Reliability Equivalence Factors in Exponentiated Exponential Distribution.

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Abstract

The reliability equivalence factors of parallel system with n independent and identical components will be discussed. The failure rates of the system components are functions of time and follow the exponentiated exponential distribution. We shall introduce two different methods to improve the system designs. The first is the reduction method and the second is called redundancy method which is composed of hot and cold duplication methods. In reduction method, it is assumed that the failure rates of some of the system components are reduced by a factor $\rho$, $0 < \rho < 1$. In the hot duplication method, it is assumed that some of the system components are duplicated in parallel while some of the system components are duplicated in parallel by a perfect switch in the cold duplication method. Therefore, one can make equivalence between the reduction method and the duplication method based on some reliability measures. In other words, the design of the system that is improved according to the reduction method should be equivalent to the design of the system according to one of the redundancy method. The comparison of the designs produce the so-called reliability equivalence factors. The survival function and $r$th moment time to failures will be used as performance measures of the system reliability to compare different system designs.

Keywords: Reliability equivalence factors; Exponentiated exponential distribution; Reduction method; The hot and cold duplication methods; $\alpha$- Fractiles.

1. Introduction

Recently a new distribution, named exponentiated exponential (EE) distribution or Generalized exponential (GE) distribution was introduced and studied quite

\[ F(x, \theta, \lambda) = (1 - e^{-\lambda x})^\theta, \quad \theta, \lambda, x > 0, \]  

with the density function

\[ f(x, \theta, \lambda) = \theta \lambda (1 - e^{-\lambda x})^{\theta - 1} e^{-\lambda x}, \quad \theta, \lambda, x > 0, \]  

where \( \theta \) is the shape parameter and \( \lambda \) is the scale parameter. When the shape parameter \( \theta \) equals one it reduces to a one-parameter exponential distribution, that is exponentiated exponential is a generalization of one-parameter exponential distribution. The exponentiated exponential distribution with shape parameter \( \theta \) and scale parameter \( \lambda \) will be denoted by \( \text{EE}(\theta, \lambda) \). It is observed in Gupta and Kundu (1999) that the \( \text{EE}(\theta, \lambda) \) can be used quite effectively in analyzing many lifetime skewed data, and the properties of the \( \text{EE}(\theta, \lambda) \) distribution are quite close to the corresponding properties of the two-parameter gamma distribution.

A reliability equivalence factor is a factor by which a characteristic of components of a system design has to be multiplied in order to reach equality of a characteristic of this design and a different design regarded as a standard. Equivalent of different system designs with respect to a reliability characteristic such as mean time to failure (MTTF) or survival function in case of no repairs is needed. Rade (1989) has introduced the concept of reliability equivalence. Rade (1990, 1991, 1993a, 1993b) has applied such concept to some simple series and parallel systems of two independent and identical components. Rade (1993a and 1993b) has used three different methods to improve the reliability of a system. In these methods it is assumed that the reliability of a system can be improved by: (i) improving the quality of one or several components by decreasing their failure rates; (ii) adding a hot component to the system; (iii) adding a cold redundant component to the system. Sarhan (2000) has introduced more generals improving methods of a system. In such methods, the reliability of a system can be improved by: (i) improving the quality of some components by reducing their failure rates by a factor \( \rho \), \( 0 < \rho < 1 \); (ii) assuming hot duplications of some components; (iii) assuming cold duplications of some components; (iv) assuming cold redundant standby components connected with some components (one for each) by random switches. Rade (1993a, 1993b) and Sarhan (2000) have used the survival function as the performance measure of the reliability system. Rade (1993a, 1993b) has calculated the reliability equivalence factors of series and parallel systems which consist of independent and identical components with constant failure rate. Sarhan et al. (2004) have studied the equivalence of different designs of a four independent and identical components series–parallel system. They have tried to deduce the reliability equivalence factors of a series–parallel system. In obtaining these factors, the reliability function and mean time to failure of the system as performance measures to compare different system designs of original system and others improved systems. Sarhan (2005) obtained the reliability equivalence factors of a parallel system with n independent and identical components.
using the general method which is established in (2000). Sarhan and his colleagues’ work assumed that the failure rates of the system’s components are constant. That means their work subject to the exponential distribution. Moreover, either the parallel or series system with \( n \) components assumed to be independent and identical. However, Sarhan (2000) obtained the MTTF in the case of independent and non-identical components series systems and some especial cases studied in parallel system. Generally speaking, to obtain the MTTF the first moment of order statistics (see, e.g., Arnold et al. (1992) and Asadi (2006) is used. That is, compute

\[
E(X_{n:n}) = \mu_{mn}\text{ for parallel system and } E(X_{1:n}) = \mu_{1n}\text{ for the series system.}
\]

In the current study we shall use the survival function and the mean time to failure to calculate the reliability equivalence factors for parallel system, consisting of \( n \) independent and identical components. The constant failure rate arises when assuming the exponential distribution and non-constant failure rate arises when considering distributions rather than the exponential distribution such as: Weibull, Erlang, Gamma, see Barakat and Abdelkader (2000, 2004), Abdelkader (2004 a and 2004 b), Abdelkader and Abotahoun (2006) and Abdelkader (2010). This paper is organized as follows. In Section 2, we present the parallel system. The survival functions of the original and improved systems are presented in Section 3. The generalized reliability equivalence factors are derived in Section 4. The b- fractiles of the original and improved systems are introduced in Section 5.

2. n-Component parallel system

The system considered here consists of \( n \) independent and identical components connected in parallel. If \( T \) is the time to failure of the component, then the probability that it will not fail in given environment before time \( t \), i.e., its reliability is

\[ R(t) = P(T > t). \]

For the exponentiated exponential distribution the reliability function is given by

\[ R(t) = 1 - (1 - e^{-\lambda t})^\theta. \]  

The most important methods for improving the system reliability are:

(i) The reduction method, this can be done by reducing the failure rates of \( r \), \( 1 \leq r \leq n \), of the components by the same factor \( \rho \), \( 0 < \rho < 1 \). This leads to increase the reliability of the components in the system.

(ii) Hot duplication method, \( m \) hot redundant standby components operate in parallel, for \( 1 \leq m \leq n \).

(iii) Cold duplication method, \( m \) cold redundant standby components are switched in to take over the function of a component or subsystem that has failed, \( 1 \leq m \leq n \).
Equivalence between the improved systems obtained according to the reduction method and the hot and cold duplication methods based on the value of the reliability function should be taken place. In the next Section the reliability functions of the original system and the improved systems obtained according to the above-described methods will be discussed.

3. The reliability functions and MTTF’s

In this section, we present the reliability (survival) functions and the MTTFs of the original and improved systems

3.1 The original system

The reliability function of the original system which consists of n s-independent components, denoted \( R(t) \), can be obtained as follows:

\[
R(t) = 1 - \prod_{i=1}^{n} [1 - R_i(t)]. \tag{4}
\]

Using Eqs. (3) and (4), the reliability function of the system can be written as

\[
R(t) = 1 - \left(1 - e^{-\lambda t}\right)^{\beta n} = \sum_{i=1}^{\beta n} (-1)^{i+1} \left(\frac{\beta n}{i}\right) e^{-i\lambda t}, \tag{5}
\]

where \( \theta \) is a positive integer.

The system MTTF is defined by

\[
MTTF = \int_{0}^{\infty} R(t) \, dt = \sum_{i=1}^{\beta n} (-1)^{i+1} \frac{\left(\frac{n\theta}{i}\right)}{i\lambda}. 
\]

Using the following relation

\[
\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} \left(\frac{n}{i}\right) = \sum_{i=1}^{n} \frac{1}{i} \approx \frac{1}{n} + \log(n), \tag{6}
\]

we can write

\[
MTTF = \frac{1}{\lambda} \sum_{i=1}^{n\theta} \frac{1}{i} \approx \frac{1}{\lambda} \left(\frac{1}{n\theta} + \log(n\theta)\right). \tag{7}
\]

3.2. The reduction method

Assume that the system is improved by improving \( r \), \( 1 \leq r \leq n \), of its components according to the reduction method. That is, the failure rates of \( r \) components are reduced from \( \lambda \) to \( \rho \lambda \), \( 0 < \rho < 1 \). Let \( R_{(r), \rho}(t) \) denote the reliability function of the system improved by reducing the failure rates of \( r \) of its components by the factor \( \rho \). One can \( R_{(r), \rho}(t) \) to be

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\[ R_{(r),\rho}(t) = 1 - [1 - R(\rho t)]^r [1 - R(t)]^{n-r} \]  
\[ = 1 - (1 - e^{-\lambda \rho t})^{\theta r} (1 - e^{-\lambda t})^{\theta (n-r)}. \]  

Using the binomial expansion, one can write \( R_{(r),\rho}(t) \) in the following form
\[ R_{(r),\rho}(t) = 1 - \sum_{j=0}^{\theta (n-r)} (-1)^j \binom{\theta (n-r)}{j} e^{-j \lambda t} \sum_{i=0}^{\theta r} (-1)^i \binom{\theta r}{i} e^{-i \lambda \rho t} = \]
\[ \sum_{j=1}^{\theta (n-r)} \frac{(-1)^{j+1}}{j \lambda} \binom{\theta (n-r)}{j} + \sum_{j=0}^{\theta (n-r)} \sum_{i=1}^{\theta r} \frac{(-1)^{i+j+1}}{(j+i) \lambda} \binom{\theta (n-r)}{j} \binom{\theta r}{i}. \]  

The system MTTF, say \( MTTF_{(r),\rho} \), is
\[ MTTF_{(r),\rho} = \int_0^\infty R_{(r),\rho}(t) \, dt \]
\[ = \frac{1}{\lambda} \sum_{j=1}^{\theta (n-r)} \frac{1}{j} + \frac{1}{\lambda} \sum_{j=0}^{\theta (n-r)} \sum_{i=1}^{\theta r} \frac{(-1)^{i+j+1}}{(j+i) \lambda} \binom{\theta (n-r)}{j} \binom{\theta r}{i}. \]  

From (8) and (11), one can write \( MTTF_{(r),\rho} \) as
\[ MTTF_{(r),\rho} = MTTF + \frac{1}{\lambda} \left\{ \sum_{j=0}^{\theta (n-r)} \sum_{i=1}^{\theta r} \frac{(-1)^{i+j+1}}{(j+i) \lambda} \binom{\theta (n-r)}{j} \binom{\theta r}{i} - \sum_{j=\theta (n-r)+1}^{\theta n} \frac{1}{j} \right\}. \]  

That is, reducing the failure rates of \( r \) of the system’s components by the same factor \( \rho \) increases the MTTF of the system by the amount \( \Delta MTTF_{(r),\rho} \), which is given by
\[ \Delta MTTF_{(r),\rho} = MTTF_{(r),\rho} - MTTF = \frac{1}{\lambda} \left\{ \sum_{j=0}^{\theta (n-r)} \sum_{i=1}^{\theta r} \frac{(-1)^{i+j+1}}{(j+i) \lambda} \binom{\theta (n-r)}{j} \binom{\theta r}{i} - \sum_{j=\theta (n-r)+1}^{\theta n} \frac{1}{j} \right\}. \]

3.3. The hot duplication method

Let us assume that the system is improved by improving \( m \), \( 1 \leq m \leq n \), of its components according to the hot duplication method. Let \( R_{(m),H}(t) \) denote the reliability function of the system improved by improving \( m \) of its components by hot duplication. One can obtain \( R_{(m),H}(t) \) to be
\[ R_{(m),H}(t) = 1 - \prod_{i=1}^{n+m} (1 - R_i(t)) = 1 - (1 - e^{-\lambda t})^{\theta (n+m)}. \]  

Using the binomial expansion, one can write \( R_{(m),H}(t) \) in the following form
The system MTTF, say $MTTF^H_{(m)}$, is given by

$$MTTF^H_{(m)} = \int_0^\infty R^H_{(m)}(t) \, dt = \frac{1}{\lambda} \sum_{i=1}^{\theta(n+m)} \frac{(-1)^{i+1}}{i} \theta(n+m) \sum_{i=1}^{\theta(n+m)} \frac{1}{i} (\theta(n+m))_i^e - i \lambda t.$$  

Relation (17) means that, the process of improving the original system by improving $m$ of its components according to hot duplication method increases the MTTF of the system by the amount $\Delta MTTF^H_{(m)}$, which is given by

$$\Delta MTTF^H_{(m)} = MTTF^H_{(m)} - MTTF = \frac{1}{\lambda} \sum_{i=\theta(n+1)}^{\theta(n+m)} \frac{1}{i}.$$  

3.4. The cold duplication method

Let us consider now that the system is improved by improving $m$, $1 \leq m \leq n$, of its components according to the cold duplication method. Let $R^C_{(m)}(t)$ denote the reliability function of the system improved by improving $m$ of its components according to cold duplication method.

The function $R^C_{(m)}(t)$ can be obtained as follows

$$R^C_{(m)}(t) = 1 - [1 - R^C_{(1)}(t)]^m[1 - R(t)]^{n-m} = 1 - [1 - R^C_{(1)}(t)]^m (1 - e^{-\lambda t})^{\theta(n-m)},$$

where $R^C_{(1)}(t)$ denotes to the reliability of a system’s component after it was improved according to cold duplication method. Following the technique given in Ref. [9], one can obtain $R^C_{(1)}(t)$ to be, at $\theta = 1$, $R^C_{(1)}(t) = (1 + \lambda t)e^{-\lambda t}$. Therefore, $R^C_{(m)}(t)$ becomes

$$R^C_{(m)}(t) = 1 - [1 - (1 + \lambda t)e^{-\lambda t}]^m[1 - e^{-\lambda t}]^{n-m}.$$
4. Reliability equivalence factors

Generally, the reliability equivalence factor is defined as that factor by which the failure rates of some of the system’s components should be reduced in order to reach equality of the reliability of another better system, see Sarhan (2005). In what follows, we present some of reliability equivalence factors of the parallel system studied here.

4.1. Hot reliability equivalence factor \( \rho_{(m),(r)}^H(\alpha) \)

The hot reliability equivalence factor, say \( \rho = \rho_{(m),(r)}^H(\alpha) \), is defined as that factor \( \rho \) by which the failure rates of \( r \) of the system’s components should be reduced in order to reach the reliability of that system which improved by improving \( m \) of the original system’s components according to hot duplication method. That is, \( \rho = \rho_{(m),(r)}^H(\alpha) \) is the solution of the following system of two equations

\[
R_{(r),\rho}(t) = \alpha, \quad R_{m}^H(t) = \alpha. \tag{25}
\]

Substituting from (14) into the second equation in (25), one can derive that

\[
e^{-\lambda t} = 1 - (1 - \alpha)^{\frac{1}{\theta(n+m)}}. \tag{26}
\]

Substituting from (9) into the first equation in (25), we get

\[
\alpha = 1 - (1 - e^{-\lambda t})^{\theta r} (1 - e^{-\lambda t})^{\theta(n-r)}. \tag{27}
\]

Using (26) and (27), we find

\[
1 - \alpha = (1 - \alpha)^{\frac{n-r}{n+m}} \{ 1 - [1 - (1 - \alpha)^{\frac{1}{\theta(n+m)}}]\}^{\theta r}. \tag{28}
\]
Solving the Equation (28) with respect to $\rho$, the hot reliability factor $\rho = \rho_{(m),(r)}^H(\alpha)$ becomes

$$\rho = \rho_{(m),(r)}^H(\alpha) = \frac{\ln \left(1 - (1-\alpha)\frac{1}{\theta(n+m)}\right)}{\ln \left(1 - (1-\alpha)\theta(n+m)\right)}$$

(29)

at $\theta = 1$, the results agree with Sarhan [24] and the results introduced in [20, 21] can be obtained as special cases from (29) when $\theta = 1, n = 2, m = 1, r = 1, 2$.

4.2. Cold reliability equivalence factor $\rho_{(m),(r)}^C(\alpha)$

The cold reliability equivalence factor, say $\rho = \rho_{(m),(r)}^C(\alpha)$, is defined as the factor $\rho$ by which the failure rates of $r$ of the system’s components should be reduced in order to reach the reliability of that system where $m$ of the original system’s components have cold duplicates. That is, $\rho = \rho_{(m),(r)}^C(\alpha)$ is the solution of the following system of two equations, at $\theta = 1$,

$$R_{(r),\rho}(t) = \alpha, \quad R_{(m)}(t) = \alpha.$$  

(30)

Using Equations (8), (20) and (30), at $\theta = 1$, we get

$$1 - \alpha = [1 - (1 + \lambda t)e^{-\lambda t}]^m(1 - e^{-\lambda t})^{n-m},$$  

(31)

$$1 - \alpha = [1 - e^{-\lambda \rho t}]^r(1 - e^{-\lambda t})^{n-r}.$$  

(32)

Solving (31) and (32), we can derive the cold reliability factor $\rho = \rho_{(m),(r)}^C(\alpha)$ as

$$\rho = \rho_{(m),(r)}^C(\alpha) = \frac{1}{\ln x} \ln \left[1 - \frac{(1-\alpha)^{1/r}}{(1-x)^{1/r}}\right],$$  

(33)

where $x$ is non-negative real solution of the following equation

$$(1 - x + x \ln x)^m (1 - x)^{n-m} + \alpha - 1 = 0.$$  

(34)

As it seems, Eq.(34) has no closed form solution. Therefore, some numerical technique is needed to get $x$ and then to calculate $\rho_{(m),(r)}^C(\alpha)$.

5. $\alpha$-Fractiles

The $\alpha$-fractiles of the original and modified systems are presented in this section. Let $L(\alpha)$ be the $\alpha$-fractile of the original system. Also, let $L_{(m)}^H(\alpha)$ and $L_{(m)}^C(\alpha)$ denote, respectively, to the $\alpha$-fractile of the modified system obtained by
improving \( m \) of the system’s components according to hot and cold duplication methods.

The fractile \( L(\alpha) \) can be found by solving the following equation with respect to \( L \)

\[
R \left( \frac{L}{\alpha} \right) = \alpha. \tag{35}
\]

Using (4) and (35), one can find out that

\[
L(\alpha) = -\ln [1 - (1 - \alpha)^{\frac{1}{\theta m}}]. \tag{36}
\]

The fractile \( L^H_{(m)}(\alpha) \) can be deduced by solving the following equation with respect to \( L \)

\[
R^H_{(m)} \left( \frac{L}{\alpha} \right) = \alpha. \tag{37}
\]

Using (14) and (37), we can obtain

\[
L^H_{(m)}(\alpha) = -\ln [1 - (1 - \alpha)^{\frac{1}{\theta (n+m)}}]. \tag{38}
\]

Finally, \( L^C_{(m)}(\alpha) \) can be obtained by solving the following equation with respect to \( L \)

\[
R^C_{(m)} \left( \frac{L}{\alpha} \right) = \alpha. \tag{39}
\]

Using (9) and (39), at \( \theta = 1 \), we can deduce

\[
1 - [1 - (1 + L)e^{-L}]^m[1 - e^{-L}]^{n-m} - \alpha = 0. \tag{40}
\]

The above Eq. (40) has no closed solution in \( L \). Thus, to find out \( L = L^C_{(m)}(\alpha) \), we have to use a numerical technique method to solve (40).

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References


