A FAST AND ACCURATE DECIMATION-IN-ANGLE HIERARCHICAL FAN-BEAM BACKPROJECTION ALGORITHM

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ABSTRACT

We introduce a fast algorithm for backprojecting images from tomographic fan-beam projections that aggregates the projections in a hierarchical structure and achieves a computational cost of $O(N^2 \log P)$, when backprojecting an $N \times N$ pixel image from $P$ projections. Like in the parallel-beam algorithm in [1], the images in the hierarchy are formed by the rotation and the adding together of other images made up of fewer projections. The low computational cost of the algorithm depends on the efficient sampling of the intermediate images in the hierarchy. Understanding the algorithm within the signal processing framework, a general scheme for sampling an image made up of projections of arbitrary geometries is introduced. While the algorithm is related to one by Nilsson [2], the Fourier domain understanding leads to a more efficient sampling scheme for the intermediate images.

1. INTRODUCTION

The 2D tomographic reconstruction problem is to reconstruct an image from a set of its line-integral projections. In the case of the fan-beam tomographic geometry, when the line-integrals are performed on the spokes of a fan, the method used to estimate the image from its projections is a weighted-filtered-backprojection algorithm [3]. The weighted backprojection operation, when performed in the traditional way, is the most computationally intensive part of this method: the backprojection of an $N$-by-$N$ pixel image from $P$ projections has a computational complexity of $O(N^2 \log P) = O(N^3)$, because $P = O(N)$. As the resolution and image matrix sizes increase, this unfavorable scaling of computation with problem size leads to longer reconstruction times.Further recent increases in data acquisition rates in modern multislice 3D CT scanners, and the emergence of real-time CT imaging have exacerbated the need for faster backprojection methods. Faster backprojection will also enable the use of iterative reconstruction, which involves successive backprojections and reprojections, to process data that is noisy, sparse, or otherwise degraded.

This work is an extension of our algorithm for the parallel-beam geometry [1]. While this fan-beam algorithm is similar to that of Nilsson [2], understanding it within the signal processing framework results in a more efficient sampling scheme, and provides a systematic means to optimize and adjust the trade off between computational cost and accuracy. The algorithm achieves its cost reduction by a hierarchical recursive decomposition/aggregation of the projection data by view angles. We classify it therefore as a decimation-

in-angle algorithm, to distinguish it from the family of decimation-
in-space hierarchical backprojection algorithms, that recursively decompose the image [4].

2. FAN-BEAM BACKPROJECTION VIA ROTATIONS

As shown in Fig. 1, the fan-beam tomographic projection, at a source-angle $\beta$, of a two-dimensional image $f(x, y)$ is defined as the set of line integrals along the rays of a fan, parameterized by $\gamma$, centered at the source position on a circle of radius $D$ from the origin. The function $f(x)$ is assumed to be zero-valued outside a disc of radius $R$.

Fig. 1. Fanbeam geometry: image radius $R$, source radius $D$, source-angle $\beta$, fan-angle $\gamma$, and distance from the source $T$.

Projections are available at a set of discrete source angles $\{\beta_p : p = 1, 2, ..., P\}$, and within each fan the angles of the rays are indexed by $\{\gamma_j : j = 1, 2, ..., J\}$. In the case of equiangular fan-beam geometry, considered here, the detectors are equally distributed on the arc of a circle centered at the source, so the fan-angles are evenly spaced. The algorithm is easily generalized to other geometries.

The reconstruction algorithm from a set of $P$ fan-beam projections involves first filtering each projection to produce a filtered fan-beam projection $W_\beta$, according to the set of $P$ source-angles $\beta(= \{\beta_1, \beta_2, ..., \beta_P\})$. Weighted-backprojection at a single angle $\beta$ is defined as:

$$
(W_\beta q)(\vec{x}) \triangleq \frac{2\pi}{P} \frac{1}{T^2(\gamma, \phi(\vec{x}), \beta)} q\left(\frac{\arctan}{D - r(\vec{x})} \sin(\beta - \phi(\vec{x}))\right)
$$

(1)
Here \((r(\vec{x}), \phi(\vec{x}))\) is the polar representation of \(\vec{x}\) and \(T\) is the distance of \(\vec{x}\) from the source. The weighted backprojection of \(P\) projections at source angles \(\beta\) is defined as

\[
(W_\beta(q_p) \mathcal{P}_{p=1}^P)(\vec{x}) = \sum_{p=1}^{P} \mathcal{W}_\beta q_p
\]

(2)

We denote zero-backprojection by \(\mathcal{W}_0\) — the special case of a single source-angle \(\beta = 0\). It is easily shown that:

\[
(W_\beta(q_p) \mathcal{P}_{p=1}^P)(\vec{x}) = \sum_{p=1}^{P} (K(-\beta_p) \mathcal{W}_0 q_p)(\vec{x})
\]

(3)

where \(K\) is the rotation operator (i.e. \((K(\theta)f)(x_1, x_2) = f(\cos \theta x_1 - \sin \theta x_2, \cos \theta x_2 + \sin \theta x_1)\)). This says that the weighted backprojection at a set of source-angles \(\{\beta_p\}\) may be performed by first zero-backprojecting \((\mathcal{W}_0)\) each projection, rotating the \(p^{th}\) backprojected image by \(-\beta_p\), and adding all \(P\) such rotated images together.

3. FAST HIERARCHICAL BACKPROJECTION

The Fast Hierarchical Backprojection Algorithm for the fan-beam geometry, similarly to the parallel-beam case [1], combines the fan-beam projections in a ternary hierarchical structure, exploiting the fact that intermediate images in the hierarchy, being formed by projections that are close to each other in source-angle \(\beta\), can be sampled sparsely. Fig. 2 shows the block-diagram of the algorithm for a set of \(P = 36\) projections. The hierarchy consists of \(L + 1\) levels, where \(P = 4 \times 3^L\) (i.e. \(L = 2\)). In the first level the projections are zero-backprojected:

\[
I_{l,m} = \mathcal{W}_0 q_m
\]

(The superscript ‘d’ in Fig. 2 stands for ‘discrete-domain’). In levels 2 through \(L\) the images are rotated and added in groups of 3:

\[
I_{l+1,m} = \mathcal{K}(\delta_{l,3m-2})I_{l,3m-2} + I_{l,3m-1} + \mathcal{K}(\delta_{l,3m})I_{l,3m},
\]

(4)

and in the final level, making use of free rotations by angles that are multiples of \(\pi/2\), four images are rotated and added:

\[
I_{L+1,1} = I_{L,1} + \mathcal{K}(\pi/2)I_{L,2} + \mathcal{K}(\pi)I_{L,3} + \mathcal{K}(-3\pi/2)I_{L,4}
\]

(5)

While this derivation assumes a particular configuration of view-angles, the algorithm is easily generalized for arbitrary sets of view-angles. Combining the above equations we find that each intermediate image is equal to the sum of a set of rotated, zero-backprojected image:

\[
I_{l,m} = \sum_{p \in N_l} \prod_{l'=1}^{l-1} \mathcal{K}(\delta_{l',m(p,l')}) \mathcal{W}_0 q_m
\]

(6)

where \(\mu(p,i) \triangleq \lfloor p/3^{l-1} \rfloor\) describes the path in the hierarchy that the \(p^{th}\) projection takes to the \(i^{th}\) level, and \(N_l = \{3^{l-1}(m-1)+1, 3^{l-1}(m-1)+2, \ldots, 3^{l-1}m\}\) is the set of indices of constituent projections of the image \(I_{l,m}\).

In the case where the source angles are uniformly distributed \((\beta_i = -\pi/2 + \Delta \beta (j - 1))\) where \(\Delta \beta = \pi/(2 \cdot 3^L)\), the intermediate rotation angles are as follows:

\[
\delta_{l,3m-2} = \begin{cases} +\Delta \beta 3^{l-1} & \text{if } i = 2 \\ 0 & \text{if } i = 1 \\ -\Delta \beta 3^{l-1} & \text{if } i = 0 \end{cases}
\]

(7)

so (6) reduces to

\[
I_{l,m} = \sum_{p = -(3^{l-1}-1)/2}^{(3^{l-1}-1)/2} K(-p\Delta \beta) \mathcal{W}_0 q_{(3^{l-1}+1)/2+p}
\]

(8)

4. EFFICIENT SAMPLING

The computational cost of the algorithm depends on the efficient sampling of intermediate images such as (8), which is the addition of several single-projection fan-beam-projected images. The Fourier-analysis techniques used in devising uniform sampling schemes in the parallel-beam case [1] are extended to devise spatially-varying sampling schemes in the fan-beam case.

The first step to designing efficient sampling schemes for an intermediate image is to understand how the frequency content of the image varies spatially. We perform a space-frequency analysis of the image (analogous to the time-frequency analysis that is well-known in the signal-processing literature) and use a quantity we call the local spectral support to encapsulate this information. This is best introduced by the following example. Consider a bandlimited function \(q(x)\) that is backprojected onto an image plane under two different geometries: parallel-beam and fan-beam.

In the first (parallel-beam) case, \(q(x)\) is backprojected onto the image-plane at an angle \(\theta = -\pi/12\) as shown in Fig. 3(a). The
backprojected image is constant along the red dashed lines. At every point in the image we are interested in finding the spatial direction in which the image varies the least (the slow direction or the direction of lowest spatial bandwidth) and the direction in which the image varies the most (the fast direction or the direction of highest spatial bandwidth). At all points in the parallel-beam image shown in Fig. 3(a), the slow direction is along the dashed lines, and the fast direction is perpendicular to it. This directionality is encapsulated by the Fourier transform of this image, as shown in Fig. 3(d). It has a support on a line segment through the origin of the frequency plane tilted at the projection-angle \( \theta \), i.e. along the fast direction of the image. The length of the line-segment is the fast-direction bandwidth of the image.

![Diagram](image-url)

**Fig. 3.** Single projection images in the space domain — (a) (b) and (c), and their respective spatial supports — (d),(e) (associated with point \( p_1 \)) and (f) (associated with point \( p_2 \)).

In the second (fan-beam) case, as shown in Fig. 3(b), \( q(x) \) is backprojected onto the image-plane along the rays of a fan (as in Equation 1, without the \( 1/T^2 \) scaling). This image is constant along the dashed lines i.e. the rays of the fan. The slow direction of the image at any point will be along the ray on which it is located, and the fast direction will be perpendicular to it. The Fourier transform of this image will no longer encapsulate this directionality, but a windowed Fourier Transform, localized to a point, will.

Consider the windowed Fourier transform of two different points \( p_1 \) (Fig. 3(b)) and \( p_2 \) (Fig. 3(c)) in the image. The Fourier transform of the image, with a window centered at point \( p_1 \), will have a spectral support as shown in Fig. 3(e) (ignoring the broadening of the spectra due to windowing in the space domain). This spectral support is a line-segment oriented in the fast direction at the point \( p_1 \). On the other hand with a window centered at point \( p_2 \), the Fourier transform will have a spectral support as shown in Fig. 3(f) i.e. the support will lie on a line-segment oriented in the fast direction at the point \( p_2 \). Notice also that because \( p_1 \) lies closer to the vertex of the fan than \( p_2 \) does, the variation (bandwidth) of the image along the fast direction will be greater at \( p_2 \) than at \( p_1 \). So the length of the line-segment in Fig. 3(e) is less than that in Fig. 3(f).

We call this idealized spectral support of the windowed Fourier transform centered at a spatial location \( \vec{p} \), the local spectral support of the image at that point. Notice that the local spectral support at different points in the parallel-backprojection image (Fig. 3(a)) will be the same as the spectral support of the global Fourier transform.

The above analysis leads to the understanding of the local spectral support at a point \( \vec{p} \) in an image that is back-projected from a single fan-beam projection. It has a negligible spatial bandwidth in the direction of the ray of the fan passing through it, while it’s bandwidth in the perpendicular direction is inversely proportionate to the distance from the vertex of the fan. When two such single-projection images are added together, so are their Fourier transforms (both global and windowed). Therefore the local spectral support at every point of the composite image is exactly the union of the local spectral support of each image at that point. The intermediate images of the hierarchical algorithm are of the form \( f = \sum_{\lambda} K(\delta_p) W_{\delta} q_{\delta} \) for some angles \( \delta_p \in [\beta_{\min}, \beta_{\max}] \). Being the addition of several single-projection images, the local spectral support at a point in the composite image is the union of the local spectral support of the individual images at that point as shown in Fig. 4. The local spectral support of the point marked by the cross-hairs in the left image, is shown in the right image. Since the constituent projections of image have source-angles distributed between \( \beta_{\min} \) and \( \beta_{\max} \), the local spectral support consists of a set of line-segments, each corresponding to a projection, through the origin of the Fourier plane oriented at angles between \( \theta_{\min} \) and \( \theta_{\max} \). (\( \Omega \) is a scaling constant that converts the frequency of the fan-beam projection to spatial frequency.) The length of a line-segment, its bandwidth, is inversely proportional to the distance of the point from the vertex of the corresponding fan.

![Diagram](image-url)

**Fig. 4.** Local Spectral Support: The image consists of fan-beam projections at source-angles ranging from \( \beta_{\min} \) to \( \beta_{\max} \). The local spectral support at the point indicated by the "cross-hairs" in the left image is illustrated in the right image.

The second step to designing efficient sampling schemes of intermediate images in the algorithm is to use the knowledge of the local spectral support at a point in the image to design the local sampling scheme at that point. Given an image \( f(\vec{x}) \) with a particular (global) spectral support, multi-dimensional sampling theory dictates the sampling matrix \( V \in \mathbb{R}^{2 \times 2} \) that will generate the lattice of sample points to efficiently sample the image. Because of the spatially-varying local spectral support, in the fan-beam case, \( V \) is a function of \( \vec{x} \). The essence of the design of a spatially-varying sampling scheme is to ensure that the local sampling pattern at a point in the image is exactly equal to the global sampling pattern that would have been used if the local spectral support at that point were the global spectral support of the image.

The sampling-design method we use to implement this principle is to numerically integrate the local sampling matrices \( V(\vec{x}) \) over the image plane, the details of which we have omitted here. The resulting sampling scheme is a grid of points that is not spatially uniform.
as in the parallel-beam case, but is spatially-varying. The grid has a
distinct fast and slow direction at every point that matches the fast
and slow directions described in the previous discussion. This lo-
cal Fourier sampling method may be used to find sampling schemes
for arbitrary projection geometries over lines, curves or planes over
arbitrary dimensions. In the case of the parallel-beam geometry, it
reduces to that discussed in [1]. The set of sampling points of in-
termediate images used by Nilsson [2] is the points of intersection
of the two extremal constituent fans. That scheme is approximately
twice as dense as the scheme prescribed here.

The resulting sampling schemes for a few intermediate images
in the fan-beam case are shown in Fig. 5. The samples lie on the rays
of a fan oriented at the source-angle of the central constituent projec-
tion of the intermediate image. As the algorithm progresses and the
range of source-angles ($\beta_{max} - \beta_{min}$ in Fig. 4) of the constituent
projections increases, so does the density of samples along the rays.
The blocks marked ↑ $U$ represent generalized up-interpolation —
the interpolation of the image onto a denser set of samples.

Fig. 5. Sampling patterns of intermediate images at two successive
levels of the algorithm. Samples lie on the rays of a fan. The density
of samples along the rays increases as the algorithm progresses.

5. SEPARABLE ROTATION AND UP-INTERPOLATION
FOR FAN-BEAM SAMPLING

The up-interpolation and rotation of an intermediate image, that is
sampled with a fan-like sampling scheme, is achieved in a two-
step process involving one-dimensional interpolations alone. As dis-
played in Fig. 6, the image, originally sampled on a sparse fan as in
Fig. 6 (a), is evaluated on a rotated dense fan as shown in Fig. 6 (d).
First, 1D interpolations are performed along the rays of the original
sparse fan onto a set of points corresponding to the intersection with
the rotated fan (Fig. 6 (b)). Then 1D interpolations are performed
along the rays of the rotated fan (Fig. 6 (c)) onto the final set of
points.

6. NUMERICAL EXPERIMENTS :

The 512 × 512 Shepp-Logan phantom is reconstructed from 972
fan-beam projections. Fig. 7 (a) shows the fast hierarchical recon-
struction of the phantom. Though the images are of comparable vi-
sual quality, examining cuts through the reconstruction shows, as in
Fig. 7 (b), that the conventional algorithm is better than the fast al-
gorithm at reconstructing edges of the phantom. The speed-up factor
in computations (counting adds and multiplies) of the fast over the
conventional method is over 10, and can be increased with further
optimizations.

7. CONCLUSION

The decimation-in-angle fast hierarchical algorithm for fan-beam
backprojection produces images comparable to the conventional method
with an order magnitude reduction in computation. Fourier-domain
analysis of the algorithm allows for the design of more efficient
schemes for the sampling of intermediate images in the algorithm.

8. REFERENCES

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