strictly positive (conservation of the biodiversity), then we have

\[
B = -AE + B^* \geq 0.
\] (11)

In other words, for the fisherman \(i\)

\[
AE^i \leq -\sum_{j=1, j \neq i}^n AE^j + B^*.
\] (12)

3. **Computing the generalized Nash equilibrium**

In this section, we restrict ourself to the case when we have only two fishermen \((n=2)\); for this case we can solve analytically the problem and give the solutions in explicit form. The general case \((n > 2)\) will be considered in the section 4.

Each fisherman trying to maximize his profit and achieve a fishing effort that is an optimal response to the effort of the other fishermen. We have a generalized Nash equilibrium where each fisherman’s strategy is optimal taking into consideration the strategies of all other fishermen. A Nash Equilibrium exists when there is no unilateral profitable deviation from any of the fishermen involved. In other words, no fisherman would take a different action as long as every other fisherman remains the same. This problem can be translated into the following two mathematical problems:

The first fisherman must solve problem \((P_1)\):

\[
\begin{align*}
\max \pi_1(E) &= <E^1, -pqAE^1 + pqB^* - c - pqAE^2 > \\
\text{subject to} & \\
AE^1 &\leq -AE^2 + B^* \\
E^1 &\geq 0 \\
E^2 &\text{is given.}
\end{align*}
\]
and the second fisherman must solve problem \((P_2)\):

\[
(P_2) \begin{cases} 
\max \pi_2(E) = < E^2, -pqAE^2 + pqB^* - c - pqAE^1 > \\
\text{subject to} \\
AE^2 \leq -AE^1 + B^* \\
E^2 \geq 0 \\
E^1 \text{ is given.}
\end{cases}
\]

We recall that \((E^1, E^2)\) is called Generalized Nash equilibrium point if and only if \(E^1\) is a solution of problem \((P_1)\) for \(E^2\) given, and \(E^2\) is a solution of problem \((P_2)\) for \(E^1\) given.

**Solving the generalized Nash equilibrium problem**: The essential conditions of Karush-Kuhn-Tucker applied to the problem \((P_1)\) require that if \(E^1\) is a solution of the problem \((P_1)\) then there exist constants \(u^1 \in IR^3_+\), \(v^1 \in IR^3_+\) and \(\lambda^1 \in IR^3_+\) such that

\[
\begin{cases} 
2pqAE^1 + c - pqB^* + pqAE^2 - u^1 + A^T \lambda^1 = 0 \\
AE^1 + v^1 = -AE^2 + B^* \\
< u^1, E^1 > = < \lambda^1, v^1 > = 0
\end{cases} \quad (KKT1)
\]

In the same way, the conditions of Karush-Kuhn-Tucker applied to the problem \((P_2)\), require that if \(E^2\) is a solution of the problem \((P_2)\) then there exist constants \(u^2 \in IR^3_+\), \(v^2 \in IR^3_+\) and \(\lambda^2 \in IR^3_+\) such that

\[
\begin{cases} 
2pqAE^2 + c - pqB^* + pqAE^1 - u^2 + A^T \lambda^2 = 0 \\
AE^2 + v^2 = -AE^1 + B^* \\
< u^2, E^2 > = < \lambda^2, v^2 > = 0
\end{cases} \quad (KKT2)
\]

It is immediately seen from \((KKT1)\) and \((KKT2)\) that

\[
\begin{cases} 
u^1 = 2pqAE^1 + c - pqB^* + pqAE^2 + A^T \lambda^1 \\
u^2 = 2pqAE^2 + c - pqB^* + pqAE^1 + A^T \lambda^2 \\
v^1 = -AE^1 - AE^2 + B^* \\
v^2 = -AE^1 - AE^2 + B^* \\
< u^i, E^i > = < \lambda^i, v^i > = 0 \quad \text{for all } i = 1, 2, 3 \\
E^i, u^i, \lambda^i, v^i \geq 0 \quad \text{for all } i = 1, 2, 3 
\end{cases} \quad (*1)
\]

\[
\begin{cases} 
u^1 = -AE^1 - AE^2 + B^* \\
< u^i, E^i > = < \lambda^i, v^i > = 0 \quad \text{for all } i = 1, 2, 3 \\
E^i, u^i, \lambda^i, v^i \geq 0 \quad \text{for all } i = 1, 2, 3 
\end{cases} \quad (*2)
\]

It is clear from equation \((*1)\) and from equation \((*2)\) that \(v^1 = v^2\).
To maintain the biodiversity of species, it is natural to assume that all biomasses remain strictly positive, that is \( B_j > 0 \) for all \( j = 1, 2, 3 \); therefore \( v^1 = v^2 > 0 \).

As the scalar product of \((\lambda^i)_{i=1,3}\) and \((\nu^i)_{i=1,3}\) is zero, so \( \lambda^i = 0 \) for all \( i = 1, 2, 3 \). In what follows of this paper, we denote by \( v = v^1 = v^2 \). So we have the following expressions

\[
\begin{cases}
  u^1 = 2pqAE^1 + pqAE^2 + c - pqB^* \\
  u^2 = pqAE^1 + 2pqAE^2 + c - pqB^* \\
  v = -AE^1 - AE^2 + B^* \\
  < u^i, E^i > = 0 \quad \text{for all } i = 1, 2, 3 \\
  E^i, u^i, v^i \geq 0 \quad \text{for all } i = 1, 2, 3
\end{cases}
\]

thus

\[
\begin{pmatrix}
  u^1 \\
  u^2 \\
  v
\end{pmatrix} =
\begin{bmatrix}
  2pqA & pqA & A^T \\
  pqA & 2pqA & 0 \\
  -A & -A & 0
\end{bmatrix}
\begin{pmatrix}
  E^1 \\
  E^2 \\
  0
\end{pmatrix} +
\begin{pmatrix}
  c - pqB^* \\
  c - pqB^* \\
  B^*
\end{pmatrix}.
\]

Let us denote by

\[
\begin{aligned}
  z &= \begin{pmatrix}
  E^1 \\
  E^2 \\
  0
\end{pmatrix},
  w &= \begin{pmatrix}
  u^1 \\
  u^2 \\
  v
\end{pmatrix},
  M &= \begin{bmatrix}
  2pqA & pqA & A^T \\
  pqA & 2pqA & 0 \\
  -A & -A & 0
\end{bmatrix}
\end{aligned}
\]

and

\[
\begin{pmatrix}
  c - pqB^* \\
  c - pqB^* \\
  B^*
\end{pmatrix}
\]

then our problem is equivalent to the Linear Complementarity Problem \(\text{LCP}(M, b)\) :

Find vectors \( z, w \in \mathbb{R}^6 \) such that \( w = Mz + b \geq 0 \), \( z, w \geq 0 \), \( z^Tw = 0 \).

To show that \(\text{LCP}(M, b)\) has a unique solution, we will use the following result:

\textbf{Théorème 3.1} : \(\text{LCP}(M, b)\) has a unique solution for every \( b \) if and only if \( M \) is a P-matrix.
Preuve 3.1: See Cottle\cite{18} and Murty\cite{8}.

Recall that: A matrix $M$ is called $P$-matrix if the determinant of every principal submatrix of $M$ is positive (see Murty\cite{17}).

The class of P-matrices generalizes many important classes of matrices, such as positive definite matrices, M-matrices, and inverse M-matrices, and arises in applications.

Note that each matrix symmetric positive definite is $P$-matrix, but the reverse is not always true.

Now we show that the matrix $M$ of our problem is $P$-matrix; which is equivalent to the existence and uniqueness of a solution of $LCP(M, b)$, therefore, the existence and uniqueness of a generalized Nash equilibrium.

Théorème 3.2: The matrix

\[
M = \begin{bmatrix} 2pqA & pqA & AT \\ pqA & 2pqA & 0 \\ -A & -A & 0 \end{bmatrix}
\]

is $P$-matrice.

Preuve 3.2: By the assumptions made in section 2 we have $a_{ii} < 0$ for all $i = 1, 2, 3$ and $\Delta > 0$ so, if we note by $(M_i)_{i=1,...,9}$ the submatrix of $M$, we obtain

\[
\begin{align*}
\text{det}(M_1) &= -2p_1q_1a_{11} > 0 \\
\text{det}(M_2) &= 4p_1q_1p_2q_2q_1K_1r_3q_2K_2\Delta > 0 \\
\text{det}(M_3) &= 8p_1q_1p_2q_2p_3q_3K_3q_1K_1q_2K_2\Delta^2 > 0 \\
\text{det}(M_4) &= -12a_{11}p_1^2q_1^2p_2q_2p_3q_3K_3q_1K_1q_2K_2\Delta^2 > 0 \\
\text{det}(M_5) &= 18p_1^2q_1^2p_2^2q_2^2p_3q_3q_1K_1r_3q_2K_2q_3K_3q_1K_1q_2K_2\Delta^3 > 0 \\
\text{det}(M_6) &= 27p_1^2q_1^2p_2q_2^2p_3q_3^2(K_3K_3q_1K_1q_2K_2\Delta^2)^2 > 0 \\
\text{det}(M_7) &= -9p_1q_1p_2^2q_2^2p_3q_3^2a_{11}(q_3K_3q_1K_1q_2K_2\Delta^2)^2 > 0 \\
\text{det}(M_8) &= 3p_1q_1p_2q_2^2p_3q_3^2q_1K_1r_3q_2K_2\Delta(q_3K_3q_1K_1q_2K_2\Delta^2)^2 > 0 \\
\text{det}(M_9) &= p_1q_1p_2q_2p_3q_3(q_3K_3q_1K_1q_2K_2\Delta^2)^3 > 0.
\end{align*}
\]
So the matrix $M$ is a $P$-matrix and therefore the linear complementarity problem $LCP(M, b)$ admits one and only one solution. This solution is given by

$$
\begin{align*}
E^1 &= \frac{1}{3} A^{-1}(B^* - \frac{c}{pq}) \\
E^2 &= \frac{1}{3} A^{-1}(B^* - \frac{c}{pq})
\end{align*}
$$

where $A^{-1}$ is the inverse of $A$, this matrix is given by

$$
A^{-1} = \begin{bmatrix}
\frac{r_1}{q_1} & \frac{c_{12}}{q_1} & \frac{c_{13}}{q_1} \\
K_1 q_1 & q_2 & q_3 \\
c_{31} & c_{32} & c_{33} \\
q_1 & q_2 & q_3
\end{bmatrix}
$$

It is interesting to compare with the when we consider only one fisherman who catches the three marine species (which are competing for space or food), then the fishing effort that maximizes the benefit of this fisherman is given by

$$
E = \frac{1}{2} \left[ \left( \frac{r_1}{K_1 q_1} + \frac{c_{21}}{q_2} + \frac{c_{31}}{q_3} \right) (B_1^* - \frac{c_1}{p_1 q_1}) \\
+ \left( \frac{c_{12}}{q_1} + \frac{r_2}{q_2} + \frac{c_{32}}{q_3} \right) (B_2^* - \frac{c_2}{p_2 q_2}) \\
+ \left( \frac{c_{13}}{q_1} + \frac{c_{23}}{q_2} + \frac{r_3}{q_3} \right) (B_3^* - \frac{c_3}{p_3 q_3}) \right]
$$

where $\text{det}(A) = q_1 q_2 q_3 K_1 K_2 K_3 / \triangle$ and $\triangle$ is given by (3).

The results are significantly different.

4. Numerical simulations of the mathematical model and discussion of the results

Now we deal with the general case by considering $n$ fishermen who catch three marine species that compete for space or food, this leads to the following generalized Nash equilibrium problem:
The fisherman \(i = 1, \ldots, n\) must solve the following problem \((P_i)_{1 \leq i \leq n}\):

\[
\begin{align*}
\max & \quad \pi_i(E) = < E^i, -pqAE^i + pqB^* - c - \sum_{k=1, k \neq i}^{n} pqAE^k > \\
\text{subject to} & \quad AE^i \leq -\sum_{k=1, k \neq i}^{n} AE^k + B^* \\
& \quad E^i \geq 0 \\
& \quad (E^k)_{1 \leq k \neq i \leq n} \text{ is given.}
\end{align*}
\]

To solve this problem we transform it into a linear complementarity problem of finding the two vectors \(z = (E^1, \ldots, E^n, 0)^T\) and \(w = (u^1, \ldots, u^n, v)^T\) satisfying \(z \geq 0, w = Mz + b \geq 0\) and \(< z, w > = 0\) where

\[
M = \begin{bmatrix}
2pqA & pqA & \ldots & pqA & A^T \\
pqA & 2pqA & \ldots & pqA & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
pqA & \ldots & pqA & 2pqA & 0 \\
-A & -A & pqA & \ldots & -A & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
c - pqB^* \\
c - pqB^* \\
\ldots \\
c - pqB^* \\
B^*
\end{bmatrix}
\]

It is very complicated to solve such a linear complementarity problem (LCP) for a large \(n\) even numerically. Many algorithms exist in the literature for solving this kind of problems (see for instance: Lenke[15], Murty[19], Kojima[13]), but for (LCP) with a large scale matrix these methods need very powerful machines to be implemented. That is why we developed algorithms ([9], [10]) more efficient for solving this problem.

We take as a case of study three marine species having the following characteristics

<table>
<thead>
<tr>
<th></th>
<th>(r_1=0.5)</th>
<th>(r_2=0.3)</th>
<th>(r_3=0.2)</th>
<th>(K_1=1000)</th>
<th>(K_2=700)</th>
<th>(K_3=600)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_{12}=2.10^{-4})</td>
<td>(c_{13}=3.10^{-4})</td>
<td>(c_{21}=10^{-5})</td>
<td>(c_{23}=2.10^{-5})</td>
<td>(c_{31}=10^{-4})</td>
<td>(c_{32}=10^{-4})</td>
<td></td>
</tr>
<tr>
<td>(q_1=0.1)</td>
<td>(q_2=0.02)</td>
<td>(q_3=0.004)</td>
<td>(p_1=40)</td>
<td>(p_2=60)</td>
<td>(p_3=120)</td>
<td></td>
</tr>
<tr>
<td>(c_1=0.05)</td>
<td>(c_2=0.10)</td>
<td>(c_3=0.20)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We’ll see how changes in the price or the number of fishermen can affect the effort of catch, the level of captures and the profits of fishermen. As
a first result we have (table 1) : an increase in price leads to an increase in fishing effort and an increase in catch levels.

<table>
<thead>
<tr>
<th>Price of the first specie</th>
<th>Price of the second specie</th>
<th>Price of the third specie</th>
<th>Total effort to catch the three species</th>
<th>Total catch of the three species</th>
</tr>
</thead>
<tbody>
<tr>
<td>02,00</td>
<td>03,00</td>
<td>006,00</td>
<td>62,89</td>
<td>530,02</td>
</tr>
<tr>
<td>05,00</td>
<td>07,50</td>
<td>015,00</td>
<td>63,34</td>
<td>532,17</td>
</tr>
<tr>
<td>08,00</td>
<td>12,00</td>
<td>024,00</td>
<td>63,45</td>
<td>532,71</td>
</tr>
<tr>
<td>10,00</td>
<td>15,00</td>
<td>030,00</td>
<td>63,49</td>
<td>532,89</td>
</tr>
<tr>
<td>14,00</td>
<td>21,00</td>
<td>042,00</td>
<td>63,53</td>
<td>533,09</td>
</tr>
<tr>
<td>18,00</td>
<td>27,00</td>
<td>054,00</td>
<td>63,55</td>
<td>533,20</td>
</tr>
<tr>
<td>20,00</td>
<td>30,00</td>
<td>060,00</td>
<td>63,56</td>
<td>533,24</td>
</tr>
<tr>
<td>30,00</td>
<td>45,00</td>
<td>090,00</td>
<td>63,59</td>
<td>533,36</td>
</tr>
<tr>
<td>40,00</td>
<td>60,00</td>
<td>120,00</td>
<td>63,60</td>
<td>533,42</td>
</tr>
</tbody>
</table>

Table 1 : An increase in price leads to an increase in fishing effort and an increase in catch levels.

Now we will see the influence of the number of fishermen on the catch levels and on the profit (see Table 2); to do so, we consider three situations :

In the first one we consider only one fisherman who catches the three marine species, to maximize the profit of this fisherman constrained by the conservation of the biodiversity of the three marine species, he must catch 137, 13, in this case his profit is equal to 7260, 31.

In the second one we consider two fishermen who catch the three marine species, to maximize the profit, each fisherman must catch 060, 96, in this case the profit of each fisherman is equal to 3226, 80, this situation reduces the catch of each fisherman by 44, 45% and reduces the profit of each fisherman by 44, 44%.

In the third situation we consider five fishermen who catch the three marine species, to maximize the profit, each fisherman must catch 015, 25, in this case the profit of each fisherman is equal to 0806, 70, this situation reduces the catch of each fisherman by 11, 12% and reduces the profit of each fisherman by 11, 11%.
So the three situations show that, when the number of fishermen is increasing, the catch and the profit of each fisherman are decreasing.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Catch/Fisherman</th>
<th>Profit/Fisherman</th>
</tr>
</thead>
<tbody>
<tr>
<td>n1</td>
<td>137,13</td>
<td>7260,31</td>
</tr>
<tr>
<td>n2</td>
<td>060,96</td>
<td>3226,80</td>
</tr>
<tr>
<td>n3</td>
<td>015,25</td>
<td>0806,70</td>
</tr>
</tbody>
</table>

Table 2: The influence of number of fishermen on the catch and profit.

On the contrary, since the number of fishermen is increasing, the total effort to catch the three species is increasing, but the total catch is decreasing and the total profit is decreasing as shown in Figure 1.

Figure 1: The influence of the fishermen number on the total catch and total profit.
Now we see that an increase in fishermen number leads to an increase in the total fishing effort and reduced the total profit as shown in Table 3.

<table>
<thead>
<tr>
<th>Fishermen number</th>
<th>Total fishing effort</th>
<th>Total Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>01 fisherman</td>
<td>34,98</td>
<td>7260,31</td>
</tr>
<tr>
<td>02 fishermen</td>
<td>46,64</td>
<td>6453,61</td>
</tr>
<tr>
<td>03 fishermen</td>
<td>52,47</td>
<td>5445,23</td>
</tr>
<tr>
<td>05 fishermen</td>
<td>58,30</td>
<td>4033,50</td>
</tr>
<tr>
<td>10 fishermen</td>
<td>63,60</td>
<td>2400,10</td>
</tr>
<tr>
<td>15 fishermen</td>
<td>65,59</td>
<td>1701,63</td>
</tr>
<tr>
<td>20 fishermen</td>
<td>66,63</td>
<td>1317,06</td>
</tr>
<tr>
<td>25 fishermen</td>
<td>67,27</td>
<td>1074,01</td>
</tr>
<tr>
<td>30 fishermen</td>
<td>67,70</td>
<td>906,59</td>
</tr>
<tr>
<td>35 fishermen</td>
<td>68,02</td>
<td>784,29</td>
</tr>
<tr>
<td>40 fishermen</td>
<td>68,25</td>
<td>691,05</td>
</tr>
<tr>
<td>45 fishermen</td>
<td>68,44</td>
<td>617,61</td>
</tr>
<tr>
<td>50 fishermen</td>
<td>68,59</td>
<td>558,27</td>
</tr>
</tbody>
</table>

Table 3: The influence of number of fishermen on the total fishing effort and total profit.

5. Conclusion and perspectives

In this work we have defined a bioeconomic equilibrium model for ‘n’ fishermen who catch three species, these species compete with each other for space or food. The natural growth of each species is modeled using a logistic law. We have calculated the fishing effort that maximizes the profit of each fisherman at biological equilibrium by using the generalized Nash equilibrium problem. The existence of the steady states and its stability are studied using eigenvalue analysis. Finally, some numerical examples are given to illustrate the results.

In this work, we have considered that the prices of marine species are constants, we consider in a future work to define functions of providing long term, where price is no longer a constant but depends on the level of effort and biomass stock of each species remaining.
Références


