LOCAL AND GLOBAL HOPF BIFURCATION IN A DELAYED HEMATOPOIESIS MODEL

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In this paper, we consider the following nonlinear differential equation

$$\frac{dx}{dt} = x(t) \left[ \frac{q}{r + x^n(t) - p} \right].$$

(1)

We first consider the existence of local Hopf bifurcations, and then derive the explicit formulas which determine the stability, direction and other properties of bifurcating periodic solutions, using the normal form theory and center manifold reduction. Further, particular attention is focused on the existence of the global Hopf bifurcation. By using the global Hopf bifurcation theory due to Wu [1998], we show that the local Hopf bifurcation of (1) implies the global Hopf bifurcation after the second critical value of the delay \( \tau \). Finally, numerical simulation results are given to support the theoretical predictions.

Keywords: Time delay; local Hopf bifurcation; global Hopf bifurcation; periodic solutions.

1. Introduction

For a long time, it has been acknowledged that delays can have very complicated impact on the dynamics of a system (see e.g. [Hale & Lunel, 1993; Kuang, 1993; Wu, 1996]). For example, delays can cause the loss of stability and induce various oscillations and periodic solutions. It is well known that periodic solutions can arise through the Hopf bifurcation in delay differential equations. However, these periodic solutions bifurcating from Hopf bifurcations are generally local. More specifically, the existence of these periodic solutions remains valid only in a small neighborhood of the critical value and are usually of small amplitudes. Therefore, we wish to know whether these nonconstant periodic solutions which are obtained through local Hopf bifurcations can continue for a large range of parameter values.

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A great deal of research has been devoted to the global existence of nonconstant periodic solutions in delayed equations (see, e.g., [Chow, 1974; Mallet-Paret & Nussbaum, 1986; Táboas, 1990; Zhau et al., 1997] and references therein). The method of showing the global existence of nonconstant periodic solutions used by the above mentioned researchers is Nussbaum’s theorem [1974], which depends in a crucial way on the concept of ejectivity and the decomposition theory of linear functional differential equations. In this paper, instead of applying Nussbaum’s theorem, we shall use a different approach, the degree theory, to study the global existence of nonconstant periodic solutions arising from the following nonlinear delayed differential equation

$$\frac{dx}{dt} = x(t) \left[ \frac{q}{r + x^n(t - \tau)} - p \right], \quad t \geq 0 \quad (2)$$

with $p, q, r, \tau \in (0, \infty)$ and $n \in N = \{1, 2, \ldots \}$. Equation (2) is one of the models proposed by Nazarenko [1976] to study the control of a single population of cells. Equation (2) has been recently investigated by several researchers. Kubiaczyk and Saker [2002] have shown that every positive solution of (2) oscillates about its positive equilibrium point $x_s = [(q/p) - r]^{1/n}$ if

$$\frac{nqx^*_n}{(r + x^*_n)^2} > \frac{1}{\epsilon},$$

and $x_s$ is locally asymptotically stable provided that

$$\frac{nqx^*_n}{(r + x^*_n)^2} < \frac{\pi}{2}.$$
Separating the real and imaginary parts, we obtain
\[ \alpha \cos \omega \tau = 0, \quad \alpha \sin \omega \tau = \omega, \] (6)
which lead to
\[ \omega = \alpha, \quad \tau_k = \frac{1}{\alpha} \left( 2k\pi + \frac{\pi}{2} \right), \quad k = 0, 1, \ldots, \] (7)
i.e. when \( \tau = \tau_k \), (5) has a pair of purely imaginary roots \( \pm \alpha i \).

Let \( \lambda_k = \eta_k(\tau) + i\omega_k(\tau) \) denote a root of (5) near \( \tau = \tau_k \), such that \( \eta_k(\tau_k) = 0 \), \( \omega_k(\tau_k) = \alpha \). Substituting \( \lambda_k \) into (5) and differentiating with respect to \( \tau \), we then have
\[ \frac{d\lambda}{d\tau} - \alpha \left( \lambda + \tau \frac{d\lambda}{d\tau} \right) e^{-\lambda \tau} = 0. \]

It follows that
\[ \frac{d\lambda}{d\tau} = \frac{\alpha \lambda e^{-\lambda \tau}}{1 - \alpha \tau e^{-\lambda \tau}}, \]
which, together with (6), implies
\[ \left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau_k} = \frac{\alpha^2 \sin \alpha \tau_k + i \alpha^2 \cos \alpha \tau_k}{1 - \alpha \tau_k \cos \alpha \tau_k + i \alpha \tau_k \sin \alpha \tau_k} \]
\[ = \frac{\alpha^2}{1 + i \alpha \tau_k}. \]
Therefore, we have that the following transversality condition holds:
\[ \frac{d}{d\tau} \text{Re}\{\lambda_k(\tau_k)\} = \frac{\alpha^2}{1 + (\alpha \tau_k)^2} > 0. \] (8)

From the above discussion, we can easily obtain the following results about the location of roots of transcendental equation (5).

**Lemma 2.1.** For Eq. (5), we have the following results.

(i) When \( \tau = \tau_k \) \( (k = 0, 1, 2, \ldots) \), Eq. (5) has a pair of simple imaginary roots \( \pm \alpha i \).
(ii) When \( \tau \in [0, \tau_0) \), all roots of Eq. (5) have negative real parts, and when \( \tau = \tau_0 \), all roots of Eq. (5), except \( \pm \alpha i \), have negative real parts. But when \( \tau \in (\tau_k, \tau_{k+1}] \), Eq. (5) has \( 2(k + 1) \) roots with positive real parts.

From Lemma 2.1 and the transversality condition (8), we may easily obtain the results about the stability of the positive equilibrium and the local Hopf bifurcation for (2).

**Theorem 2.2.** If \( q > pr \), then we have

(i) the positive equilibrium \( x_* \) is asymptotically stable for \( \tau \in [0, \tau_0) \), and unstable for \( \tau > \tau_0 \); and
(ii) (2) undergoes a Hopf bifurcation at the positive equilibrium \( x_* \) when \( \tau = \tau_k \), for \( k = 0, 1, 2, \ldots \).

**Remark 2.3.** In Theorem 2.2, we not only obtain the sufficient condition for asymptotical stability of the positive equilibrium of (2), which is the same as that given in [Kubiaczyk & Saker, 2002], but also get the conditions assuring the existence of local Hopf bifurcations.

### 3. Direction and Stability of the Hopf Bifurcation

In the previous section, we have shown that (2) undergoes Hopf bifurcation at \( \tau = \tau_k \), \( k = 0, 1, \ldots \).

As pointed out in [Hassard et al., 1981], it is interesting to determine the direction, stability and period of these periodic solutions bifurcating from the equilibrium \( x_* \). In this section, we derive the explicit formulae which determine these factors at the critical value \( \tau_k \) using normal form and center manifold theories due to Hassard et al. [1981].

For convenience, let \( x(t) = y(\tau t) \) and \( \tau = \tau_k + \mu \). Then (3) can be written as an FDE in \( C = C([-1, 0], R) \) as
\[ \dot{x}(t) = L_\mu(x_t) + f(\mu, x_t), \] (9)
where \( x_t(\theta) = x(t + \theta) \in C \), and \( L_\mu : C \to R \), \( f : R \times C \to R \) are given respectively by
\[ L_\mu(\phi) = - (\tau_k + \mu) \alpha \phi(-1), \] (10)
and
\[ f(\mu, \phi) = (\tau_k + \mu) \left[ \alpha_1 \phi(0) \phi(-1) \right. \]
\[ + \frac{1}{2!} \alpha_2 \phi(0) \phi^2(-1) + \frac{1}{2!} \alpha_2 x_* \phi^2(-1) \]
\[ + \frac{1}{3!} \alpha_3 x_* \phi^3(-1) + O(\phi^4) \right], \] (11)
where \( \alpha = \frac{aq \alpha x_*^2}{(r + x_*^2)^2} \), \( \alpha_k = \frac{(\delta^k g(0))/\delta y^k}{(k = 1, 2, 3) \text{ and } g(y) = g/(r + (y + x_*)^2)}. \)

By the Riesz representation theorem, there exists a function \( \eta(\theta, \mu) \) of bounded variation for \( \theta \in [-1, 0] \), such that
\[ L_\mu \phi = \int_{-1}^{0} d\eta(\theta, 0) \phi(\theta) \quad \text{for } \phi \in C. \]
In fact, we can choose
\[ \eta(\theta, \mu) = (\tau_k + \mu) \alpha \delta(\theta + 1), \]
where $\delta$ is the dirac delta function. For $\phi \in C^1([-1,0], R)$, define
\[
A(\mu)\phi = \begin{cases}
\frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\
\int_{-1}^{0} d\eta(\mu,s)\phi(s), & \theta = 0.
\end{cases}
\]
and
\[
R(\mu)\phi = \begin{cases}
0, & \theta \in [-1,0), \\
f(\mu,\phi), & \theta = 0.
\end{cases}
\]
Then (9) is equivalent to
\[
\dot{x}_t = A(\mu)x_t + R(\mu)x_t,
\]
where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1,0]$.
For $\psi \in C^1([0,1], (R^*)^*)$, define
\[
A^*\psi(s) = \begin{cases}
-\frac{d\psi(s)}{ds}, & s \in (0,1], \\
\int_{-1}^{0} d\eta(t,0)\psi(-t), & s = 0.
\end{cases}
\]
and a bilinear inner product
\[
\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi,
\]
(13)
where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and $A^*$ are adjoint operators. By the discussion in Sec. 2, we know that $ \pm i\tau k \alpha $ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of $A^*$.

By the direct computation, it can be verified that $q(\theta) = e^{i\tau k \alpha \theta}$ is an eigenvector of $A(0)$ corresponding to $i\tau k \alpha$, and $q^*(s) = De^{i\tau k \alpha s}$ is an eigenvector of $A^*$ corresponding to $-i\tau k \alpha$. Furthermore,
\[
\langle q^*(s), q(\theta) \rangle = 1, \quad \langle q^*(s), \bar{q}(\theta) \rangle = 0,
\]
where
\[
D = \frac{1}{1 + i\tau k \alpha}.
\]

In the remainder of this section, we will follow the ideas and use the same notations as given in [Hassard et al., 1981] to first compute the coordinates describing the center manifold $C_0$ at $\mu = 0$. Let $x_t$ be the solution of Eq. (9) when $\mu = 0$. Define
\[
z(t) = \langle q^*, x_t \rangle, \\
W(t, \theta) = x_t(\theta) - 2\text{Re}\{z(t)q(\theta)\}.
\]
(14)
On the center manifold $C_0$ we have
\[
W(t, \theta) = W(z(t), \bar{z}(t), \theta),
\]
where
\[
W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots,
\]
z and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\bar{q}^*$. Note that $W$ is real if $x_t$ is real. We consider only real solutions. For the solution $x_t \in C_0$ of (9), since $\mu = 0$, we have
\[
\dot{z}(t) = i\tau k \alpha z(t) + (q^*(t), f(0, W(z, \bar{z}, \theta) + 2\text{Re}\{zq(\theta)\})) + i\tau k \alpha z(t) + \bar{q}^*(0)f(0, W(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\})
\]
def
\[
= i\tau k \alpha z(t) + \bar{q}^*(0)f_0(z, \bar{z}).
\]
We rewrite this equation as
\[
\dot{z}(t) = i\tau k \alpha z(t) + g(z, \bar{z})
\]
with
\[
g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z})
\]
\[
= g_{20}z^2 + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2}
\]
\[
+ g_{21}\frac{z^2\bar{z}}{2} + \cdots.
\]
(15)

Note that $x_t(\theta) = W(t, \theta) + zq(\theta) + \bar{q}q(\theta)$ and $g(\theta) = e^{i\tau k \alpha \theta}$. Thus, we may obtain
\[
g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = \bar{D}\tau k \left[ a_0x(t)x(t) - 1 + \frac{1}{2}a_2x^2(t) - 1 + \frac{1}{2}a_3x^3(t) - 1 + \cdots \right]
\]
\[
= \left\{ 2a_0e^{-i\tau k \alpha} + \alpha_2c_2e^{-2i\tau k \alpha}\right\} \frac{z^2}{2}
\]
\[
+ (2a_1\text{Re}e^{i\tau k \alpha} + \alpha_2c_2)e\bar{z}
\]
\[
+ (2a_1e^{i\tau k \alpha} + \alpha_2c_2e^{2i\tau k \alpha})\bar{z}^2
\]
\[
+ [a_0(2W_{11}(-1) + W_{20}(-1) + 2e^{-i\tau k \alpha}W_{11}(0) + e^{i\tau k \alpha}W_{20}(0)) + \alpha_2c_2(2 + e^{-2i\tau k \alpha})
\]
\[
+ \alpha_2c_2(2e^{-i\tau k \alpha}W_{11}(-1) + e^{i\tau k \alpha}W_{20}(-1)
\]
\[
+ \alpha_3c_2e^{-i\tau k \alpha}z^2 + \cdots \right\}
Comparing the coefficients with (15), we have
\[ g_{20} = \overline{D} \tau_k (2 \alpha_1 e^{-i \tau_k \alpha} + \alpha_2 x_* e^{-2i \tau_k \alpha}); \]
\[ g_{11} = \overline{D} \tau_k (2 \alpha_1 \text{Re}\{e^{i \tau_k \alpha}\} + \alpha_2 x_*) ; \]
\[ g_{02} = \overline{D} \tau_k (2 \alpha_1 e^{i \tau_k \alpha} + \alpha_2 x_* e^{2i \tau_k \alpha}); \]
\[ g_{21} = \overline{D} \tau_k \{\alpha_1 [2W_{11}(-1) + W_{20}(-1)]
+ 2e^{-i \tau_k \alpha}W_{11}(0) + e^{i \tau_k \alpha}W_{20}(0)]
+ \alpha_2 (2 + e^{-2i \tau_k \alpha}) + \alpha_2 x_*[e^{-i \tau_k \alpha}W_{11}(-1)
+ e^{i \tau_k \alpha}W_{20}(-1)] + \alpha_3 x_* e^{-i \tau_k \alpha}\}. \]
Note that \( g_{21} \) contains \( W_{20}(\theta) \) and \( W_{11}(\theta) \) which need to be computed.
From (12) and (14), we have
\[ \dot{W} = \dot{z} - \dot{\varphi} - \overline{\varphi} \]
\[ = \begin{cases} AW - 2 \text{Re}\{\varphi^*(0)f_0q(\theta)\}, & \theta \in [-1, 0), \\
AW - 2 \text{Re}\{\varphi^*(0)f_0q(0)\} + f_0, & \theta = 0. \end{cases} \]
def \[= AW + H(z, \overline{z}, \theta), \]
\[ (17) \]
where
\[ H(z, \overline{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) \overline{z}z 
+ H_{02}(\theta) \frac{\overline{z}^2}{2} + \cdots. \]
(18)
Expanding the above series and comparing the corresponding coefficients, we obtain
\[ (A - 2i \tau_k \alpha)W_{20}(\theta) = -H_{20}(\theta), \]
\[ AW_{11}(\theta) = -H_{11}(\theta), \ldots \]
(19)
From (18), we know that for \( \theta \in [-1, 0), \)
\[ H(z, \overline{z}, \theta) = -\overline{\varphi}^*(0)f_0q(\theta) - q^*(0)\overline{T}_{02}\overline{\varphi}(\theta) 
= -gq(\theta) - \overline{g}\overline{\varphi}(\theta). \]
(20)
Comparing the coefficients with (19) gives that
\[ H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \]
(21)
and
\[ H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \]
(22)
It follows from (19) and (21) that
\[ \dot{W}_{20}(\theta) = 2i \tau_k \alpha W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \]
Note that \( q(\theta) = e^{i \tau_k \alpha \theta} \), hence
\[ W_{20}(\theta) = \frac{ig_{20}}{\tau_k \alpha} e^{i \tau_k \alpha \theta} + \frac{ig_{02}}{3\tau_k \alpha} e^{-i \tau_k \alpha \theta} 
+ E_1 e^{2i \tau_k \alpha \theta}. \]
(23)
Similarly, from (19) and (22), we can obtain
\[ W_{11}(\theta) = -\frac{ig_{11}}{\tau_k \alpha} e^{i \tau_k \alpha \theta} 
+ \frac{ig_{11}}{\tau_k \alpha} e^{-i \tau_k \alpha \theta} + E_2. \]
(24)
In what follows, we shall seek appropriate \( E_1 \) and \( E_2 \). From the definition of \( A \) and (20), we obtain
\[ \int_{-1}^{0} \frac{d\eta(\theta)}{\tau_k \alpha}W_{20}(\theta) = 2i \tau_k \alpha W_{20}(0) - H_{20}(0) \]
and
\[ \int_{-1}^{0} \frac{d\eta(\theta)}{\tau_k \alpha}W_{11}(\theta) = -H_{11}(0), \]
(25)
(26)
where \( \eta(\theta) = \eta(0, \theta) \). From (17), we have
\[ H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta) 
+ 2\tau_k \alpha e^{-i \tau_k \alpha} + \tau_k \alpha_2 x_* e^{-2i \tau_k \alpha}. \]
(27)
Substituting (23) and (27) into (25), we obtain
\[ E_1 = \frac{2\tau_k \alpha e^{-i \tau_k \alpha} + \tau_k \alpha_2 x_* e^{-2i \tau_k \alpha}}{2i \tau_k \alpha - \int_{-1}^{0} e^{2i \tau_k \alpha \theta} d\eta(\theta)}, \]
which, together with the definition of \( \eta(\theta) \), leads to
\[ E_1 = \frac{2\alpha_1 e^{-i \tau_k \alpha} + \alpha_2 x_* e^{-2i \tau_k \alpha}}{\alpha(2i + e^{-2i \tau_k \alpha})}. \]
Similarly, substituting (24) and (28) into (26), we get
\[ E_2 = \frac{2\tau_k \alpha_1 e^{i \tau_k \alpha} + \tau_k \alpha_2 x_*}{2i \tau_k \alpha - \int_{-1}^{0} e^{2i \tau_k \alpha \theta} d\eta(\theta)} \]
\[ = \frac{2\alpha_1 e^{i \tau_k \alpha} + \alpha_2 x_*}{\alpha (2i + e^{2i \tau_k \alpha})}. \]
Therefore, all \( g_{ij} \) in (17) have been expressed in terms of the parameters and the delay given in (9).
Furthermore, we can compute the following values:
\[ c_1(0) = \frac{i}{2\tau_k \alpha} \left( g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \]
\[ \mu_2 = \frac{\text{Re}\{c_1(0)\}}{\text{Re}\{\lambda_{\tau_k}(\theta)\}}, \]
\[ \beta_2 = 2\text{Re}\{c_1(0)\}, \]
\[ T_2 = \frac{\text{Im}\{c_1(0)\} + \mu_2 \text{Im}\{\lambda_{\tau_k}(\theta)\}}{\tau_k \alpha}, \]
which determine the quantities of bifurcating periodic solutions on the center manifold at the critical value \( \tau_k \), i.e. \( \mu_2 \) determines the directions of the Hopf bifurcation: if \( \mu_2 > 0 \) (\( \mu_2 < 0 \)), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for \( \tau > \tau_k \) (\( \tau < \tau_k \)); \( \beta_2 \) determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if \( \beta_2 < 0 \) (\( \beta_2 > 0 \)); and \( T_2 \) determines the period of the bifurcating periodic solutions: the period increases (decreases) if \( T_2 > 0 \) (\( T_2 < 0 \)).

Remark 3.1. Note that when \( \tau > \tau_0 \), the corresponding characteristic equation of (3) has at least one root with positive real part. Therefore, when \( \tau = \tau_k \) (\( k \geq 1 \)), the nontrivial periodic solutions bifurcating from the positive equilibrium must be unstable in the phase space, even though they are stable on the center manifold. But when \( \tau = \tau_0 \), the stability of bifurcating periodic solutions on the center manifold is equivalent to that of bifurcating periodic solutions in the whole phase space.

4. Global Existence of Periodic Solutions

In this section, we shall study the global continuation of periodic solutions bifurcating from the positive equilibrium \( x_* \). Throughout this section, we follow closely the notations used in [Wu, 1998].

To simplify notations, we may rewrite (2) as the following functional differential equation

\[
\dot{x}(t) = F(x(t), \tau, T),
\]

where \( x(t) \in C([-\tau, 0], R) \) and \( F(x(t), \tau, T) = x(t)[(q/(r + x(t)^p)) - p] \). Here we still suppose that \( q > pr \). It is obvious that when \( n = 2k \), (29) has three equilibria \( \bar{x}_1 \equiv 0, \bar{x}_2 \equiv x_* \), and \( \bar{x}_3 \equiv -x_* \), while (29) has two equilibria \( \bar{x}_1 \equiv 0, \bar{x}_2 \equiv x_* \) for \( n = 2k + 1 \). Following the work of Wu [1998], we need to define

\[
\mathbf{X} = C([-\tau, 0], R),
\]

\[
\sum = Cl\{(x, \tau, T) \in \mathbf{X} \times R \times R^+; \quad x(T) = 0 \}
\]

is a \( T \)-periodic solution of (29),

\[
N = \{(x, \tau, T); F(x, \tau, T) = 0 \}.
\]

Let \( \ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})} \) denote the connected component of \( (x_*, \tau_k, 2\pi/\alpha) \) in \( \sum \), where \( \tau_k \) is defined in (7). From Theorem 2.2, we know that \( \ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})} \) is nonempty. For the benefit of the readers, we first state the global Hopf bifurcation theory due to Wu [1998] for functional differential equations.

**Lemma 4.1** [Wu, 1998]. Assume that \( (x_*, \tau, T) \) is an isolated center satisfying the hypotheses \( (A_1)-(A_4) \) in Wu [1998, p. 4814]. Denote by \( \ell(x_*, \tau, T) \) the connection of \( (x_*, \tau, T) \) in \( \sum \). Then either

(i) \( \ell(x_*, \tau, T) \) is unbounded, or
(ii) \( \ell(x_*, \tau, T) \) is bounded, \( \ell(x_*, \tau, T) \cap N \) is finite and

\[
\sum_{(x_*, \tau, T) \in \ell(x_*, \tau, T) \cap N} \gamma_m(x_*, \tau, T) = 0
\]

for all \( m = 1, 2, \ldots \), where \( \gamma_m(x_*, \tau, T) \) is the mth crossing number of \( (x_*, \tau, T) \) if \( m \in J(x_*, \tau, T) \), or it is zero if otherwise.

Clearly, if (ii) in Lemma 4.1 is not true, then \( \ell(x_*, \tau, T) \) is unbounded. Thus, if the projections of \( \ell(x_*, \tau, T) \) onto \( x \)-space and onto \( T \)-space are bounded, then the projection of \( \ell(x_*, \tau, T) \) onto \( T \)-space is unbounded. Further, if we can show that the projection of \( \ell(x_*, \tau, T) \) onto \( T \)-space is away from zero, then the projection of \( \ell(x_*, \tau, T) \) onto \( T \)-space must include interval \( [\tau, \infty) \). In the following, we follow this ideal to prove our main results about the global Hopf bifurcation.

**Lemma 4.2.** When \( q > pr \), Eq. (29) has no nontrivial \( \tau \)-periodic solutions.

**Proof.** For a contradiction, suppose that Eq. (29) has \( \tau \)-periodic solution. Then the ordinary differential equation

\[
\dot{x}(t) = x(t) \left[ \frac{q}{r + x(t)^p} - p \right]
\]

has a periodic solution.

When \( n = 2k \), (30) has three steady states \( x(t) = 0, x(t) = x_* \), and \( x(t) = -x_* \). From (30), it follows that when either \( 0 < x(t) < x_* \) or \( x(t) < -x_* \), \( \dot{x}(t) > 0 \), and \( \dot{x}(t) < 0 \) as either \( x(t) > x_* \) or \( -x_* < x(t) < 0 \). Therefore, the ordinary differential equation (30) with \( n = 2k \) has no nontrivial periodic solution.

However, when \( n = 2k + 1 \), (30) has two steady states \( x(t) = 0 \) and \( x(t) = x_* \). Suppose that \( x(t) \) is a nontrivial periodic solution of (30). From

\[
x(t) = x(0) \exp \left\{ \int_0^t \left[ \frac{q}{r + x(s)^p} - p \right] ds \right\}
\]

we have either \( x(t) > 0 \) or \( x(t) < 0 \). Notice that \( \dot{x}(t) > 0 \) for \( 0 < x(t) < x_* \) and \( \dot{x}(t) < 0 \) for
In addition, from (2) and (31), we may obtain

\[ 0 = \frac{q}{r + x^n(\xi)} - p, \]

where \( x(\xi) = \min\{x(t)\} \). Thus, we get \( x(\xi) = \left[ (q - pr)/p \right]^{\frac{1}{n}} > 0 \), which contradicts \( x(t) < 0 \). This implies that (30) with \( n = 2k + 1 \) has no negative periodic solution. Thus, the ordinary differential equation (30) with \( n = 2k + 1 \) has no nontrivial periodic solution, too.

From the above discussion, we known that (30) has no nontrivial periodic solution. This fact implies that Eq. (29) has no nontrivial \( \tau \)-periodic solutions. \( \blacksquare \)

**Lemma 4.3.** If \( q > pr \), then all the nontrivial periodic solutions of (29) are uniformly bounded for \( \tau \in [0, \tau^*] \).

**Proof.** For a periodic function \( x(t) \), we define

\[ x(\xi) = \min\{x(t)\}, \quad x(\eta) = \max\{x(t)\}, \quad (31) \]

Let \( x(t) \) be a periodic solution of (2). Then we obtain

\[ x(t) = x(0) \exp \left\{ \int_0^t \left[ \frac{q}{r + x^n(s - \tau)} - p \right] ds \right\}, \]

which implies that either \( x(t) \equiv 0 \) or \( x(t) \neq 0 \). Thus, it is sufficient to consider the following two cases.

**Case 1.** When \( x(t) > 0 \), it follows from (2) that

\[ \dot{x}(t) < \frac{q - pr}{r} x(t). \]

By using the comparison theorem of ordinary differential equation, we obtain

\[ x(t) < x(t - \tau) \exp \left\{ \frac{(q - pr)\tau}{r} \right\}, \quad \text{for} \; t > \tau. \quad (32) \]

In addition, from (2) and (31), we may obtain

\[ 0 = \frac{q}{r + x^n(\eta - \tau)} - p, \]

which implies

\[ x(\eta - \tau) = \left[ \frac{q - pr}{p} \right]^{\frac{1}{n}}. \quad (33) \]

Therefore, from (32) and (33), we have

\[ x(\eta) < x(\eta - \tau) \exp \left\{ \frac{(q - pr)\tau}{r} \right\} = \left[ \frac{q - pr}{p} \right]^{\frac{q - pr}{p}} \exp \left\{ \frac{(q - pr)\tau}{r} \right\}. \quad (34) \]

**Case 2.** When \( x(t) < 0 \) and \( n = 2k \), we set \( y(t) = -x(t) > 0 \). Then (2) can be rewritten as

\[ \dot{y}(t) = y(t) \left[ \frac{q}{r + y^n(t - \tau)} - p \right]. \]

Thus, by the same arguments as Case 1, we get

\[ y(\eta) < \left[ \frac{q - pr}{p} \right]^{\frac{1}{n}} \exp \left\{ \frac{(q - pr)\tau}{r} \right\}, \]

which means

\[ x(\xi) < - \left[ \frac{q - pr}{p} \right]^{\frac{1}{n}} \exp \left\{ \frac{(q - pr)\tau}{r} \right\}. \quad (35) \]

On the other hand, if \( x(t) < 0 \) and \( n = 2k + 1 \), then from (2) and (31), we get

\[ 0 = \frac{q}{r + x^n(\xi - \tau)} - p. \]

It follows that

\[ x(\xi - \tau) = \left[ \frac{q - pr}{p} \right]^{\frac{1}{n}} > 0, \]

which obviously contradicts the assumption \( x(t) < 0 \). Thus, Eq. (2) with \( n = 2k + 1 \) has no negative periodic solution.

From (34) and (35), it follows that if \( x(t) \) is a nonconstant periodic solution of (2), then

\[ - \left[ \frac{q - pr}{p} \right]^{\frac{1}{n}} \exp \left\{ \frac{(q - pr)\tau}{r} \right\} < x(t) < \left[ \frac{q - pr}{p} \right]^{\frac{1}{n}} \exp \left\{ \frac{(q - pr)\tau}{r} \right\}, \]

which implies that all the nontrivial periodic solutions of (2) are uniformly bounded. The proof is completed. \( \blacksquare \)

**Remark 4.4.** Although the boundaries of periodic solutions depend on the delay \( \tau \), they are uniformly bounded provided that \( \tau \) is bounded.

**Theorem 4.5.** If \( q > pr \), then, for each \( \tau > \tau_k \) \( (k \geq 1) \), Eq. (2) has at least \( k \) periodic solutions, where \( \tau_k = (1/\alpha)(2k\pi + (\pi/2)). \)
Proof. It is sufficient to prove that the projection of \( \ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})} \) onto \( \tau \)-space is \([\tau, \infty)\) for each \( k \geq 1 \), where \( \tau \leq \tau_k \).

The characteristic equation of (29) at an equilibrium \( \mathbf{\pi} \) takes the following form

\[
\Delta(\mathbf{\pi}, \tau, T)(\lambda) = \lambda I - DF(\mathbf{\pi}, \tau, T)(e^{\lambda T} I),
\]

\[
= \lambda - \frac{\partial F(\mathbf{\pi}, \tau, T)}{\partial x(t)} - \frac{\partial F(\mathbf{\pi}, \tau, T)}{\partial x(t-\tau)} e^{-\lambda \tau}.
\]

From [Wu, 1998], we know that \((\mathbf{\pi}, \tau, T)\) is called a center if \(F(\mathbf{\pi}, \tau, T) = 0\) and \(\Delta(\mathbf{\pi}, \tau, T)(2\pi/T) = 0\). A center \((\mathbf{\pi}, \tau, T)\) is said to be isolated if it is the only center in some neighborhood of \((\mathbf{\pi}, \tau, T)\). It follows from (36) that

\[
\Delta(0, \tau, T)(\lambda) = \lambda - \frac{q - pr}{r},
\]

which has no purely imaginary roots. Thus, we conclude that (29) does not have a center in the form of \((0, \tau, T)\). In addition, it is easy to verify that when \(n = 2k\),

\[
\Delta(-x_*, \tau, T)(\lambda) = \Delta(x_*, \tau, T)(\lambda) = \lambda + \alpha e^{-\lambda \tau}.
\]

On the other hand, from the discussion about the local Hopf bifurcation in Sec. 2, it is easy to verify that \((x_*, \tau_k, 2\pi/\alpha)\) and \((-x_*, \tau_k, 2\pi/\alpha)\) are isolated centers. However, when \(n = 2k + 1\), (29) has only single isolated center \((x_*, \tau_k, 2\pi/\alpha)\). It follows from Sec. 2 that there exist \(\varepsilon > 0\), \(\delta > 0\) and a smooth curve \(\lambda : (\tau_k - \delta, \tau_k + \delta) \rightarrow C\) such that \(\Delta(\lambda(\tau)) = 0, |\lambda(\tau) - \alpha| < \varepsilon\) for all \(\tau \in [\tau_k - \delta, \tau_k + \delta]\) and

\[
\lambda(\tau_k) = \alpha i, \quad \frac{d\text{Re}\{\lambda(\tau)\}}{d\tau} \bigg|_{\tau = \tau_k} > 0.
\]

Let

\[
\Omega_{\varepsilon, \frac{2\pi}{\alpha}} = \{ (\eta, T) : 0 < \eta < \varepsilon, |T - \frac{2\pi}{\alpha}| < \varepsilon \}.
\]

It is easy to verify that on \([\tau_k - \tau, \tau_k + \tau] \times \partial \Omega_{\varepsilon, \frac{2\pi}{\alpha}}, \Delta(x_*, \tau, T) (\eta + \frac{2\pi}{T} i) = 0\) if and only if \(\eta = 0\),

\[
\tau = \tau_k, \quad T = \frac{2\pi}{\alpha}.
\]

Therefore, the hypotheses \((A_1) \sim (A_4)\) in [Wu, 1998] are satisfied for \(m = 1\). Moreover, if we define

\[
H^{\pm}(x_*, \tau_k, \frac{2\pi}{\alpha}, \eta, T) = \Delta(x_*, \tau_k \pm \delta, T) (\eta + i \frac{2\pi}{T}),
\]

then we have the crossing number of the isolated center \((x_*, \tau_k, 2\pi/\alpha)\) as follows

\[
\gamma \left( x_*, \tau_k, \frac{2\pi}{\alpha} \right) = \text{deg}_{B} \left( H^{-}(x_*, \tau_k, \frac{2\pi}{\alpha}, \Omega_{\varepsilon, \frac{2\pi}{\alpha}}) \right) = -1,
\]

where \(\text{deg}_{B}\) denotes the Brouwer degree. Similarly, we also have

\[
\gamma \left( -x_*, \tau_k, \frac{2\pi}{\alpha} \right) = -1.
\]

Thus, we have

\[
\sum_{(\mathbf{\pi}, \tau, T) \in \ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}} \gamma(\mathbf{\pi}, \tau, T) < 0,
\]

where \((\mathbf{\pi}, \tau, T)\), in fact, has either the form as \((-x_*, \tau_k, 2\pi/\alpha)\) \((k = 0, 1, \ldots)\) or the form as \((x_*, \tau_k, 2\pi/\alpha)\) \((k = 0, 1, \ldots)\). It follows from Lemma 4.1 that the connected component \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\) through \((x_*, \tau_k, 2\pi/\alpha)\) in \(\Sigma\) is unbounded. From (7), we have

\[
\tau_k = \frac{1}{\alpha} \left( 2k\pi + \frac{\pi}{2} \right) > \frac{2\pi}{\alpha}, \quad \text{for } k > 1.
\]

Thus, there exists an \(n_k \in N\) such that

\[
\frac{\tau_k}{n_k} < \frac{2\pi}{\alpha} < \tau_k.
\]

Now we prove that the projection of \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\) onto \(\tau\)-space is \([\tau, \infty)\), where \(\tau \leq \tau_k\). Clearly, it follows from the proof of Lemma 4.2 that Eq. (2) with \(\tau = 0\) has no nontrivial periodic solution. Hence, the projection of \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\) onto \(\tau\)-space is away from zero.

For a contradiction, we suppose that the projection of \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\) onto \(\tau\)-space is bounded. This means that the projection of \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\) onto \(\tau\)-space is included on a interval \((0, \tau^*)\). Noticing \(2\pi/\alpha < \tau_k\) and applying Lemma 4.2 we have \(T < \tau^*_s\) for all \((x(t), \tau, T)\) belonging to \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\). This implies that the projection of \(\ell_{(x_*, \tau_k, \frac{2\pi}{\alpha})}\) onto \(T\)-space
is bounded. Then, applying Lemma 4.3 we get that the connected component \( \Omega_{(x_*, \tau_0, \Delta x)} \) is bounded. This leads to a contradiction, which completes the proof.

5. A Numerical Example

From the algorithm given in Sec. 3, we know that if the values of \( p, q \) and \( \tau \) are known, then we can compute the values of \( \mu_2 \) and \( \beta_2 \), which determine the stability and direction of periodic solutions bifurcating from the positive equilibrium at the critical point \( \tau_k \). In this section, we present some numerical results of the following system:

\[
\dot{x}(t) = x(t) \left[ \frac{2}{1 + x^2(t-\tau)} - 1 \right],
\]

which has a positive equilibrium \( x_* \equiv 1 \). It follows from (4) that \( \tau_0 \approx 1.5708 \), \( \tau_1 \approx 7.85398 \), \( \tau_2 \approx 14.1372 \), . . . By Theorem 2.2, we know that the positive equilibrium \( x_* \) is stable when \( \tau < \tau_0 \), as illustrated by computer simulations, as shown in Fig. 1, for \( \tau \approx 1.4 \). When \( \tau \) passes through \( \tau_0 \), \( x_* \) loses its stability and a Hopf bifurcation occurs at \( \tau > \tau_k \). Furthermore, from Sec. 3, we can determine the stability and direction of periodic solutions bifurcating from the positive equilibrium at the critical point \( \tau_k \). For instance, when \( \tau = \tau_0 \approx 1.5708 \), we find \( c_1(0) \approx -0.689745 + 0.723412i \), indicating that \( \mu_2 > 0 \) and \( \beta_2 < 0 \). Therefore, bifurcating periodic solutions exist at least for the value \( \tau \) slightly larger than \( \tau_0 \) and the corresponding periodic orbits are orbitally, asymptotically stable, as depicted in Fig. 2, where \( \tau = 1.7 \). For a larger delay \( \tau = 9 \), which is relatively far away from \( \tau_1 \), the bifurcating periodic solutions still exist (see Fig. 3). Numerical simulation results also verify theoretical investigation for the global existence of periodic solutions discussed in Sec. 4.

6. Discussion

In this paper, we investigate local and global Hopf bifurcations in a delayed hematopoiesis model. We show that, as the delay \( \tau \) increases, the positive equilibrium \( x_* \) loses its stability and a sequence of Hopf bifurcations (local) occur at \( x_* \). For a local Hopf bifurcation, using center manifold reduction and normal form theories, we derive explicit formulae determining stability, direction and period of bifurcating periodic solutions. Further, existence of periodic solutions for \( \tau \) far away from the local Hopf bifurcation values is also established. More specifically, we show that, for each \( \tau \in [\tau_1, \infty) \), Eq. (2) has at least a periodic solution. Unfortunately, the following questions have not been investigated completely by the authors.

Fig. 1. The positive equilibrium Eq. (37) is stable when \( \tau = 1.4 < \tau_0 \).
Fig. 2. A periodic solution Eq. (37) is orbitally, asymptotically stable when $\tau = 1.7$.

Fig. 3. A large periodic solution Eq. (37) for $\tau_1 < \tau < 9 < \tau_2$.

1. For any $\tau \in [\tau_0, \tau_1)$, it is unknown whether Eq. (2) has a periodic solution.
2. From the global Hopf bifurcation theory due to Wu [1998], we obtain the existence of periodic solutions for a large range $\tau \in [\tau_1, \infty)$, but this approach seems to generate little information for the stability.

Here, these two problems are left open.

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References


