Abstract—Acting as a supervision or optimization module for a local controller, indirect iterative learning control (ILC) can improve closed-loop performance along the batch direction. In particular, if the set-point for the local controller is updated, this type of indirect ILC is termed set-point-related (SPR) indirect ILC. SPR indirect ILC has shown excellent performance; however, there is no reported work on stability analysis for this method. Based on a 2-dimensional (2D) framework, a sufficient condition for asymptotical stability of SPR indirect ILC is derived in this paper.

I. INTRODUCTION

Inspired by human learning, researchers have tried to find schemes to implement learning ability in the automatic control of dynamic systems. One such scheme is known as iterative learning control (ILC). In 1978, the initial explicit formulation of ILC was presented in Japanese [1]. In 1984, Arimoto et al. first introduced this method in English [2]. These contributions are widely regarded as the origin of ILC. During the past several decades, ILC has been applied widely in batch processes [3]-[4] and continuous processes with periodic inputs [5]-[6].

In general, there are two ways to utilize ILC: 1) to determine the control signal directly, a method known as direct ILC [7]; 2) to update some parameters for the local controller, a method that is known as indirect ILC [7]. For clarity, the main structures of direct and indirect ILC are illustrated in Fig. 1.

As shown in Fig. 1, there are two loops in the case of indirect ILC: the inner loop is the local controller; while the outer loop is the ILC. In this case, ILC acts as a supervision or optimization module for the closed-loop system under the local controller. The following two issues are essential for indirect ILC: what algorithms are used to design the local controller, and which parameters of the local controller are updated by ILC. Generally speaking, ILC can be used to update the set-point [8], control gain [9], weight [10], and other parameters [11]-[12] for the local controller. Therefore, \( q(t,k) \) in Fig. 1 could be set-point, control gain, weight, or other parameters.

Compared with the direct form, indirect ILC has some distinct advantages. First, it is not necessary to change the existing process structure, and only an outer loop module is required to update selected parameters for the local controller. Second, indirect ILC has better robustness properties than the direct form; this is because direct ILC must have a feedforward term, which is sensitive to variations in the batch direction, but a feedforward term is not necessary for the local controller of the indirect algorithm. Furthermore, the development of the indirect form is very advanced: stability and robustness are not the only requirement for control design, and some optimization schemes are involved to improve the performance. However, the reported results on indirect ILC are scarce, thereby motivating the current work.

In the review paper [7], 207 articles from the Web of Science that featured “iterative learning control” in the title were reviewed, and only 16 of them were determined to focus on indirect ILC. Among the 16 indirect-ILC-related papers, ILC was used to update the set-point for the local control in only two works [8],[13]. Motivated by research opportunities, the authors finished two studies [6],[14] in this field recently, in which both of the local controllers were selected as model predictive control (MPC). For short, an indirect ILC for adjusting the set-point of the local controller is termed as set-point-related (SPR) indirect ILC.

In the existing works on SPR indirect ILC, the stability of the closed-loop system is not proved yet. Obviously, stability analysis for SPR indirect ILC is very important for both
theory development and practical implementation. This issue was addressed in this work. To focus on ILC, the local controller was designed by using a simple algorithm – P-type control. For simplicity, a single-input single output system was studied in this work. The stability of the closed-loop system was proved in a 2-dimensional (2D) framework.

II. PROBLEM FORMULATION

A. System Statement

Consider the following single-input single-output batch process,
\[
\begin{align*}
    x(t+1, k) &= Ax(t, k) + Bu(t, k) \\
    y(t, k) &= Cx(t, k) \\
    x(0, k) &= x_0; \quad t = 0, 1, \ldots T-1; \quad k = 1, 2, \ldots
\end{align*}
\]

where \( t \) denotes time; \( k \) denotes batch index; \( x(t, k) \in \mathbb{R}^n \), \( u(t, k) \in \mathbb{R}^1 \), and \( y(t, k) \in \mathbb{R}^1 \) represent, respectively, the states, output, and input of the process at time \( t \) of the \( k \)th batch run; \( \{A, B, C\} \) are the system matrices with appropriate dimensions; \( x_0 \) is the identical initial condition for each batch.

The control objective is to determine a control law such that the output of the process tracks a given target, \( y_s(t) \), as closely as possible. The tracking error is defined as:
\[
e(t, k) = y_s(t) - y(t, k)
\]

Some additional assumptions on system (1) are required:

**Assumption 1.** System (1) is output controllable. Therefore, there exists a matrix \( K \) such that the closed-loop system
\[
x(t+1, k) = (A - BK) x(t, k)
\]
is asymptotically stable. In other words, \( \lambda_{\text{max}}(A - BK) < 1 \), where \( \lambda_{\text{max}}(*) \) denotes the maximum absolute value of eigenvalues for matrix *.

**Assumption 2.** \( \text{Re}\left\{C \left[I - z_i^{-1} (A - BK) C \right]^{-1} BK \right\} \neq 0 \)
always holds for all \( |z_i| \leq 1 \), where \( \text{Re}(*) \) denotes the real part of the complex number *.

B. The Simplest Feedback Controller

Because the focal point for this work is ILC, the local controller was designed by using the simplest structure, as shown below
\[
u_i(t, k) = K \left[y_s(t) - y(t, k)\right] = K e(t, k)
\]
This controller is referred to as the P-type controller.

Notice that the target \( y_s(t) \) was used as the set-point for the P-type controller in (3). In fact, choosing the target as the set-point is the most common way in control design. Exploiting the repetitive nature of the process, ILC can be used to optimize the set-point for the local controller and then to improve the closed-loop performance.

C. ILC-based P-type Control (Indirect ILC)

In this section, a novel P-type control is introduced,
\[
u_i(t, k) = K \left[y_s(t) - y(t, k)\right] = K e(t, k)
\]

The set-point \( y_s(t, k) \) could be different in various batches, and it is updated by using ILC, as shown below,
\[
y_s(t, k) = y_s(t, k-1) + L e(t+1, k-1)
\]
This is a typical P-type ILC [15], where \( L \) is the learning gain matrix. The essence of (5) is optimizing the set-point using the tracking performance in the previous batch.

The following analysis will prove that an ILC-based set-point can improve the P-type controller’s performance if an approximate \( L \) is designed.

III. STABILITY ANALYSIS

In the previous section, two controllers, a P-type controller and an ILC-based P-type controller, were presented. In this section, the stability of these controllers will be analyzed in a 2-dimensional framework. First, the following notation is introduced,
\[
\Delta \xi(t, k) \equiv \xi(t, k) - \xi(t, k-1), \quad \xi = x, y, u, \ldots
\]

where \( \Delta \xi \) is the variation of \( \xi \) in batch direction.

Some Lemmas on the stability of 2D systems are now introduced.

**Lemma 1** [16, P161]. A Roesser’s system,
\[
\begin{align*}
    x^h(i+1, j) &= A_{11} x^h(i, j) + A_{12} x^v(i, j) \\
    x^v(i, j+1) &= A_{21} x^h(i, j) + A_{22} x^v(i, j)
\end{align*}
\]

is asymptotically stable if and only if
\[
d(z_i^{-1}, z_2^{-1}) = \text{det} \begin{bmatrix} I - z_i^{-1} A_{11} & -z_1^{-1} A_{12} \\ -z_2^{-1} A_{21} & I - z_2^{-1} A_{22} \end{bmatrix} \neq 0
\]
for all \( |z_i| \leq 1, |z_2| \leq 1 \). Where \( x^h \) is the horizontal state vector; \( x^v \) is the vertical state vector; \( \text{det}(*) \) denotes the determinant of matrix *.

**Lemma 2** [16, P158]. It is always true that
\[ d(z_i^{-1}, z_i^{-1}) = \det \left( I - z_i^{-1} A_i \right) \times \]
\[ \det \left[ I - z_i^{-1} \left( A_{22} + A_{21} (z_i I - A_1)^{-1} A_2 \right) \right] \]  

(9)

\[ d(z_i^{-1}, z_i^{-1}) \text{ is denoted the denominator polynomial for system (7).} \]

A. 2D Formulation for P-type Control

From (1), (2), (3), and (6), it can be shown that

\[ \Delta x(t + 1, k) = A \Delta x(t, k) + B \Delta u_i(t, k) = A \Delta x(t, k) + B \Delta y(t, k) \]
\[ = A \Delta x(t, k) - B \Delta y(t, k) = (A - BKC) \Delta x(t, k) \]  

and

\[ e(t + 1, k) = e(t + 1, k - 1) - \Delta y(t + 1, k) \]
\[ = e(t + 1, k - 1) - CA \Delta x(t, k) \]
\[ = e(t + 1, k - 1) - C(A - BKC) \Delta x(t, k) \]  

(11)

Eq. (10) and (11) comprise a 2D system,

\[ \Sigma_1 : \left[ \begin{array}{c} \Delta x(t + 1, k) \\ e(t + 1, k) \end{array} \right] = \left[ \begin{array}{cc} A - BKC & 0 \\ -C(A - BKC) & 1 \end{array} \right] \left[ \begin{array}{c} \Delta x(t, k) \\ e(t + 1, k - 1) \end{array} \right] \]  

(12)

Equation (12) is a typical Roesser’s model denoting

\[ x^s(t + 1, k) = \Delta x(t + 1, k) \] \quad and \quad \[ x^r(t, k + 1) = e(t + 1, k) \]  

Therefore, the batch process (1) under P-type controller (2) can be transferred to a Roesser’s 2D model (12). Analyzing the stability of the batch process (1) under P-type controller is equivalent to analyzing the stability of Roesser’s 2D systems (12). Its denominator polynomial is

\[ d_s(z_i^{-1}, z_i^{-1}) = \det \left[ I - z_i^{-1} (A - BKC) \right] \times \det \left[ I - z_i^{-1} I \right] \]  

(13)

If we set \( z_i^{-1} = 1 \), then \( d_s(z_i^{-1}, z_i^{-1}) = 0 \). According to Lemma 1, system (12) is not asymptotically stable.

Because \( x(0, k) = x_0 \), one knows that \( \Delta x(0, k) \equiv 0 \). Based on the first line of (12), it is true that \( \Delta x(t, k) \equiv 0 \). From the second line of (12), one knows that \( e(t + 1, k) = e(t + 1, k - 1) \), in other words, the P-type controller cannot improve the performance from batch to batch. In fact, it is well-known that the P-type controller always produces steady-state tracking errors. The stability analysis in the 2D framework validates this conclusion.

B. 2D Formulation for ILC-based P-type Control

Under ILC-based P-type controller (4) and (5), system (1) can be transformed as

\[ \Delta x(t + 1, k) = A \Delta x(t, k) + B \Delta u_i(t, k) \]
\[ = A \Delta x(t, k) + B \Delta y_i(t, k) - B \Delta y(t, k) \]
\[ = (A - BKC) \Delta x(t, k) + BK LE(t + 1, k - 1) \]

and

\[ e(t + 1, k) = e(t + 1, k - 1) - C \Delta x(t + 1, k) \]
\[ = (I - CBKL) e(t + 1, k - 1) - C (A - BKC) \Delta x(t, k) \]  

(15)

Therefore, a 2D system can be obtained as follows:

\[ \Sigma_2 : \left[ \begin{array}{c} \Delta x(t + 1, k) \\ e(t + 1, k) \end{array} \right] = \left[ \begin{array}{cc} A - BKC & BKL \\ -C(A - BKC) & I - CBKL \end{array} \right] \left[ \begin{array}{c} \Delta x(t, k) \\ e(t + 1, k - 1) \end{array} \right] \]  

(16)

Analyzing the stability of the batch process (1) under ILC P-type controller is equivalent to analyzing the stability of Roesser’s 2D system (16). Its denominator polynomial is

\[ d_s(z_i^{-1}, z_i^{-1}) = \det \left[ I - z_i^{-1} (A - BKC) \right] \times \det \left[ I - z_i^{-1} \left( (A - BKC) I - BK L \right) \right] \]  

(17)

where

\[ CBKL + C (A - BKC) ((z_i I - (A - BKC))^{-1} BK L \]
\[ = CBKL + C \left( z_i (A - BKC)^{-1} - I \right)^{-1} BK L \]
\[ = C \left[ I + \left( z_i (A - BKC)^{-1} - I \right)^{-1} \right] BK L \]
\[ = C \left( z_i (A - BKC)^{-1} - I \right)^{-1} \left[ (z_i (A - BKC)^{-1} - I) + I \right] BK L \]
\[ = C \left[ I - z_i^{-1} (A - BKC) \right]^{-1} BK L \]  

(18)

Combining (17) and (18), one obtains

\[ d_s(z_i^{-1}, z_i^{-1}) = \det \left[ I - z_i^{-1} (A - BKC) \right] \times \det \left[ I - z_i^{-1} \left( I - C z_i^{-1} (A - BKC) \right)^{-1} BK L \right] \]

(19)

Based on Assumption 1, one knows that \( \lambda_{\max} (A - BKC) < 1 \); hence,

\[ \det \left[ I - z_i^{-1} (A - BKC) \right] \neq 0, \text{ for all } |z_i^{-1}| \leq 1. \]  

(20)

**Lemma 3.** Given Assumption 2, there exists \( L \) such that

\[ \det \left[ I - z_2^{-1} \left( I - C z_2^{-1} (A - BKC) \right)^{-1} BK L \right] \neq 0 \]

for all \( |z_1^{-1}| \leq 1, |z_2^{-1}| \leq 1 \).  

(21)
Proof. It is clear that $\text{Re}\left\{C\left[I-z_1^{-1}(A-BK)\right]^{-1}BK\right\}$ is continuous in the closed and bounded set $|z_1^{-1}|\leq 1$. As introduced in page 325 of [17], if a function is continuous in a closed and bounded set, then it is bounded therein. Therefore, there is $\alpha_1$ and $\alpha_2$ such that

$$\alpha_i \leq \text{Re}\left\{C\left[I-z_1^{-1}(A-BK)\right]^{-1}BK\right\} \leq \alpha_2, \text{ for all } |z_1^{-1}| \leq 1.$$  \hspace{1cm} (22)

According to Assumption 2 and Bolzano’s Intermediate Value Theorem [17], $\alpha_1$ and $\alpha_2$ have the same sign. Without loss of generality, it is assumed that $2 \alpha_2 > \alpha_1 > 0$.

Similarly, the imaginary part is also bounded; hence, there is $\beta > 0$ such that

$$\left|\text{Im}\left\{C\left[I-z_1^{-1}(A-BK)\right]^{-1}BK\right\}\right| \leq \beta, \text{ for all } |z_1^{-1}| \leq 1. \hspace{1cm} (23)$$

As shown in Fig. 2, $C\left[I-z_1^{-1}(A-BK)\right]^{-1}BK$ is within the dashed circle. A value of $L$ is needed to make

$$C\left[I-z_1^{-1}(A-BK)\right]^{-1}BKL$$

within the dashed frame. For any point, $E$, in the upper side of the dashed frame, its real part is $\alpha \leq H \leq \alpha_2$ and its imaginary part is $|E-H| = \beta$. Because triangles $0EH$ and $01G$ are similar,

$$\frac{|G|}{|E|} = \frac{H}{\sqrt{H^2 + \beta^2}}$$

Therefore,

$$l(H) = \frac{|F|}{|E|} = \frac{2|G|}{|E|} = \frac{2H}{H^2 + \beta^2} \hspace{1cm} (24)$$

Calculating the derivative of $l(H)$,

$$l'(H) = \frac{2\alpha_2 - 2\beta^2}{(H^2 + \beta^2)^2}$$  \hspace{1cm} (25)

One knows that $l'(H) > 0$ when $H < \beta$ and $l'(H) < 0$ when $H > \beta$. Hence, the minimum point of $l(H)$ in the interval $[\alpha_1, \alpha_2]$ much be achieved at the endpoints. That is to say, if

$$0 < L < \min \left\{ \frac{2\alpha_1}{\alpha_1^2 + \beta^2}, \frac{2\alpha_2}{\alpha_2^2 + \beta^2} \right\},$$

then one has

$$\left|I - C\left[I-z_1^{-1}(A-BK)\right]^{-1}BKL\right| < 1, \text{ for all } |z_1^{-1}| \leq 1. \hspace{1cm} (26)$$

Similarly, if $\alpha_1 < \alpha_2 < 0$, then one can choose

$$0 > L > \max \left\{ \frac{2\alpha_1}{\alpha_1^2 + \beta^2}, \frac{2\alpha_2}{\alpha_2^2 + \beta^2} \right\}.$$  \hspace{1cm} (27)

In summary, if the learning gain is chosen as

$$\begin{cases} 
0 < L < \min \left\{ \frac{2\alpha_1}{\alpha_1^2 + \beta^2}, \frac{2\alpha_2}{\alpha_2^2 + \beta^2} \right\}, & \text{if } \alpha_2 > 0 \\
0 > L > \max \left\{ \frac{2\alpha_1}{\alpha_1^2 + \beta^2}, \frac{2\alpha_2}{\alpha_2^2 + \beta^2} \right\}, & \text{if } \alpha_1 < \alpha_2 < 0
\end{cases} \hspace{1cm} (27)$$

where $\alpha_1, \alpha_2$ were defined in (22) and $\beta$ was defined in (23), then (21) holds. This finishes the proof. \hfill \blacksquare

Summarizing the previous results in this section, the main conclusion of this work can be obtained.

**Theorem 1.** Given a batch process (1) that satisfies Assumptions 1 and 2, there is a matrix $L$ (as shown in (27)), such that the ILC-based P-type controller (4) and (5) asymptotically stabilizes this process.

**Proof.** Integrating (20) and (21), one obtains that

$$d_2(z_1^{-1}, z_2^{-1}) \neq 0, \text{ for all } |z_1^{-1}| \leq 1, |z_2^{-1}| \leq 1. \hspace{1cm} (28)$$

Invoking Lemma 1, system (16) is asymptotically stable, in other words, $\|e(t,k)\| \rightarrow 0$ as $t, k \rightarrow \infty$. \hfill \blacksquare

For a given $t$, one have that $\|e(t,k)\| \rightarrow 0$ as $k \rightarrow \infty$. Therefore, an ILC-based set-point can improve the tracking performance in the batch direction.

Another advantage of an ILC-based set-point is excellent...
robustness to repetitive disturbances. This issue will be discussed in the following section.

C. Performance with Repetitive Disturbances

Assume that there is a repetitive disturbance, \( w(t, k) \equiv w(t) \), in the batch process. The batch process becomes

\[
\begin{align*}
    x(t + 1, k) &= Ax(t, k) + Bu(t, k) + w(t) \\
    y(t, k) &= Cx(t, k)
\end{align*}
\]  

(29)

Under P-type controller (3), the closed-loop system is

\[
\begin{align*}
    x(t + 1, k) &= (A - BK) x(t, k) + BK Y_r(t) + w(t) \\
    y(t, k) &= Cx(t, k)
\end{align*}
\]  

(30)

By using z-transform, the above system can be described as

\[
y = C \left[ zI - (A - BK) \right]^{-1} (BK Y_r + w)
\]

Hence,

\[
e = \left[ I - C \left[ zI - (A - BK) \right]^{-1} BK \right] Y_r \\
- C \left[ zI - (A - BK) \right]^{-1} w
\]  

(31)

The influence of \( w \) on the tracking performance is clear.

However, for the ILC-based P-type controller, it is easy to prove that \( w \) does not change system (16). Therefore, Theorem 1 holds even though there are repetitive disturbances, thus, the tracking performance still can be improved in the batch direction.

For simplicity, a batch process was considered in this work. In fact, for a continuous process with repetitive disturbances, ILC-based set-point could also be used to eliminate the effect of disturbances and improve the control performance [6].

IV. SIMULATION RESULTS

Injection molding, which is a typical batch process, consists of three main phases: filling, packing, and cooling [18]. In the filling phase, the injection velocity should be controlled to follow a given profile to ensure product quality. In this section, P-type control and ILC-based P-type control are applied to injection velocity control to compare their performance.

The injection velocity response to the proportional valve has been identified as an autoregressive model [19]:

\[
P(z^{-1}) = \frac{1.69 z^{-1} + 1.419 z^{-2}}{1 - 1.582 z^{-1} + 0.5916 z^{-2}}
\]  

(32)

where \( z^{-1} \) is the backward shift operator. The state-space representation of the above model is introduced as

\[
\begin{align*}
    x(t + 1, k) &= \begin{bmatrix} 1.582 & -0.5916 \\ 1 & 0 \end{bmatrix} x(t, k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t, k) \\
    y(t, k) &= \begin{bmatrix} 1.69 & 1.419 \end{bmatrix} x(t, k)
\end{align*}
\]

(33)

The target in this work takes the following form:

\[
Y_r(t) = \begin{cases} 0.15 \times t, & \text{for } 0 \leq t < 100 \\ 15, & \text{for } 100 \leq t \leq 200 \end{cases}
\]  

(34)

In this section, the gain matrices were designed as \( K = 0.1 \) and \( L = 0.1 \). To evaluate the tracking performance, the average tracking error (ATE) is defined as

\[
ATE(k) = \frac{\sum_{i=0}^{200} |e(t, k)|}{201}
\]  

(35)
A smaller $ATE(k)$ values indicates a better tracking performance in the $k$th batch. The control results under the P-type controller and the ILC-based P-type controller were shown in Fig. 3. Since the learning has not started yet, the ILC-based P-type controller in the first batch is the same as the P-type controller. From Fig. 3, one can see that the ILC can improve the control performance from batch to batch and the performance under ILC-based P-type controller is always better than that under P-type controller after the first batch. To see the performance improvement clearly, the comparison of ATE values in 20 batches is shown in Fig. 4.

Now, it is assumed that there are repetitive disturbances in the batch process, as shown in (29), and the disturbances are $\omega(t) = 0.5 \sin(t/10)$. The output responses in this case are shown in Fig. 5 and ATE comparison are given in Fig. 6. According to Fig. 4 and Fig. 6, two cases can be compared: 1) ILC-based P-type controller with disturbances; 2) P-type controller without disturbance. After batch 13, the tracking performance in the first case is better than that in the second case, which demonstrates the excellent performance of indirect ILC in handling repetitive disturbances.

V. CONCLUSION

Choosing a P-type controller as a local control, and using ILC to update the set-point for the local control, an indirect ILC scheme has been proposed for batch processes. The process under the indirect ILC was transformed to a Roesser’s 2D model. Based on this 2D framework, a sufficient condition for asymptotical stability of the indirect ILC was introduced and proved. This is the first work on stability analysis for SPR indirect ILC.

REFERENCES


Fig. 5. The output responses under the P-type controller and the ILC-based P-type controller.

Fig. 6. Comparison of average tracking error (ATE) for the P-type controller and the ILC-based P-type controller.