On Ranking Opportunity Sets in Economic Environments

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This paper examines how freedom of choice as reflected in an agent’s opportunity sets can be measured in economic environments where opportunity sets are non-empty and compact subsets of the non-negative orthant of the n-dimensional real space. Several plausible axioms are proposed for this purpose. It is then shown that, under different sets of axioms, one can represent the ranking of compact opportunity sets by different types of real-valued functions with intuitively plausible properties. Journal of Economic Literature Classification Numbers: D63, D70.

1. INTRODUCTION

The purpose of this paper is to examine how freedom of choice as reflected in an agent’s opportunity set (i.e., the set of all options available to him/her) can be measured in economic environments. Consider an agent who may be faced with different opportunity sets in different circumstances. Given a specific opportunity set, the agent has to choose exactly one of the options belonging to the set. Each opportunity set offers him/her some freedom of choice. How does one rank these different sets in terms of the degrees of freedom of choice that they offer to the agent? This is a problem that has...
been discussed in a series of recent papers (see, among others, Arrow [2], Foster [4], Pattanaik and Xu [11, 12], Puppe [13], and Sen [16, 17]). However, most of these papers share one basic feature: they visualize the possible opportunity sets as finite subsets of some given universal set of alternatives. While, in many ways, the assumption of a finite opportunity set makes the problem technically more tractable, such tractability has a cost insofar as, in economic contexts, opportunity sets are often infinite sets. For example, the opportunity set of a competitive consumer is a budget set which normally contains an infinite number of consumption bundles. Much of the analysis in the recent axiomatic literature on freedom cannot be used in this context since such analysis is almost always based on the assumption that the opportunity sets are finite. The main purpose of this paper is to extend that analysis to the case involving infinite opportunity sets. In many economic problems, opportunity sets are customarily assumed to be subsets of the $n$-dimensional real space, with certain specified properties such as closure, boundedness etc. Following this tradition, we shall take the class of all compact subsets of the $n$-dimensional non-negative real space to be the class of all possible opportunity sets in our framework (we discuss this assumption further in Section 3), and consider the problem of ranking these opportunity sets in terms of the freedom of choice that they offer to the agent. In particular, we explore the implications of alternative sets of plausible axioms for such ranking of opportunity sets.

The plan of the paper is as follows. In Section 2, we briefly discuss the intuition regarding freedom that underlies our formal analysis. While this intuition has been discussed elsewhere (see Jones and Sugden [6], and Pattanaik and Xu [11, 12]), we feel that it may be worthwhile recapitulating here some of the points which are crucial for understanding the substantive content of many of our axioms. Section 3 lays down our basic notation and definitions. In Section 4, the problem of ranking compact opportunity sets in terms of freedom is posed in a framework where preferences do not figure in the informational basis of such ranking. In this section, we introduce several axioms, which constitute restrictions on the ranking of opportunity sets in terms of freedom of choice, and we prove that, given these axioms, one can represent the ranking of opportunity sets by a real-valued function satisfying several interesting properties. Given this representation, the ranking of opportunity sets can be interpreted as a ranking based on the "size" of the opportunity sets under consideration. Section 5 extends the analysis of Section 4 by incorporating a richer informational structure that includes preferences. This necessitates a change in the set of axioms used in the preceding section. A modified set of axioms leads to a new real-valued representation with a different intuitive significance. Section 6 contains some concluding remarks.
2. FREEDOM OF CHOICE AND ITS NON-UTILITARIAN VALUE

As we have noted, by admitting infinite opportunity sets, we depart from the formal framework of much of the existing axiomatic literature on the freedom of choice and its measurement. However, our basic conceptual concern is the same as that of many other earlier writers (see, for example, Jones and Sugden [6], Pattanaik and Xu [12], and Sen [17]) insofar as we focus on the non-utilitarian value of the freedom of choice.

The conventional theory of welfare economics, which has tended to view social welfare as being determined exclusively by the utilities of the individuals in the society, fails to accommodate the notion of freedom of choice and its importance for judgments about social welfare. If the competitive market mechanism in an economy is replaced by a command system of planning and every consumer is ordered to consume the same commodity bundle that she chose earlier from her budget set in the former competitive economy, then the consumption bundle, and, hence, the utility, of each consumer would remain unchanged. In that case, conventional welfare economics, which bases social welfare judgments exclusively on individual utilities would consider the two situations to be indistinguishable in terms of social welfare. Yet, most of us would feel that the replacement of the competitive market mechanism by the command system, which collapses the set of alternative consumption bundles available to a consumer to a single point, reduces the consumers' freedom of choice, and this reduction in the individuals' freedom of choice should enter into the evaluation of social welfare. We feel that the opportunity set (which, in our example, happens to be the budget set) of the agent has a significance independently of the option actually chosen by the agent from that opportunity set. Even if the agent may actually choose option $x$ from the opportunity set $A$, the availability of the other options in $A$ reflects the freedom of choice that the agent enjoys and this freedom is curtailed when the opportunity set is constrained to be $\{x\}$.

None of this, of course, denies that freedom may serve as an instrument for preference-satisfaction. An expansion of the opportunity set may be valuable to the agent because it may increase her utility by enabling her to attain an option that is more preferred in terms of her existing preferences. Alternatively, the individual may be uncertain about the preferences that she would have at the time when she would actually have to make a choice (see Arrow [2] and Kreps [8]). Given such uncertainty, a larger opportunity set provides greater flexibility and consequently a higher level of expected utility. The role of freedom of choice as an instrument for achieving higher utility or "expected utility" is undoubtedly important. However, a significant part of the recent literature (see, for example, Jones and Sugden [6], Sen [16, 17], Pattanaik and Xu [11, 12] and Sugden [19]) has focused on the...
non-utilitarian value of freedom. The view taken here is that freedom of choice has a value independently of the level of utility (or, expected utility) that it enables the agent to achieve. It is this conception of the value of freedom that underlies the analysis in this paper.

Even when one focuses on the non-utilitarian reasons for valuing freedom, one can think of different approaches to the problem of ranking opportunity sets in terms of freedom. For our purpose, it will be useful to distinguish two alternative approaches in the recent literature, which have highlighted different aspects of the notion of freedom. First, we have what may be called the non-preference-based (NPB) approaches (see, for example, Steiner [18] and Pattanaik and Xu [11]), where preferences do not play any role in the ranking of opportunity sets. Second, we have the preference-based (PB) approaches, where preferences of some type constitute an integral part of the information on which the ranking of opportunity sets is based (see, for example, Jones and Sugden [6], Sen [16], Foster [4], Pattanaik and Xu [12], and Sugden [19]). The NPB approach in the existing literature has emphasized the size of the opportunity set (that is, the quantity of available options) as the crucial determinant of freedom, while the PB approach has brought to the discussion the notion of the quality of the available options, such quality being judged in terms of the relevant preferences. We believe that, at this stage of the development of the subject, useful insights can be gained from each of these two approaches. Accordingly, we shall explore both these avenues in this paper. In Section 4, we consider an NPB approach for ranking compact subsets of the $n$-dimensional non-negative real space, while, in Section 5, we consider a PB approach to the same problem.

3. THE BASIC NOTATION AND DEFINITIONS

Let $\mathbb{R}_+^n$ be the non-negative orthant of the $n$-dimensional real space. The points in $\mathbb{R}_+^n$ will be denoted by $x, y, z, a, b, ...$ and will be called alternatives. The alternatives can be interpreted as commodity bundles in the conventional sense, as bundles of functionings (see Sen [14, 15]), or as bundles of characteristics (see Lancaster [9] and Gorman [5]). One can also think of other broader interpretations of alternatives. Let $H, J$ and $K$ denote, respectively, the set of all non-empty subsets of $\mathbb{R}_+^n$, the set of all bounded subsets of $\mathbb{R}_+^n$, and the set of all compact subsets of $\mathbb{R}_+^n$.

We assume that the set of all (mutually exclusive) options which may be available to the agent at any given point of time is an element of $H$. Such a set will be called the agent’s opportunity set. The opportunity sets will be denoted by $A, B, C, X$ etc.

Our central concern is with the problem of ranking the different opportunity sets, with which the agent may be faced, in terms of the freedom of
choice that they offer to the agent. What are the elements of $H$ that one would like to rank in this fashion? It is possible to argue that, even if a particular element $A$ of $H$ will never actually materialize as the opportunity set of the agent under consideration, one can conceive of $A$ as the agent's opportunity set, and, therefore, there is no reason why one should exclude the conceptual problem of assessing the freedom that $A$ would offer to the agent, were $A$ to be the agent's opportunity set. A concrete example may help clarify the point. In the standard theory of competitive consumers' behavior, where the alternatives are commodity bundles, certain sets of alternatives can never actually figure as the consumer's opportunity set as determined by the consumer's initial endowment and the prices. Thus, in the two commodity case shown in Fig. 1, the set $A$ can never actually be the competitive consumer's budget set.

However, one can argue that there is no conceptual reason why one should not consider the hypothetical case where the consumer is faced with the problem of having to choose an alternative from the set $A$, and assess the extent of freedom of choice that he/she would enjoy in such a hypothetical situation (for a somewhat similar argument in the theory of revealed preference, see the well-known paper of Arrow [1]). If one accepts this line of reasoning, then one may be inclined to take $H$ as the set of all possible opportunity sets that one should seek to rank in terms of freedom of choice. But this makes the problem technically less tractable (unlike in Arrow [1] where a corresponding assumption drastically simplified the technical structure). Nor would such a demanding framework seem to be required from a pragmatic point of view. In most economic contexts, the opportunity set that the agent actually faces is always bounded. Also, in most such contexts, it seems reasonable to assume that the opportunity set of the agent is always closed. Since in most economic environments, the class of opportunity sets that can actually materialize is a subclass of the class of all non-empty and compact subsets of $\mathbb{R}^n$, for all practical purposes, there would not be much loss if we could rank only non-empty and compact

![Figure 1](image-url)
subsets of $\mathbb{R}_+^n$. Therefore, throughout this paper, we focus on $K$, the class of all compact subsets of $\mathbb{R}_+^n$. $K$, of course, contains the empty set $\emptyset$. For technical convenience, we do not exclude the empty set. As we mention later, we adopt the convention that the empty set represents the least amount of freedom.

For all $A \in J$, let $\alpha(A)$ denote the closure of $A$. Hence, for all $A \in J$, $\alpha(A)$ is always a compact set and, therefore, belongs to $K$.

Let $\succeq$ be an ordering over $K$. For all $A, B \in K$, $[A \succ B]$ will be interpreted as “$A$ offers at least as much freedom as $B$,” $>$ and $\sim$, respectively, are the asymmetric and symmetric part of $\succeq$.

A representation of $\succeq$ is defined to be a function $\phi: K \to \mathbb{R}_+$ such that for all $A, B \in K$, $A \succeq B$ if $\phi(A) \geq \phi(B)$.

**Definition 3.1.** A representation $\phi$ of $\succeq$ is said to be

(3.1.1) additive if for all $A, B \in K$, $A \cap B = \emptyset \Rightarrow \phi(A \cup B) = \phi(A) + \phi(B)$;

(3.1.2) countably additive if for all $A_1, A_2, \ldots \in K$, $\bigcap_{i=1}^\infty A_i = \emptyset \Rightarrow \sum_{i=1}^\infty \phi(A_i)$;

(3.1.3) sub-additive if for all $A, B \in K$, $A \cap B = \emptyset \Rightarrow \phi(A \cup B) \leq \phi(A) + \phi(B)$;

(3.1.4) countably sub-additive if for all $A_1, A_2, \ldots \in K$, $\bigcap_{i=1}^\infty A_i = \emptyset \Rightarrow \sum_{i=1}^\infty \phi(A_i)$.

4. THE RANKING OF OPPORTUNITY SETS IN THE NPB FRAMEWORK

Some Basic Axioms

**Definition 4.1.** $\succeq$ satisfies

(4.1.1) Non-triviality if, for all $A \in K$, $A \succeq \emptyset$; and there exists $X \in K$ such that $X \succ \emptyset$.

(4.1.2) Denseness if, for all $A, B \in K - \{\emptyset\}$ such that $A \succeq B$, there exists an $A' \subseteq A$ such that $A' \in K$, $A' \not\succeq \emptyset$ and $A' \sim B$.

(4.2.3) Archimedean Property if, for all $A, B \in K$, $[A \succ B \succ \emptyset]$, then there exists $C_1, \ldots, C_m \in K$ such that $C_1 \sim \cdots \sim C_m \sim B$ and $B \uplus C_1 \cup \cdots \cup C_m \succeq A$.

An ordering over a set $Z$ is a binary relation $R$, defined over $Z$, which is reflexive (for all $x \in Z, xRx$), connected (for all distinct $x, y \in Z, xRy$ or $yRx$) and transitive (for all $x, y, z \in Z, xRy$ and $yRz$, then $xRz$).
Independence \( i f f \), for all \( A, B, C \in J \), \( i f \ A \cap C = B \cap C = \emptyset \), then \( \pi(A) \geq \pi(B) \) \( i f f \ \pi(A \cup C) \geq \pi(B \cup C) \).

The property of Non-triviality has two parts. The first part, which requires that every opportunity set should offer at least as much freedom as the empty set, is based on an implicit convention under which the empty set is assumed to represent the least amount of freedom. To make our problem a non-trivial one, the second part of Non-triviality requires that there exists at least one opportunity set that offers the individual strictly more freedom than the empty set.

The intuition of Denseness can be explained as follows. Suppose \( A \) offers at least as much freedom as \( B \). Then, provided \( A \) can be broken into as small pieces as one likes, one can start with \( A \), and, if necessary, by throwing out suitable bits of \( A \), one can finally arrive at a subset of \( A \) that offers the same amount of freedom as \( B \). It thus requires that opportunity sets are finely divisible. It is interesting to note that one implication of Denseness is that, for all \( x \) and \( y \), \( \{ x \} \sim \{ y \} \). This particular implication of Denseness was earlier introduced as an independent condition by Jones and Sugden [6] and Pattanaik and Xu [11] and was called indifference of no-choice situations (INS) by Pattanaik and Xu [11]. The justification usually given for INS is that, when the agent is faced with a singleton feasible set, he/she has no real freedom of choice, all singleton sets offer the agent the same degree of freedom (for a critique of INS, the reader may refer to Sen [16, 17]).

Archimedean Property is a commonly used mathematical property. It requires that, for all opportunity sets \( A \) and \( B \) in \( K \), with \( A \) offering more freedom than \( B \) and \( B \) offering more freedom than the empty set, there always exists a finite number of opportunity sets such that each of these opportunity sets offers the same amount of freedom as \( B \), and the union of all these opportunity sets and \( B \) offers at least as much freedom as \( A \). The intuition of Archimedean Property is clear: if \( A \) offers more freedom than \( B \) and \( B \) offers more freedom than the empty set, then one can always add to \( B \), one at a time, opportunity sets each of which offers exactly the same amount of freedom as \( B \), so that after a finite number of such additions, the enlarged opportunity set will eventually offer at least as much freedom as \( A \). In other words, the opportunity sets are required to be tight.

Finally, Independence is a standard independence property used in the literature (see, for example, Pattanaik and Xu [11] and Sen [16]) and has its appeal and plausibility.

A Representation Theorem in the NPB Framework

We now explore the implications of the four properties introduced in Definition 4.1. We show that, together, they are sufficient for a countably
additive representation of the freedom ranking \( \succeq \) (see Theorem 4.2 below). We first state and prove the result, and then comment on its significance.

**Theorem 4.2.** Let \( \succeq \) on \( K \) satisfy Non-triviality, Denseness, Archimedean Property and Independence. Then \( \succeq \) has a countably additive representation \( \phi \) such that
\[
\phi(\emptyset) = 0, \quad \text{and there exists } X \in K \text{ such that } \phi(X) > 0, \quad \cdots (4.1)
\]
and
\[
\text{for all } A, B \in K, B \subseteq A \Rightarrow \phi(B) \leq \phi(A). \quad \cdots (4.2)
\]

We proceed to the proof of this theorem via: (a) several lemmas (Lemma 4.3, 4.4, 4.5, 4.6 and 4.7); (b) a definition (Definition 4.8) which introduces the notion of an essential structure due to Krantz et al. [7]; and (c) a theorem (Theorem 4.9) which is due to Krantz et al. [7] and which we state without giving the proof.

In Lemmas 4.3, 4.4, 4.5, 4.6, and 4.7, it is assumed that the ordering \( \succeq \) over \( K \) satisfies Non-triviality, Denseness, Archimedean Property and Independence.

**Lemma 4.3.** If \( A, B \in K \) and \( B \subseteq A \), then \( A \not\succ B \).

**Proof.** Assume \( A, B \in K \) and \( B \subseteq A \). Consider \( A - B \). Note that \( B \cap (A - B) = \emptyset \). By Non-triviality, \( \pi(A - B) \not\succ \emptyset \). Hence, by Independence, from \( B \sim A \) we must have \( \pi(B \cup (A - B)) \not\succ B \), i.e., \( A \succeq B \).

**Lemma 4.4.** If \( A, B \in K \) and \( B \subseteq A \), then \( A \succ B \) iff \( \pi(A - B) \succ \emptyset \).

**Proof.** The proof is similar to that of Lemma 4.3.

**Lemma 4.5.** If \( A, B \in K \) and \( B \subseteq A \), then \( B \cap \pi(A - B) \sim \emptyset \).

**Proof.** Assume \( A, B \in K \) and \( B \subseteq A \). Now suppose \( B \cap \pi(A - B) \succ \emptyset \). Note that \( (B - \pi(A - B)) \cap (B \cap \pi(A - B)) = \emptyset \), \( (B - \pi(A - B)) \cup (B \cap \pi(A - B)) = B - \pi(A - B) \cup (B \cap \pi(A - B)) = B \), and \( \pi(B - \pi(A - B)) = B \). By Independence, we have \( \pi(B - \pi(A - B)) \cup (B \cap \pi(A - B)) \succ \pi(B - \pi(A - B)) \cup \emptyset \), that is, \( B \not\succ B \), a contradiction. Therefore, \( B \cap \pi(A - B) \sim \emptyset \).

**Lemma 4.6.** If \( A, B, C, D \in K \), and \( A \cap B \sim C \cap D \sim \emptyset \), then \( [A \sim C \text{ and } B \sim D] \Rightarrow [A \cup B \sim C \cup D] \) and \( [A \succeq C \text{ and } B \succeq D] \Rightarrow [A \cup B \succeq C \cup D] \).

**Proof.** Assume \( A, B, C, D \in K \), and \( A \cap B \sim C \cap D \sim \emptyset \). Let \( A' = A - D \) and \( D' = D - A \). Note that \( A' \cap B = A' \cap D = A \cap D' = C \cap D' = \emptyset \) and
A ⊔ D' = A' ⊔ D. If A ⪰ C and B ⪰ D, then, by Independence, A ⊔ \pi(D') ⪰ C ⊔ \pi(D') and \pi(A') ⊔ B ⪰ \pi(A') ⊔ D. Since \geq is an ordering, noting that \pi(A') ⊔ D = \pi(A' ⊔ D) = A ⊔ \pi(D') = \pi(A ⊔ D'), we then have \pi(A' ⊔ B) ⪰ \pi(C ⊔ D'). Note that (A ∩ D) ∩ (A' ⊔ B) = (A ∩ D) ∩ (C ⊔ D') = ∅. By Independence, A ⊔ B = \pi(A' ⊔ B) ⊔ (A ∩ D) = \pi(C ⊔ D') ⊔ (A ∩ D) = C ⊔ D. Similarly, it can be shown that if A ⊳ C and B ⊳ D, then A ⊔ B ⊳ C ⊔ D.

**Lemma 4.7.** If A, B, C, D ∈ K and A ∩ B ∼ ∅, then [A ⊳ C and B ⊳ D] ⇒ [A ⊔ B ⊳ C ⊔ D].

**Proof.** Let A, B, C, D ∈ K, A ∩ B ∼ ∅, A ⊳ C and B ⊳ D. Consider D' = D − C. From Lemma 4.3, D ⊳ \pi(D'). Since \geq is an ordering, noting B ⊳ D and D ⊳ \pi(D'), we have B ⊳ \pi(D'). If we can show that C ∩ \pi(D') ∼ ∅, then the result follows as a simple consequence of Lemma 4.6. We now show that C ∩ \pi(D') ∼ ∅. Since C ∩ \pi(D') ≤ C, by Lemma 4.5 it follows that [C ∩ \pi(D')] ∩ (C − [C ∩ \pi(D')]) ∼ ∅. Noting that \pi(C − [C ∩ \pi(D')]) = C, C ∩ \pi(D') ∼ ∅ is then obtained.

**Definition 4.8** (see Krantz et al. [7]). Let A be a nonempty set, ≽ a binary relation on A, ⊳ the asymmetric part of ≽, B a nonempty subset of A × A, and + a binary function from B into A. The quadruple ⟨A, ≽, ⊳, +⟩ is an extensive structure with no essential maximum if the following six properties are satisfied for all a, b, c ∈ A:

(i) ≽ is an ordering over A;

(ii) If (a, b) ∈ B and (a × b, c) ∈ B, then (b, c) ∈ B, (a, b × c) ∈ B, and (a × b) × c ≽ a × (b × c);

(iii) If (a, c) ∈ B and a ⊳ b, then (c, b) ∈ B and a × c ⊳ b × c;

(iv) If a ⊳ b, then there exists d ∈ A such that (b, d) ∈ B and a ⊳ b × d;

(v) If (a, b) ∈ B, then a × b ⊳ a;

(vi) Every strictly bounded standard sequence is finite (a₁, ..., aₙ, ...) is a standard sequence if for n = 2, ..., [aₙ = aₙ₋₁ ⊳ a₁], and it is strictly bounded if for some b ∈ A and for all aₙ in the sequence, b ⊳ aₙ).

**Theorem 4.9** (Krantz et al. [7]). Let ⟨A, ≽, ⊳, +⟩ be an extensive structure with no essential maximum. Then there exists a function ϕ: A → Rₜ such that for all a, b ∈ A

(a) a ⊳ b iff ϕ(a) ≥ ϕ(b),

and

(b) if (a, b) ∈ B, then ϕ(a × b) = ϕ(a) + ϕ(b).
If another function $\psi'$ satisfies (a) and (b), then there exists a positive number $\beta$ such that, for all non-maximal $a \in A$, $\psi'(a) = \beta \psi(a)$.

**Proof of Theorem 4.2.** For all $A \in K$, let $E(A)$ denote the equivalence class that includes $A$. Let $E(K) = \{E(A): A \in K \text{ and } A \notin E(\emptyset)\}$. By Non-triviality, there exists $X \in K$ with $X \not= \emptyset$. Hence, $E(X) \in E(K)$. If $E(K)$ contains one and only one element $E(X)$, then, for all $A \in K$, we let $\phi(A) = 0$ if $A \sim X$, and $\phi(A) = 1$ if $A \sim X$. In that case, $\phi$ constructed in this fashion will meet all the requirements for $\phi$ specified in the conclusion of Theorem 4.2. Henceforth, we assume that, for some $C \in K$, $E(K)$ contains $E(C)$ and $E(C) \not= E(X)$. Now define:

$$\Sigma = \{(E(A), E(B)): A \not= \emptyset, B \not= \emptyset, \text{ and there exist}$$

$$A' \in E(A), B' \in E(B) \text{ such that } A' \cap B' \sim \emptyset\}.$$ 

By assumption, there exists $A \in K$ such that $E(A) \not= E(X)$ and $A \not= \emptyset$. Without loss of generality, assume that $X > A$. Then, by Denseness, there exist $A' \in K$ with $A' \subseteq X$ and $A' > A$. Since $\geq$ is an ordering and $X > A$, it then follows $X > A'$. By Lemma 4.4, $a(X - A') > \emptyset$, and by Lemma 4.5, $E(A') \not= \emptyset$. Since $A' > A \not= \emptyset$, $a(X - A') > \emptyset$, and $A' \cap a(X - A') \sim \emptyset$, $E(A') \not= \emptyset$. Therefore, $\Sigma$ is non-empty. We then define the binary operation $\cdot$ on $\Sigma$ by letting

$$E(A) \cdot E(B) = E(A) \cup E(B), \quad \text{if } A \cap B \sim \emptyset.$$ 

By Lemma 4.6, the binary operator $\cdot$ is well-defined. Let $\geq_E$ be the induced binary relation on $E(K)$. We now prove that $(E(K), \geq_E, \cdot)$ is an extensive system without a maximal element (where $\geq_E$ is the induced freedom ranking of $\geq$ on the set of equivalence classes).

1. $\geq_E$ is obviously a linear ordering of $E(K)$ since, by Ordering, $\geq$ is an ordering of $K$.

2. Suppose $(E(A), E(B)) \in \Sigma$, $(E(A) \cdot E(B), E(C)) \in \Sigma$, $A \not= \emptyset$, $B \not= \emptyset$, $A \cap B \sim \emptyset$. By definition of $\Sigma$, there exist $D \in E(A) \cup E(B)$ and $C' \in E(C)$ such that $D \cap C' \not= \emptyset$. Since $B \subseteq A \cup B$ and $A \not= \emptyset$, Lemma 4.4 implies that $D \sim A \cup B \succ B$. This, together with $C' \sim C$ and $C' \cap D \sim \emptyset$, implies that, by Denseness, there exist $B', C' \in K$ such that $B' \in E(B)$, $C' \in E(C)$ and $B' \cap C' \sim \emptyset$. Thus, $(E(B), E(C)) \in \Sigma$.

Next, we establish that $(E(A), E(B) \cdot E(C)) \in \Sigma$. Note that $A \not= \emptyset$, $B \not= \emptyset$, $A \cap B \sim \emptyset$, and $D \sim A \cup B \succ B'$. From $C' \sim C$ and $C' \cap D \sim \emptyset$, by Lemma 4.6, there exist $B' \in E(B)$, $C'' \in E(C)$, and $G \in K$ with $G \sim D \cup C'$, $B' \cup C'' \not= G$, $B' \cap C'' \not= \emptyset$, $B' \sim B'$, and $C'' \sim C'$, if $D \sim a(G - C'')$, since $C' \sim C''$ and $C' \cap D \sim \emptyset$. Lemma 4.6 yields $D \cup C' \sim a(G - C'') \cup C'' = G \sim (D \cup C')$, which contradicts with the assumption that $\geq$ is an ordering.
The supposition that \( n(G - C^m) > D \) leads to a similar contradiction. Hence, \( D \sim n(G - C^m) \). Now, suppose \( A' > A \), where \( A' = n(G - (B' \cup C^m)) \).

It can be shown easily that \( A' \cap B' \sim A' \cap (B' \cup C^m) \sim \emptyset \). Since \( B' \sim B \), \( A' \cap B' \sim \emptyset \), Lemma 4.6 implies that \( D \sim n(G - C^m) = (A' \cup B') \sim (A \cup B) \sim D \), which is another contradiction. And if \( A > A' \), then \( D \sim (A \cup B) > (A' \cup B') \sim D \), which is also a contradiction. So, \( A > A' \). Since \( A' \in E(A), B' \cup C^m \in E(B) \cdot E(C) \), and \( A' \cap (B' \cup C^m) \sim \emptyset \), it is shown that \( (E(A), E(B) \cdot E(C)) \in \Sigma \). The assertion that \( (E(A) + E(B)) \cdot E(C) \ni E(A) \cdot (E(B) \cdot E(C)) \) follows from the associativity of the union \( \cup \) operator.

(3) Suppose that \((E(A), E(C)) \in \Sigma \) and \( E(A) \ni E(B) \), and without loss of generality, \( A \cap C \sim \emptyset \). If \( E(A) = E(B) \), there is nothing to show. If \( E(A) \ni E(B) \), i.e., \( A > B \), since \( C \sim C \) and \( A \cap C = \emptyset \), Denseness implies the existence of \( B' \subseteq A \) such that \( B' \in E(B) \) and \( B' \cap C \sim \emptyset \). So \((E(C), E(B)) \in \Sigma \), and with \( A > B' \), by Lemma 4.6, \( E(A) \cdot E(C) \ni E(B) \cdot E(B') \).

(4) If \( E(A) \ni E(B) \), by Denseness, from \( A > B \) and \( \emptyset \sim \emptyset \) where \( A \in E(A), B \in E(B) \), there exist \( A' \in E(A), B' \in E(B) \) such that \( B' \subseteq A' \), \( A > A' \), \( B \sim B' \). Let \( C = n(A' - B') \), then \( E(A) = E(A') = E(B') \cdot E(C) = E(B) \cdot E(C) \).

(5) Suppose that \((E(A), E(B)) \in \Sigma \), where \( A \cap B \sim \emptyset \), \( A \ni \emptyset \), and \( B \ni \emptyset \). Since \( B \ni \emptyset \), it then follows from Lemma 4.4, \( A \cup B > A \), and so \( E(A) \cdot E(B) > E(A) \).

(6) Finally, from Archimedean Property and Lemma 4.7, it follows easily that \( \{n : E(B) \ni nE(A)\} \) is finite, where \( nE(A) = (n - 1)E(A) \cdot E(A) \) and \( 1E(A) = E(A) \).

By Theorem 4.9, there is a positive real-valued function \( \pi \) on \( E(K) \) such that
\[
E(A) \ni E(B) \quad \text{iff} \quad \pi(E(A)) \ni \pi(E(B)),
\]
and
\[
\text{for} \quad (E(A), E(B)) \in \Sigma, \pi(E(A) \cdot E(B)) = \pi(E(A)) \cdot \pi(E(B)).
\]

Select that \( \pi \) and an \( X \in K \) for which \( \pi(X) = 1 \) with \( X \ni \emptyset \) and, for \( A \in K \), define
\[
\phi(A) = \begin{cases} \pi(E(A)), & \text{if} \quad A > \emptyset, \\ 0, & \text{if} \quad A \ni \emptyset. \end{cases}
\]

It is easy to see that \( \phi \) is additive and satisfies (4.1). By Lemma 4.3, for all \( A, B \in K \), such that \( B \subseteq A, A > B \), and, hence, \( \phi(A) \ni \phi(B) \).
We now show that $\phi$ is countably additive. To show this, we first prove that

for all opportunity sets $A_1, A_2, \ldots \in K$, and all $B \in K$,

$$\left[ \text{if } A_i \subseteq A_{i+1} \text{ for every positive integer } i, \bigcup_{i=1}^{\infty} A_i \in K, \right.$$  

and $B \supseteq A_i$ for all $i$, then $B \supseteq \bigcup_{i=1}^{\infty} A_i$, \right.$$ \ldots (4.3)

Let $A_1, A_2, \ldots \in K$ be such that $A_i \subseteq A_{i+1}$ for every positive integer $i$ and $\bigcup_{i=1}^{\infty} A_i, B \in K$ be such that $B \supseteq A_i$ for all $i$. Denote $\bigcup_{i=1}^{\infty} A_i$ as $A$. Consider $\Gamma = \{ \emptyset, A, A_i, i = 1, 2, \ldots \}$. Note that the union of every class of sets in $\Gamma$ is a set in $\Gamma$ and the intersection of every finite class of sets in $\Gamma$ is a set in $\Gamma$. Thus, $(A, \Gamma)$ form a topological space and every element in $\Gamma$ is an open set in this space. To see that $A$ is compact in $(A, \Gamma)$, we note that every net in $A$ in the topological space $(A, \Gamma)$ has an accumulation point in $A$. This is because every net in $A$ in the topological space $(A, \Gamma)$ is a net in $A$ in the Euclidean space and vice versa and $A$ is compact in the Euclidean space.

Then, by virtue of the compactness of $A$ in $(A, \Gamma)$, which is equivalent to that every open cover of $A$ has a finite subcover, and noting that $A = \bigcup_{i=1}^{\infty} A_i$, which says that $\{ A_i, i = 1, 2, \ldots \}$ is an open cover of $A$, there exists a finite subcover $\{ C_1, \ldots, C_m \}$ so that $A = \bigcup_{i=1}^{m} C_i$. Since $\{ C_i, i = 1, \ldots, m \}$ is a subcover, each $C_i$ belongs to $\{ A_i, i = 1, 2, \ldots \}$. Let $C_i$ be so arranged that $C_i \subseteq C_{i+1}$ for $i = 1, \ldots, m-1$. Thus, $\bigcup_{i=1}^{m} C_i = C_m$. From $A = \bigcup_{i=1}^{m} C_i$, clearly, $A = C_m$. If $B \supseteq A_i$ for all $i$, it must be true that $B \supseteq C_m = A$. Hence, (4.3) holds.

Now, let $A_1, A_2, \ldots \in K$ be pairwise disjoint and $\bigcup_{i=1}^{\infty} A_i \in K$. Since $\phi$ is additive, $\sum_{i=1}^{n} \phi(A_i) = \phi(\bigcup_{i=1}^{n} A_i) \leq \phi(\bigcup_{i=1}^{\infty} A_i)$ for every $n$. Hence, $\sum_{i=1}^{\infty} \phi(A_i) \leq \phi(\bigcup_{i=1}^{\infty} A_i)$. Suppose that for some $\{ A_i \}$, $\sum_{i=1}^{\infty} \phi(A_i) < \phi(\bigcup_{i=1}^{\infty} A_i)$. Let $\phi(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{\infty} \phi(A_i) = \epsilon > 0$. Consider two cases. First, suppose that for some $A_k$ in $\{ A_i \}$, $\epsilon > \phi(A_k) > 0$. Let

$$B_p = \bigcup_{i=1}^{q} A_i \quad \text{and} \quad B = \left( \bigcup_{i=1}^{\infty} A_i \right) - A_k.$$

\[3\] This property was first proposed by Villegas [20] in the discussion of qualitative probability theory.

\[4\] A net in a topological space $(\Gamma, Z)$ is a function from a directed set $D$ into $Z$, where a directed set $D$ is a set $D$ on which there is defined a reflexive and transitive binary relation, denoted by $\leq$, with the property that whenever $a, b \in D$, there is a $c \in D$ with $a \leq c$ and $b \leq c$.

Let $\{ x_i \}$ be a net in the topological space $(\Gamma, Z)$. The point $x \in Z$ is called an accumulation point of the net iff for each neighborhood $U$ of $x$, for all given $n \in D$, there is an $m \in D$ with $n \leq m$ and $x_m \in U$. Then, $Z$ is compact if and only if every net in $Z$ has an accumulation point in $Z$. See, for example, Ash [3].
Note that \( \{ A_i \} \) are pairwise disjoint, \( \bigcup_{i=1}^{\infty} A_i \in K \), and \( B \) is compact. For each \( q \), \( B_q \subseteq B_{q+1} \), and \( \phi(B_q) \leq \sum_{i=1}^{\infty} \phi(A_i) = \phi(\bigcup_{i=1}^{\infty} A_i) - \epsilon \leq \phi(\bigcup_{i=1}^{\infty} A_i) - \phi(A_k) = \phi(B) \). Thus, \( B \preceq B_q \) for all \( q \). But since \( A_k \succ \emptyset \), we have
\[
\bigcup_{q=1}^{\infty} B_q = \bigcup_{i=1}^{\infty} A_i = B \cup A_k > B,
\]
a violation of (4.3).

The second case is where no such \( A_k \) exists. Then clearly, \( \phi(A_i) = 0 \) for all but finitely many \( i \). Let \( I = \{ i : \phi(A_i) = 0 \} \) and let \( A = \bigcup_{i \in I} A_i \). By the additivity of \( \phi \), \( \phi(\bigcup_{i=1}^{\infty} A_i) - A = \sum_{i \in I} \phi(A_i) = \sum_{i \neq k} \phi(A_i) = \phi(\bigcup_{i=1}^{\infty} A_i) - \epsilon \). Therefore, \( \phi(A) = \epsilon \). Let \( C_q = \bigcup_{i=q+1}^{\infty} A_i \). Then, \( \phi(C_q) = 0 \) follows from the additivity of \( \phi \); so we have
\[
C_q \subseteq C_{q+1}, \emptyset \succ C_q,
\]
but \( \bigcup_{q=1}^{\infty} C_q = A \succ \emptyset \), violating (4.3) again. Hence, \( \phi \) is countably additive.

Remark 4.10. For all \( A \in K \), let \( \text{vol}(A) \) denote the volume of \( A \) and let \( \text{mass}(A) \) denote the mass of \( A \). The orderings over \( K \), induced by \( \text{vol}(\cdot) \) and \( \text{mass}(\cdot) \), both satisfy Non-triviality, Denseness, Archimedean Property and Independence. Therefore, our Theorem 4.2 is not vacuous.

Remark 4.11. In Theorem 4.2, the countably additive representation \( \phi \) of \( \succ \), which satisfies (4.1) and (4.2), has a natural interpretation as an index of the “size” of the different opportunity sets. Given this interpretation of \( \phi \), Theorem 4.2 tells us that, if the ranking \( \succ \) satisfies Non-triviality, Denseness, Archimedean Property and Independence, then, essentially, it can be viewed as a ranking based on the size of the opportunity sets. Theorem 4.2 constitutes an extension of Pattanaik and Xu’s [11] result on the cardinality-based ordering for the case where the universal set contains a finite number of alternatives. To make this clearer, we note the following straightforward result, the proof of which we have omitted.

Let \( K' \) be the set of all subsets of a given finite, non-empty set of points in \( \mathbb{R}^n_+ \). Then, an ordering \( \succ \) on \( K' \) satisfies the counterparts of Non-triviality, Denseness, Archimedean Property and Independence if and only if, for all \( A, B \in K' \), \( A \succ B \) iff \( |A| \geq |B| \).

(Note that the counterparts of Non-triviality, Denseness, Archimedean Property and Independence need to be defined for \( K' \).)

Remark 4.12. In Theorem 4.2, the ordering \( \succ \) is assumed to satisfy four properties, Non-triviality, Denseness, Archimedean Property and Independence. Examples 4.13, 4.14 and 4.15 show that, for each of the properties,
Non-triviality, Denseness and Independence, it is possible to construct an ordering over $K$, which violates that property while satisfying the other three properties figuring in Theorem 4.2. However, we have not been able to construct an example where an ordering $\geq$ over $K$ violates Archimedean Property but satisfies Non-triviality, Denseness and Independence; nor have we been able to derive Archimedean Property from Non-triviality, Denseness and Independence. Therefore, the issue of whether such an example can be constructed remains an open problem.

Example 4.13. Define the binary relation $\geq_1$ over $K$ as follows: for all $A, B \in K$, $A \sim_1 B$. Note that $\geq_1$ is an ordering that satisfies Denseness, Archimedean Property and Independence, but violates Non-triviality. For this ordering, clearly, (4.1) is not satisfied.

Example 4.14. Let $N(0, \varepsilon) = \{ x \in \mathbb{R}^n_+ \mid d(x, 0) \leq \varepsilon \}$, where $\varepsilon > 0$ and $d(x, 0)$ is the (Euclidean) distance between $x$ and 0. Define the binary relation $\geq_2$ over $K$ as follows: for all $A, B \in K$, if $\{ \text{vol}(A) > \text{vol}(B) \}$ or $\{ \text{vol}(A) = \text{vol}(B) > 0 \}$, then $A \sim_2 B$, and if $\{ \text{vol}(A) = \text{vol}(B) \}$ or $\{ \text{vol}(N(0, \varepsilon) \cap B) > 0 \}$, then $A \sim_2 B$. It can be checked that $\geq_2$ is an ordering that satisfies Non-triviality, Archimedean Property and Independence, but violates Denseness. For this ordering, due to its lexicographic nature, there exists no representation.

Example 4.15. Define the binary relation $\geq_3$ over $K$ as follows: for all $A, B \in K$, $A \geq_3 B$ if $\delta(A) \geq \delta(B)$ where $\delta(C) = 0$ if $C$ is finite and $\delta(C) = 1$ if $C$ is infinite for all $C \in K$. Then, $\geq_3$ is an ordering that satisfies Non-triviality, Denseness, and Archimedean Property, but violates Independence. Clearly, $\delta(\cdot)$ defined above is not additive.

Remark 4.16. Clearly, Non-triviality is a necessary condition for the existence of a countably additive representation of $\geq$ that satisfies (4.1) and (4.2). On the other hand, neither Denseness nor Independence is necessary for the existence of a countably additive representation $\phi$ of $\geq$ that satisfies (4.1) and (4.2). This can be checked easily via the following example. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous, strictly increasing, and $f(x) < \infty$ for all $x \in \mathbb{R}^+$. Then, for all $A \in K$, let $\phi^*(A) = \sum_{x \in A} f(x)$. Clearly, $\phi^*$ is countably additive and satisfies (4.1) and (4.2). But the ordering $\geq^*$ induced by $\phi^*$ fails to satisfy Denseness and Independence. To see why $\geq^*$ fails to satisfy Denseness, consider $\{(1, 1, \ldots, 1)\}$ and $\{(2, 2, \ldots, 2)\}$. Since $f$ is strictly increasing, it is clear that $\{(2, 2, \ldots, 2)\} \geq^* \{(1, 1, \ldots, 1)\}$. However, this violates Denseness, since, as noted in Section 4, Denseness would require $\{(2, 2, \ldots, 2)\} \sim^* \{(1, 1, \ldots, 1)\}$. The violation of Independence by
\( \geq^* \) is clear by considering the following. Let \( A = \{ a = (a_1, \ldots, a_n) \in \mathbb{R}_n^+ | a_i \leq 1 \text{ for all } i = 1, \ldots, n \}, \ B = \{2, 2, \ldots, 2\} \) and \( C = \{3, 3, \ldots, 3\}\). Then, according to \( \geq^* \), we have \( C \succ^* B \) and \( C \sim^* B \cup A \), which violates Independence. Now, consider \( \phi : K \rightarrow \mathbb{R}^+ \) such that \( \phi(\emptyset) = 0; \) for every non-empty finite set \( A \in K, \ \phi(A) = |A|; \) and for every infinite set \( A \in K, \ \phi(A) = \infty. \) The ordering \( \geq^* \) induced by \( \phi \) violates Archimedean Property though \( \phi \) satisfies (4.1) and (4.2).

5. PREFERENCES AND THE RANKING OF OPPORTUNITY SETS

In this section, we consider a richer informational structure, where preferences are admitted as a relevant consideration in ranking opportunity sets. This, of course, raises an immediate question. What types of preferences should one use in the context of ranking opportunity sets? Jones and Sugden [6] (see also Pattanaik and Xu [12]) have argued that, in the context of ranking opportunity sets in terms of freedom of choice, one should use the notion of preferences “reasonable” people. To recapture the main argument, assume that given the opportunity set \( A = \{x, y, z, \ldots\} \), \( x \) is the agent’s best option in \( A \) in terms of his existing preference ordering as well as in terms of any other preference ordering that, he expects, he may have in the future. Jones and Sugden [6] argue that, even in this case, a switch from the opportunity set \( A \) to the opportunity set \( \{x\} \) will reduce the agent’s freedom if a person, in our agent’s circumstances, could reasonably choose \( y \) or \( z \) from \( A \). Thus, if one can think of a reasonable person choosing \( y \) from \( A \) in the circumstances of our agent, then a transition from the opportunity set \( A \) to the opportunity set \( \{x\} \) would reduce the agent’s freedom even though \( x \) is the uniquely best option in \( A \) in terms of the agent’s present preference ordering, as well as in terms of any of the preference orderings that, the agent thinks, he may have in the future. From this standpoint, which we find attractive, it is the preferences that reasonable people may have in the agent’s situation, rather than the agent’s actual or possible future preferences, which are relevant for the ranking of opportunity sets. Given this view of freedom, we shall start with the primitive notion of a given set of preference orderings over the universal set of alternatives, this set being interpreted as the set of all possible orderings that a reasonable person can have in the agent’s situation (equivalently, the given set of orderings can be interpreted as the set of all possible orderings that the agent can reasonably have in his situation). Note that, though we have chosen to interpret the reference set of orderings as the set of preference orderings that reasonable people may have in the agent’s circumstances, the formal analysis that follows in the rest of this section is also compatible with other possible interpretations of this reference set. For
example, if one wanted to do so, one could interpret the reference set of orderings as the set of orderings of all individuals belonging to the same society as the agent under consideration; such alternative interpretations of the reference set of orderings are compatible with the formal analysis given below.

Some Additional Notation and Definitions

Let $\varphi$ denote our reference set of orderings over $\mathbb{R}_n^+$; as noted above, $\varphi$ will be interpreted as the set of all possible preference orderings (at least as good as) over $\mathbb{R}_n^+$, that a reasonable person may have. In typical economic contexts, it is usually assumed that preference orderings are continuous and monotonic. We shall adhere to this tradition. Let $\varphi$ be the set of all continuous and monotonic orderings over $\mathbb{R}_n^+$; $\varphi$ is to be interpreted as the set of all orderings over $\mathbb{R}_n^+$ that reasonable people may have. The elements of $\varphi$ will be denoted by $R, R', R_1, R_2, R_i, \ldots$. For all $R \in \varphi$, $P$ will denote the asymmetric factor of $R$.

The analysis that follows, will make extensive use of the max of the different opportunity sets in $K$. The reference set, $\varphi$ of orderings will seldom figure directly in our formal analysis. However, insofar as the max of an opportunity set is defined in terms of the orderings in $\varphi$, $\varphi$ will play an important, though indirect, role in our analysis.

For all $A \in K$, let $\max(A)$ denote the set $\{a \in A \mid a$ is $R$-greatest in $A$ for some $R \in \varphi\}$.

Remark 5.1. Since $\varphi$ is the set of all continuous and monotonic orderings over $\mathbb{R}_n^+$, it is clear that, for every $A \in K$,

(i) $\max(A) = \{(x_1, \ldots, x_n) \mid$ there does not exist $y = (y_1, \ldots, y_n) \in A$, such that $[(y_i \geq x_i$ for all $i) \text{ and } (y_j > x_j \text{ for some } j)]\}$, and

(ii) if $A$ is non-empty, then $\max(A)$ is non-empty and compact, and, hence, belongs to $K$.

Thus, for every $A \in K$, $\max(A)$ is the undominated surface of $A$ and is compact.

Remark 5.2. For every $A \in K$, if $\max(A) = A$, then $\{\max(B) \mid A \in K\}$.

An ordering $R$ over $\mathbb{R}_n^+$ is monotonic iff, for all $x, y \in \mathbb{R}_n^+$, $[x_i \geq y_i$ for all $i \in \{1, \ldots, n\}$ and $x_j > y_j$ for some $j \in \{1, \ldots, n\}]$ implies $[xRy \text{ and not } yRx]$. The results of this section hold for a more restricted set of preferences. In particular, the additional requirement that the preference orderings be convex would not affect our results.
Some Additional Axioms

Given our informationally richer framework which includes $\varphi$, the set of all orderings that reasonable people may have, Independence no longer seems to be an attractive property. Consider Fig. 2, where we show three non-empty, compact subsets, $A$, $B$ and $C$, of $\mathbb{R}_+^2$, such that $C$ and $A \cup B$ are disjoint. Since, by assumption, the set of all continuous and monotonic orderings over $\mathbb{R}_+^2$ is the set of all orderings over $\mathbb{R}_+^2$, that a reasonable person in the agent's position can possibly have, it is clear that, for every ordering $R$ that a reasonable person may have, there exists a point $x$ in $A$ which is better, in terms of $R$, than all the points in $C$. In this sense $C$ is dominated by $A$. However $B$ does not dominate $C$ in this sense (nor does $C$ dominate $B$).

Then it may be argued that adding $C$ to $A$ does not increase the freedom of the agent though adding $C$ to $B$ does increase the agent's freedom. In general, the appeal of independence property seems to be much less in our present PB framework than in our earlier NPB framework. Therefore, we replace it by two properties which take into account the orderings in $\varphi$.

First, consider two opportunity sets $A$ and $B$ such that for every ordering of a reasonable person, there is some alternative in $A$ that is ranked higher than all the alternatives in $B$. Then, it seems highly plausible that the addition of $B$ to $A$ will not increase the agent's freedom of choice already offered by $A$. A second intuition that we can derive by considering a reasonable person's preferences is the following. Consider two opportunity sets $C$ and $D$ with the property that $C \succeq D$ and $\max(C) = C$; in other words, opportunity set $C$ offers at least as much freedom as opportunity set $D$ and every alternative in $C$ is ranked at least as high as every other alternative in $C$ according to some ordering in $\varphi$ (each and every alternative in $C$ can be chosen by a reasonable person). Now, a third opportunity set $E$ with the property of $E = \max(E)$ and $\max(E \cup C) = E \cup C$ is added to $C$. Since each and every alternative in $E \cup C$ can be a best element according
to some ordering in $\varphi$ for the agent, intuitively, the addition of $E$ to $C$ will not reduce the degree of freedom of choice already offered by $C$ and $E \cup C \geq D$ will continue to hold. There are no doubt other intuitions based on the notion of a reasonable person's preferences in the present framework. For our purpose, the two intuitions that we have just discussed are sufficient and they are captured in the properties introduced in the following definition.

**Definition 5.3.** The ordering $\succeq$ satisfies:

1. **Dominance** if, for all $A \in K$, and all $B \in J$, if, [for all $b \in B$, $b \notin \max(A \cup \{b\})$] and $A \cup B \in K$, then $A \sim A \cup B$;

2. **Composition** if, for all $A, B, C, D \in J$, if $[A \cap C = B \cap D = \emptyset$, $\max(A \cup C) = A \cup \pi(C)$, $\max(B \cup D) = B \cup \pi(D)]$, then $[A \succeq B$ and $\pi(C) \succeq \pi(D)] \Rightarrow [A \cup \pi(C) \succeq B \cup \pi(D)]$, and $[(A \succeq B$ and $\pi(C) \succeq \pi(D))] \Rightarrow [A \cup \pi(C) \succeq B \cup \pi(D)]$.

Dominance is a natural and intuitive way of incorporating information about reasonable persons' preferences. It requires that, if an opportunity set $B$ is such that, for every possible preference ordering in $\varphi$, at least one alternative in $A$ will be ranked strictly above each alternative $b$ in $B$, then the freedom offered by $A$ is exactly the same as the freedom offered by $A \cup B$. The intuition of Dominance is straightforward. Suppose, $A$ and $A \cup B$ are both compact and, for every $b$ in $B$, $b$ does not belong to $\max(A \cup \{b\})$. Then, it is clear that, for every $b$ in $B$ and every preference ordering $R$ in $\varphi$, all $R$-greatest alternatives in $A$ will be strictly preferred, in terms of $R$, to $b$ (recall that, since $A$ is compact, and, by our assumption, $R$ is continuous, an $R$-greatest alternative in $A$ will always exist). This, of course, implies that, in the presence of all the alternatives in $A$, no reasonable person would ever choose any alternative belonging to $B$. Dominance stipulates that, in this circumstance, adding $B$ to $A$ does not add to the agent's freedom. This accords well with our intuition.

Composition is another natural and intuitive way of considering information about a reasonable person's preferences. Composition is a somewhat weaker version of a property proposed by Sen [16], and subsequently discussed and used by Pattanaik and Xu [12], in the context of ranking opportunity sets in terms of freedom of choice.7 Sen's version requires that, given $A \cap C = B \cap D = \emptyset$, if $[A \succeq B$ and $C \succeq D]$, then $[A \cup C \succeq B \cup D]$, and, if $[A \succeq B$ and $C \succeq D]$, then $[A \cup C \succeq B \cup D]$. Our property weakens Sen's property by restricting the applicability of the property to the case where every alternative in $A \cup C$ can be considered a best alternative in

7 See Krantz et al. [7] for a discussion of a similar axiom in a different context-qualitative probability theory.
A \cup C$ according to some preference ordering in $\phi$, and every alternative in $B \cup D$ can also be considered a best alternative in $B \cup D$ according to some preference ordering in $\phi$. In doing so, it avoids some pitfalls arising from the Independence property.\footnote{Note, however, Composition and Independence are formally two independent properties.}

**A Representation Theorem in the PB Framework**

We now explore the corresponding implication of replacing Independence in Theorem 4.2 by Dominance and Composition. The main theorem of this section, Theorem 5.4 below, shows that Non-triviality, Denseness, Archimedian Property, Dominance and Composition together ensure the existence of a countably sub-additive representation of $\succeq$, which has several plausible properties.

**Theorem 5.4.** Let $\succeq$ on $K$ satisfy Non-triviality, Denseness, Archimedian Property, Dominance and Composition. Then $\succeq$ has a countably sub-additive representation $\phi$ such that

\[
\phi(\emptyset) = 0, \text{ and there exists } X \in K \text{ such that } \phi(X) > 0 \tag{5.1};
\]

\[
\text{for all } A \in K, \ \phi(A) = \phi(\max(A)) \tag{5.2};
\]

\[
\text{for all } A, B \in K, \text{ if } \max(A) = A \text{ and } B \subseteq A, \text{ then } \phi(B) \leq \phi(A) \tag{5.3};
\]

\[
\text{for all } A, B \in K, \text{ if } A \cap B \sim \emptyset \text{ and } \max(A \cup B) = A \cup B, \text{ then } \phi(A) + \phi(B) = \phi(A \cup B) \tag{5.4};
\]

\[
\text{and for all } A_1, A_2, \ldots \in K, \text{ such that } A_i \cap A_j = \emptyset \text{ for all distinct } i \text{ and } j, \\
\left[ \bigcup_{i=1}^{\infty} A_i \in K \text{ and } \bigcup_{i=1}^{\infty} A_i = \max(\bigcup_{i=1}^{\infty} A_i) \right] \Rightarrow \phi(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \phi(A_i). \tag{5.5}
\]

We omit the proof of Theorem 5.4 since it is very similar to the proof of Theorem 4.2.

**Remark 5.5.** For all $A \in K$, let $a(A)$ denote the area of the undominated surface of $A$. Let $f: \mathbb{R}_+^n \to \mathbb{R}_+$ be continuous and let $a_f(A) := \int a(A) f(a(A)) \, da$. Then, it is easy to check that each of area$(\cdot)$ and $a_f(\cdot)$ induces an ordering $\succeq$ over $K$, which satisfies the properties specified by Theorem 5.4. Thus, our Theorem 5.4 is not vacuous.

**Remark 5.6.** Note that the representation $\phi$ referred to in Theorem 5.4 is such that $\phi(A) = \phi(\max(A))$, which implies that $A \sim \max(A)$. Thus, under Theorem 5.4, $\max(A)$ can be viewed as the effective opportunity set corresponding to $A$, so that the ranking of any two opportunity sets $C$ and
\( D \) is essentially determined by the ranking of the two max sets corresponding to these two opportunity sets. The ranking of the max sets, in its turn, can be viewed as a size-based ranking, \( \phi(\max(A)) \) being interpreted as the size of \( \max(A) \) (note that, by (5.3) and (5.4), [for all \( A \) and \( B \in K \), if \( \max(B) \subseteq \max(A) \), then \( \phi(\max(A)) \geq \phi(\max(B)) \)], and [if \( \max(A) \cap \max(B) = \emptyset \), and \( \max(A \cup B) = \max(A) \cup \max(B) \), then \( \phi(\max(A)) \cup \phi(\max(B)) = \phi(\max(A)) + \phi(\max(B)) \)]. Thus, in our preference-based approach, the consideration of the quality of the available options, as judged in terms of the preference orderings in \( \varphi \), serves to throw out first all the dominated alternatives in each of the opportunity sets under consideration. The surviving max sets then constitute the basis of the ranking of the opportunity sets.

Remark 5.7. The intuitive reason why \( \phi \) is sub-additive rather than additive in Theorem 5.4 can be seen from the following example. Given two disjoint opportunity sets \( A \) and \( B \), the addition of \( B \) to \( A \) does not increase the agent’s freedom beyond the freedom offered by \( A \), if, for every ordering \( R \) in \( \varphi \), every option in \( B \) is less preferred than some option in \( A \).

Remark 5.8. In Theorem 5.4, the ordering \( \succ \) is assumed to satisfy five properties, Non-triviality, Denseness, Archimedean Property, Dominance and Composition. Examples 5.9, 5.10, 5.11 and 5.12 show that, for each of the properties, Non-triviality, Denseness, Dominance and Composition, one can construct an ordering over \( K \), which violates that property but satisfies the other four properties figuring in Theorem 5.4. However, the issue of whether an ordering over \( K \) can violate Archimedean Property while satisfying the other four properties in Theorem 5.4 remains an open problem.

Example 5.9. Define \( \succ \) as follows: for all \( A, B \in K \), \( A \sim_p B \). Note that \( \succ \) is an ordering that satisfies Denseness, Archimedean Property, Dominance and Composition, but violates Non-triviality. For this ordering, clearly, (5.1) is not satisfied.

Example 5.10. Let \( L(\varepsilon) =: \{ x \in \mathbb{R}_+^n : x_1 + \cdots + x_n = \varepsilon \} \), where \( \varepsilon > 0 \). Define \( \succ_b \) as follows: for all \( A, B \in K \), if \( \text{area}(\max(A)) > \text{area}(\max(B)) \) or \( (\text{area}(\max(A)) = \text{area}(\max(B)) > 0 \text{ and area}(L(\varepsilon) \cap \max(A)) > 0 \text{ but } \text{area}(L(\varepsilon) \cap \max(A)) = 0) \), then \( A \succ_b B \), and if \( (\text{area}(\max(A)) = \text{area}(\max(B)) \text{ and } (\text{area}(L(\varepsilon) \cap \max(A)) = \text{area}(L(\varepsilon) \cap \max(B)) = 0 \text{ or } \text{area}(L(\varepsilon) \cap \max(B)) > 0 \text{ and area}(L(\varepsilon) \cap \max(A)) > 0)) \), then \( A \sim_b B \). It can be checked that \( \succ_b \) is an ordering that satisfies Non-triviality, Archimedean Property, Dominance, and Composition, but violates Denseness. For this ordering, due to its lexicographic nature, there exists no representation.
For all \( A, B \in K \), let \( \succeq \) be defined as follows: \( A \succeq B \) iff \( \delta(A) \geq \delta(B) \) where \( \delta(C) = 0 \) if \( \max(C) \) is finite and \( \delta(C) = 1 \) if \( \max(C) \) is infinite for all \( C \in K \). Then, \( \succeq \) is an ordering that satisfies Non-triviality, Denseness, Archimedean Property, and Dominance, but violates Composition. Clearly, \( \delta(\cdot) \) defined above is not additive.

For all \( A, B \in K \), let \( \succeq_d \) be defined as follows: \( A \succeq_d B \) iff \( \text{vol}(A) \geq \text{vol}(B) \). Then, \( \succeq_d \) generates an ordering, satisfies Non-triviality, Denseness and Composition, but violates Dominance. For this ordering, clearly, (5.2) is violated since \( \text{vol}(\max(A)) = 0 \) for all \( A \in K \).

Remark 5.13. It is easy to check that both Non-triviality and Dominance are necessary for the existence of a countably sub-additive representation of \( \succeq \) that satisfies (5.1), (5.2), (5.3), (5.4) and (5.5). On the other hand, Denseness and Composition are not necessary for the existence of a countably sub-additive representation of \( \succeq \) that satisfies (5.1), (5.2), (5.3), (5.4) and (5.5). To see it, consider the following example. Let \( f: \mathbb{R}_+^n \to \mathbb{R}_+ \) be continuous, strictly increasing, and \( f(x) < \infty \) for all \( x \in \mathbb{R}_+^n \). Then, for all \( A \in K \), let \( \phi^*(A) = \sum_{x \in \max(A)} f(x) \). Clearly, \( \phi^* \) is countably subadditive and satisfies (5.1)–(5.5). It can be checked that the ordering \( \succeq^* \) induced by \( \phi^* \) fails to satisfy Denseness and Composition. Now, consider \( \phi^*: K \to \mathbb{R}_+ \) such that \( \phi^*(\emptyset) = 0 \); for every non-empty \( A \in K \), if \( \max(A) \) is a finite set, then \( \phi^*(A) = |\max(A)| \), and, if \( \max(A) \) is an infinite set, then \( \phi^*(A) = \infty \). The ordering \( \succeq^* \) induced by \( \phi^* \) violates Archimedean Property, though \( \phi^* \) satisfies (5.1), (5.2), (5.3), (5.4) and (5.5).

Before concluding this section, we would like to clarify certain aspects of our Theorem 5.4. Note that Denseness implies INS and that the intuition underlying Dominance is basically the same as the intuition of Jones and Sugden’s [6] principle of *Addition of Insignificant Options* (AIO) which requires that, if we add to a set \( A \) an option \( x \) not in \( A \), such that \( x \) will never be chosen by any reasonable person in the presence of the all the options in \( A \), then the freedom of the agent remains the same. This has important intuitive implications. An elegant result due to Jones and Sugden [6] (see also Sugden [19]) shows that no ordering \( \succeq \) over opportunity sets can satisfy INS, AIO and the principle of *Addition of Significant Options* (ASO) which requires that if, starting with an opportunity set \( A \), we add an element \( x \) not in \( A \), such that \( x \) will be chosen from \( A \cup \{x\} \) by some reasonable person, then such addition will increase the freedom of the agent under consideration. Since Denseness implies INS and Dominance retains the intuition of AIO, given the properties of Denseness and Dominance used in Theorem 5.4 we should expect to see a violation of the spirit of ASO. In the context of Theorem 5.4, it is easy to see how such violation takes place. Consider Fig. 3, where we have only two commodities.
Let $A$ be the line segment $xy$ and $B$ be the line segment $zw$. By Dominance $A \sim \{x\}$ and $[A \cup B] \sim \{w\}$; and, by Denseness which implies INS, $\{x\} \sim \{w\}$. Hence, by transitivity, $[A \cup B] \sim A$, which goes against the intuition of ASO. Thus, the violation of the spirit of ASO is inevitable under every ordering satisfying the properties figuring in Theorem 5.4. Also, note that an ordering $\succeq$ on $K$ which satisfies Non-triviality, Denseness, Archimedean Property, Dominance and Composition, and, hence, (5.1) through (5.5), may involve an even stronger form of violation of ASO insofar as it can permit a reduction in the freedom of the agent when we add to a set $A$ an element $x$ not in $A$, such that $\text{max}(A \cup \{x\}) = \{x\}$.

This raises the issue of how disturbing one considers the violation of ASO. This issue has been discussed in some detail by Jones and Sugden [6] and Sugden [19]. For example, Sugden [19, section 6] distinguishes two distinct aspects of freedom in a preference-based approach, namely, the range of opportunity and the scope of significant choosing. The latter aspect, the scope of significant choosing, is closely linked to Mill's [10] argument that the very act of choosing in a non-trivial fashion is, in itself, valuable, since it develops certain important human faculties. Note that the notion of significant choosing or choosing in a non-trivial fashion refers to the act of making choices on the basis of serious deliberation, and, in that sense, when one chooses from a set $\{x, y, z\}$ such that no reasonable person would ever choose $y$ or $z$ in the presence of $x$, the act of choosing is not significant or non-trivial. Then, based on this notion of freedom of choice, the set $\{x, y, z\}$ offers the same amount of freedom of choice as the set $\{x\}$. Thus, intuitively, the scope of significant or non-trivial choosing corresponding to a set $A$ is embodied by $\text{max}(A)$, that is, the set of all options in $A$ that reasonable persons may choose from $A$. Without taking the position that one should give priority to the scope of significant choosing,

\[\text{This shows that a superset does not always offer at least as much freedom as its subset.}\]
Sugden persuasively argues that, if one is concerned with the scope of significant choosing, then one may not find the violation of ASO particularly disturbing. On the other hand, if one is concerned with the range of opportunity (for example, Sen [16, 17] is primarily concerned with this aspect), then, one may like to retain the intuition of both ASO and AIO, and discard INS and, hence, Denseness. Our Theorem 5.4 may be viewed as an attempt to explore the consequences of a preference-based approach that seeks to capture the scope of significant choosing and, therefore, retains INS and AIO at the cost of sacrificing ASO.

6. CONCLUDING REMARKS

Much of the present literature on the ranking of opportunity sets in terms of freedom assumes that these sets are all subsets of a finite universal set. In seeking to extend the analysis to economic contexts, where opportunity sets are usually infinite sets, we focused on the problem of ranking compact subsets of the n-dimensional (non-negative) real space. We explored the problem in two distinct frameworks: the first did not incorporate any information about preferences, while the second was based on a reference set of preference orderings, which, following Jones and Sugden [6], we interpreted as the set of all orderings that a reasonable person may have. In each framework, we explored the implication of certain axioms or properties for the freedom ranking of opportunity sets and derived a “representation theorem” that showed that, under appropriate axioms, the ranking had some intuitively interesting properties.

Finally, it may be worth noting that, in this paper, we have not explored an important issue. Throughout our paper, we have assumed Denseness. However, as we have seen in Fig. 3, if Denseness is to be retained, then we cannot have both Dominance and the property (for convenience, call it Strong Monotonicity) that, if two disjoint sets $A$ and $B$ are such that no reasonable person would choose an option belonging to $A$ in the presence of all the options belonging to $B$, then $A \sqcup B > A$. Though Strong Monotonicity, which is a stronger version of ASO, may be dispensable when modeling the scope of significant choosing, when we seek to capture what Sugden [19] calls the range of opportunity, both Dominance and Strong Monotonicity seem to be compelling properties while Denseness seems dispensable. Therefore, it is important to analyze how one can capture the notion of the range of opportunity by discarding Denseness while retaining both Dominance and Strict Monotonicity. This is a line of enquiry which we have not pursued in this paper and which deserves independent investigation.
REFERENCES