This paper mainly considers the smooth complex spline approximation of Cauchy-type integral operators over an open arc. First, the smoothness of the operators is investigated, then some properties of complex splines are discussed, and finally the error estimates of the approximation are given.

1. INTRODUCTION

The approximation for Cauchy-type singular integrals and the numerical solution of the corresponding equations have been studied in many papers, and a series of important results have been obtained [3, 7, 8, 13]. However, the rates of convergence for the approximation are not very satisfactory. In fact, the best rates have not been revealed. The reason may be that, as mappings, the smoothness of the integrals is not investigated thoroughly. For example, paper [7] discusses the following operator:

$$\hat{A}(\varphi)(t) \equiv a(t)w_2(t)\varphi(t) - \frac{b(t)}{\pi} \int_{-1}^{1} \frac{w_2(\tau)\varphi(\tau)}{\tau - t} d\tau, \quad t \in [-1, 1], \tag{1.1}$$

where $a \in H[-1, 1]$, $b$ is a polynomial and $w_2$ is a weight function constructed from $a$ and $b$ (for details, see [7] or [6]), and proves that if $w_2 \in H^{(1)}[-1, 1]$ and $\varphi \in C^{m,\mu}[-1, 1]$ for $0 < \gamma; \mu < 1$, then $\hat{A}\varphi \in H^{(1)}[-1, 1]$ where $\lambda = \min(\gamma, \mu)$. This means that the smoothness of $\hat{A}\varphi$ depends on $w_2$. But in fact, as an operator, $\hat{A}$ is sufficiently smooth and its smoothness is not affected by $w_2$. This can be illustrated when $a(t) \equiv 0$ and $b(t) \equiv -1$ in (1.1). In this case, $\hat{A}$ becomes

$$T(\varphi)(t) \equiv \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - \tau^2}\varphi(\tau)}{\tau - t} d\tau, \quad t \in [-1, 1] \tag{1.2}$$
and it is bounded on $C^{m,\mu}[-1, 1]$ because we can easily convert it to a periodic singular integral or a Cauchy-type integral on a unit circle (cf. [15, 18]).

Usually, the Cauchy-type singular integrals are of the form

$$T_G(\varphi)(t) \equiv \frac{1}{\pi} \int_G \frac{\varphi(\tau)}{\tau - t} \, d\tau, \quad t \in \Gamma,$$

(1.3)

where $\Gamma$ is a smooth or piecewise smooth oriented curve. If $\Gamma$ is closed, as a mapping, $T_G$ is bounded on $C^{m,\mu} (\Gamma)$, and if $\Gamma$ is an axis, $T_G$ has the similar property [14, 15]. However, if $\Gamma$ is an open arc, the property disappears. Instead, we consider its “weighted type”. This is also a natural approach to the solution of singular integral equations over an open arc.

Generally speaking, by a simple transformation, our problems on an open arc can be converted to those on an interval, and then the problems may be solved more conveniently. But this will bring about at least two problems: the first is that the kernels of those integrals may become more complicated and it will then be more difficult for some treatments, such as numerical evaluations; the second is that a smooth function on the arc, as well as the related smooth operators, may become less or even not smooth on the interval if the arc is not perfectly smooth, and it will then impair the rates of its approximation. On the other hand, it is known that complex splines have a lot of advantages over other functions such as polynomials in approximation, especially in the approximation for those functions defined on arbitrary curves. So it will be more meaningful to discuss the singular integral operators and their complex spline approximation directly on arcs.

In this paper, our discussion focuses on the Cauchy singular integral operators of the form

$$A_w(\varphi)(t) \equiv a(t)w(t)\varphi(t) + \frac{b(t)}{\pi i} \int_G \frac{w(\tau)\varphi(\tau)}{\tau - t} \, d\tau, \quad t \in \Gamma,$$

(1.4)

which are derived from the theory of singular integral equations. Here we assume $a, b \in H(\Gamma)$ satisfying

$$a^2(t) - b^2(t) \neq 0, \quad t \in \Gamma$$

and $\Gamma$ is a smooth or piecewise smooth oriented open arc from $\alpha$ to $\beta$. The function $w$ is constructed as follows (cf. [5, 6, 12]).

Let $G(t) = \frac{a(t) + b(\overline{t})}{a(t) - b(\overline{t})}$, and choose a continuous branch of $G(t)$ such that

$$0 \leq \theta(t) < 1,$$

(1.5)

where $\theta(t) = \frac{\arg G(t)}{2\pi}$. Let $[y]$ denote the integral part of the real number $y$ and

$$\kappa = -[-\theta(\beta)].$$

(1.6)
Define the canonical function \([12, 15]\)

\[
X(z) = (\beta - z)^{-\kappa} \exp \left[ \frac{1}{2\pi i} \int_{\Gamma} \ln G(\tau) \, d\tau \right], \quad z \notin \Gamma
\]

and now let

\[
w(t) = \left[ (a(t) - b(t)) X^+(t) \right]^{-1}, \quad t \in \Gamma.
\]

It is easy to verify that \(w(t) \in H(\Gamma)\) and \(w(t) \neq 0, t \in \Gamma \setminus \{x, \beta\}\), which is similar to a weight function.

From the above construction, we know the smoothness of the function \(w\) is poor. However, we will show in this paper that the operator \(A_w\) is very smooth. Furthermore, if \(w(x) = w(\beta) = 0\) and \(b \in C^{m,\mu}(\Gamma)\), \(A_w\) is a bounded operator on \(C^{m,\mu}(\Gamma)\). Besides, we make a further study of complex interpolating splines and obtain some interesting properties. And then, there follow the results of the complex spline approximation for the operator \(A_w\). All these appear to be very attractive.

Throughout the paper, we assume that: \(0 < \mu < 1\), \(m\) is a nonnegative integer, \(\Gamma = \alpha \beta\), \(c_0\) is a constant such that the arc length \(|t_1 t_2| \leq c_0 |t_1 - t_2|\) for all \(t_1, t_2 \in \Gamma\), \(C(\Gamma)\) and \(C^m(\Gamma)\) denote the spaces of continuous and \(m\)-times continuously differentiable complex-valued functions on \(\Gamma\), respectively, \(||\psi|| \equiv \max_{t \in \Gamma} |\psi(t)|\), \(||\psi||_{C^m} \equiv \sum_{k=0}^{m} ||\psi^{(k)}||\), the modulus of continuity for \(\psi \in C(\Gamma)\) is denoted by \(\omega(\psi, \pi), M_{\mu}(\psi) \equiv \sup_{0 < \tau < \pi} \left| \frac{\omega(\psi, \tau)}{\tau^\mu} \right|\), \(C^{m,\mu}(\Gamma) \equiv \{\psi \in C^m(\Gamma) : M_{\mu}(\psi^{(m)}) < \infty\}\), \(H^\mu(\Gamma) \equiv C^{0,\mu}(\Gamma)\), \(H(\Gamma) \equiv \bigcup_{0 < \mu < 1} H^\mu(\Gamma)\), \(||\psi||_{H^\mu} \equiv ||\psi|| + M_{\mu}(\psi), ||\psi||_{C^m,\mu} \equiv ||\psi||_{C^m} + M_{\mu}(\psi^{(m)})\), and \(c\) is an absolute positive constant taking different value in different place.

The rest of the paper is organized as follows. In the next section, we show the smoothness of the operator \(A_w\). Some properties of complex splines are discussed in Section 3 and the results of spline approximation for \(A_w\) are given in Section 4. In Section 5 we illustrate an application for the approximation and remark on the case \(\mu = 1\), constant \(c\) and some other applications. The proof of Lemma 2.2 is longer and thus is put in the final section.

### 2. SMOOTHNESS OF SINGULAR INTEGRAL OPERATORS

In this section, we start our discussion for the smoothness of operator \(A_w\) from the introduction of some operators and the proofs of two lemmas.

Let \(u \in C(\Gamma)\). Define the operator \(S_{u,m}\) as

\[
S_{u,m}(f)(t) = m! \int_{\Gamma} u(t) \frac{f(\tau) - T_{m}(f)(\tau, t)}{(\tau - t)^{m+1}} \, d\tau, \quad f \in C^{m,\mu}(\Gamma),
\]
where \( T_m(f)(\tau, t) = f(t) + f'(t)(\tau - t) + \cdots + \frac{f^{(m)}(t)}{m!}(\tau - t)^m \) and \( T_0(f)(\tau, t) = f(t) \). If \( m = 0 \), \( S_{u,0}(f) \) is written as \( S_u(f) \).

Let \( \Psi_k(\tau, t) = k!u(\tau)\frac{f(\tau - T_k(f)(\tau, t))}{(\tau - t)^k+1} \), \( f \in C^{m,\mu}(\Gamma) \). It is easy to verify that

\[
\frac{\partial}{\partial t} \Psi_{k-1}(\tau, t) = \Psi_k(\tau, t)
\]  

(2.1)

and

\[
\sup_{t \in \Gamma} \int_{\Gamma} |\Psi_k(\tau, t)||d\tau| < \infty,
\]  

(2.2)

and using dominant convergence theorem, we obtain

\[
\frac{d^k}{dt^k} S_u(f)(t) = S_{u,k}(f)(t), \quad t \in \Gamma
\]  

(2.3)

for \( k = 1, 2, \ldots, m \).

If \( k < m \), then

\[
|f(\tau) - T_k(f)(\tau, t)| = \left| \frac{1}{k!} \int_t^\tau (\xi - \tau)^k f^{(k+1)}(\xi) \, d\xi \right|
\]

\[
\leq c \frac{1}{k!} ||f^{(k+1)}|| |\tau - t|^{k+1}
\]

(2.4)

and if \( k = m \),

\[
|f(\tau) - T_m(f)(\tau, t)| = \frac{1}{(m-1)!} \int_t^\tau (\xi - \tau)^{m-1} [f^{(m)}(\xi) - f^{(m)}(t)] \, d\xi
\]

\[
\leq c \frac{m!}{m!} \Omega(f^{(m)}, |\tau - t| |\tau - t|^m)
\]

(2.5)

for \( \tau, t \in \Gamma \), therefore

\[
|S_{u,k}(f)(t)| < c \int_{\Gamma} |u(\tau)||f^{(k+1)}||d\tau| < c ||u|| ||f^{(k+1)}||, \quad t \in \Gamma
\]

(2.6)

for \( k = 1, 2, \ldots, m - 1 \), and

\[
|S_{u,m}(f)(t)| < c ||u|| \int_0^1 \frac{\Omega(f^{(m)}, y)}{y} \, dy, \quad t \in \Gamma.
\]

(2.7)

Thus, we have
**Lemma 2.1.** For the operator $S_u$, there hold (2.3) and the following estimates:

\[
\left| \frac{d^k}{dt^k} S_u(f) \right| \leq c \|u\| \|f^{(k+1)}\|, \quad k = 0, 1, \ldots, m - 1, \tag{2.8}
\]

if $f \in C^m$ and

\[
\left| \frac{d^m}{dt^m} S_u(f) \right| \leq c \|u\| \int_0^1 \frac{\omega(f^{(m)}; y)}{y} dy, \tag{2.9}
\]

if $f^{(m)}$ satisfies Dini condition, i.e., $\int_0^1 \frac{\omega(f^{(m)}; x)}{x} dx < \infty$.

We also have

**Lemma 2.2.** There is a constant $c > 0$ depending on $m, \mu$ and $\Gamma$ such that

\[
\omega \left( \frac{d^m}{dt^m} S_u(f), x \right) \leq c \|u\| M_{\mu}(f^{(m)}) x^\mu (1 + |\ln x|) \tag{2.10}
\]

or

\[
\omega \left( \frac{d^m}{dt^m} S_u(f), x \right) \leq c \left( \|u\| + \int_0^1 \frac{\omega(u, y)}{y} dy \right) M_{\mu}(f^{(m)}) x^\mu, \tag{2.11}
\]

if $u$ satisfies Dini condition and $u(a) = u(b) = 0$, where $f \in C^{m, \mu}(\Gamma)$ and $x > 0$.

Since the proof is longer, we put it in the last section.

From Lemmas 2.1 and 2.2, it follows that

\[
\|S_u(f)\|_{C^m, \nu} \leq c N(u) \|f\|_{C^{m, \mu}}, \tag{2.12}
\]

where $0 < \nu < \mu$ and $N(u) = \|u\|$ or $0 < \nu \leq \mu$ and $N(u) = \|u\| + \int_0^1 \frac{\omega(u, x)}{x} dx$ if $u$ satisfies Dini condition and $u(a) = u(b) = 0$.

We rewrite the operator $A_w$ as

\[
A_w(\varphi)(t) = A^0_w(\varphi)(t) - \frac{1}{\pi i} \int_G w(\tau) \frac{b(\tau) - b(t)}{\tau - t} \varphi(\tau) d\tau, \quad t \in \Gamma, \tag{2.13}
\]

where

\[
A^0_w(\varphi)(t) = a(t) w(t) \varphi(t) + \frac{1}{\pi i} \int_G w(\tau) \frac{b(\tau) \varphi(\tau)}{\tau - t} d\tau, \quad t \in \Gamma \tag{2.14}
\]
and let
\[
K(\varphi(t)) = \frac{1}{\pi i} \int_{\Gamma} w(\tau) \frac{b(\tau) - b(t)}{\tau - t} \varphi(\tau) \, d\tau, \quad t \in \Gamma
\]
or
\[
K(\varphi(t)) = \frac{1}{\pi i} S_{w\varphi}(b).
\] (2.15)

Thus,
\[
A_w = A^0_w - K.
\] (2.16)

**Theorem 2.3.** There exists a constant \( c > 0 \) such that
\[
\left| \frac{d^k}{dt^k} A^0_w(\varphi) \right| \leq c \| \varphi \|_{C^{k+1}}, \quad k = 0, 1, \ldots, m - 1
\] (2.17)
for \( \varphi \in C^m \) and
\[
\| A^0_w(\varphi) \|_{C^{m,v}} \leq c \| \varphi \|_{C^{m,\mu}}
\] (2.18)
for \( \varphi \in C^{m,\mu}(\Gamma) \), where \( 0 < v < \mu \). If \( b(x)w(x) = b(\beta)w(\beta) = 0 \), then (2.18) is valid for \( v = \mu \).

**Proof.** \( A^0_w \) can be written as
\[
A^0_w(\varphi)(t) = A^0_w(1)(t)\varphi(t) + \frac{1}{\pi i} \int_{\Gamma} w(\tau)b(\tau)[\varphi(\tau) - \varphi(t)] \frac{d\tau}{\tau - t}, \quad t \in \Gamma
\]
or
\[
A^0_w(\varphi)(t) = p(t)\varphi(t) + \frac{1}{\pi i} S_{w\varphi}(\varphi)(t),
\] (2.19)
where \( p(t) = A^0_w(1)(t) \). But
\[
p(t) = p.p.\left( \frac{1}{X(z)}, t \right)
\] (2.20)
is a \( \kappa \)-degree polynomial \( (p(t) \equiv 0 \text{ if } \kappa < 0) \) (cf. [16]), and it is obvious that
\[
\| (p\varphi)^{(k)} \| \leq c \| \varphi \|_{C^{k}}, \quad k = 0, 1, \ldots, m
\] (2.21)
for \( \varphi \in C^m(\Gamma) \) and
\[
M_v((p\varphi)^{(m)}) \leq c \| \varphi \|_{C^{m,\mu}}, \quad 0 \leq v \leq \mu
\] (2.22)
or

$$\|p\phi\|_{C^{m,\mu}} \leq c\|\phi\|_{C^{m,\mu}}$$  \hspace{1cm} (2.23)$$

for $\phi \in C^{m,\mu}(\Gamma)$, where the constant $c$ depends on $a(t)$, $b(t)$ and $m$. On the other hand, $w_b$ satisfies Dini condition, so that there hold (2.8)–(2.10) for $S_{w_b}$ with $u = w_b$ and there also holds (2.11) if $w(\alpha)b(\alpha) = w(\beta)b(\beta) = 0$. Thus we obtain (2.17) and (2.18) from (2.19). The proof is complete.  

Let $b \in C^{m,\mu}(\Gamma)$. From Lemmas 2.1 and 2.2,

$$\left| \frac{d^k}{dt^k} K(\phi) \right| \leq c'\|w\phi\| \|b^{(k+1)}\| \leq c\|\phi\|, \hspace{1cm} k = 0, 1, \ldots, m - 1$$  \hspace{1cm} (2.24)$$

and

$$\|K(\phi)\|_{C^{m,\mu}} \leq c\|w\phi\| \|b\|_{C^{m,\mu}} \leq c\|w\| \|b\|_{C^{m,\mu}} \|\phi\|, \hspace{1cm} 0 < \nu < \mu$$  \hspace{1cm} (2.25)$$

for $\phi \in C(\Gamma)$, and if $w(\alpha) = w(\beta) = 0$, then

$$\|K(\phi)\|_{C^{m,\mu}} \leq c\left(\|w\phi\| + \int_0^1 \frac{w(\nu, y)}{y} \text{d}y \right) \|b\|_{C^{m,\mu}} \leq c\left(\|\phi\| + \int_0^1 \frac{\omega(\nu, y)}{y} \text{d}y \right)$$  \hspace{1cm} (2.26)$$

for $\phi \in H^{\mu}(\Gamma)$, where we have used the fact that

$$\omega(w\phi, y) \leq \|w\|\omega(\phi, y) + \|\phi\|\omega(w, y).$$

Thus, we obtain

**Theorem 2.4.** Let $b \in C^{m,\mu}(\Gamma)$. Then there is a constant $c > 0$ such that

$$\left| \frac{d^k}{dt^k} A_w(\phi) \right| \leq c\|\phi\|_{C^{k+1}}, \hspace{1cm} k = 0, 1, \ldots, m - 1$$  \hspace{1cm} (2.27)$$

for $\phi \in C^{m}(\Gamma)$ and

$$\|A_w(\phi)\|_{C^{m,\nu}} \leq c\|\phi\|_{C^{m,\mu}}$$  \hspace{1cm} (2.28)$$

for $\phi \in C^{m,\mu}(\Gamma)$, where $0 < \nu < \mu$. If $w(\alpha) = w(\beta) = 0$, then (2.28) is valid for $\nu = \mu$.

**Example 2.1.** Consider the operator $\dot{A}$ in (1.1) of Section 1. Since $a$, $b$ are real functions, we have $w_2(\pm 1)b(\pm 1) = 0$ ([16]). And $b$ is a polynomial, so that $\dot{A}$ is bounded on $C^{m,\mu}(\Gamma)$ according to Theorem 2.4. An interesting
result is that the operator
\[ \mathbf{A}^0(\varphi)(t) \equiv a(t)w_2(t)\varphi(t) - \frac{1}{\pi} \int_{-1}^{1} \frac{w_2(\tau)b(\tau)\varphi(\tau)}{\tau - t} d\tau, \quad t \in [-1, 1] \]  
(2.29)
is bounded on \( C^{m,\mu}(\Gamma) \) where we only require \( a, b \in H[-1, 1] \). It appears that its smoothness does not depend on \( a \) and \( b \).

3. SOME PROPERTIES OF COMPLEX SPLINES

Complex splines have many good properties (for details, see [1, 2, 4, 10, 11]). In order to discuss the approximation of singular integral operators, we need to have a further investigation for the splines. Here we only refer to the linear and cubic interpolating splines.

Let
\[ \Delta : \alpha = t_0 < t_1 < \cdots < t_N = \beta \]
be a partition of \( \Gamma \), \( \Gamma_j = t_{j-1}t_j \), \( h = \max_{1 \leq j \leq N} |\Delta t_j| \), and \( f^{(r)} = f^{(r)}(t_j) \), where \( t_{j-1} < t_j \) means that \( t_{j-1} \) precedes \( t_j \), \( \Delta t_j = t_j - t_{j-1} \) and \( f \in C^{m}(\Gamma) \). We denote the linear and cubic interpolating splines of \( f \) by \( s_1(f) \) or \( s_1 \) and \( s_3(f) \) or \( s_3 \), respectively.

For linear interpolating spline, we have
\[ s_1(t) = \frac{t_j - t}{\Delta t_j} f_{j-1} + \frac{t - t_{j-1}}{\Delta t_j} f_j, \quad t \in \Gamma_j, \quad j = 1, 2, \ldots, N \]
and
\[ ||s_1(f) - f|| \leq c_0 \omega(f, h). \]  
(3.1)

Let \( t, t' \in \Gamma \), \( t' < t \) and \( |t - t'| > 0 \). Then there are \( k \) and \( j \), \( 0 \leq k \leq j \leq N \), such that \( t \in \Gamma_j \) and \( t' \in \Gamma_k \). If \( k < j \), we have
\[ |s_1(t) - s_1(t')| \leq |s_1(t) - f_{j-1}| + |f_{j-1} - f_k| + |f_k - s_1(t')|. \]

According to the property of modulus of continuity,
\[ |f_{j-1} - f_k| \leq \omega(f, |t_{j-1} - t_k|) \]
\[ \leq \left(1 + \frac{|t_{j-1} - t_k|}{|t - t'|}\right) \omega(f, |t - t'|), \]
\[ |s_1(t) - f_{j-1}| \leq \frac{|t - t_{j-1}|}{|\Delta t_j|} \omega(f, |\Delta t_j|) \]
\[ \leq \frac{|t - t_{j-1}|}{|\Delta t_j|} \left( 1 + \frac{|\Delta t_j|}{|t - t'|} \right) \omega(f, |t - t'|) \]
\[ \leq \left( c_0 + \frac{|t - t_{j-1}|}{|t - t'|} \right) \omega(f, |t - t'|) \]
\[
\text{and similarly,} \quad |f_k - s_1(t')| \leq \left( c_0 + \frac{|t_k - t'|}{|t - t'|} \right) \omega(f, |t - t'|).
\]
So we have
\[
|s_1(t) - s_1(t')| \leq \left( 1 + 2c_0 + \frac{|t - t_{j-1}| + |t_{j-1} - t_k| + |t_k - t'|}{|t - t'|} \right) \omega(f, |t - t'|) \leq (1 + 3c_0) \omega(f, |t - t'|).
\]

If \( k = j \), then
\[
|s_1(t) - s_1(t')| \leq \frac{|t - t'|}{|\Delta t_j|} \omega(f, |\Delta t_j|) \leq \frac{|t - t'|}{|\Delta t_j|} \left( 1 + \frac{|\Delta t_j|}{|t - t'|} \right) \omega(f, |t - t'|) \leq (1 + c_0) \omega(f, |t - t'|).
\]
Thus, we have proved that
\[
\omega(s_1, x) \leq (1 + 3c_0) \omega(f, x), \quad x \geq 0. \tag{3.2}
\]
From (3.1) and (3.2), we conclude that
\[
\omega(s_1 - f, x) \leq c \omega(f, x^\gamma h^{1-\gamma}), \quad x \geq 0, \tag{3.3}
\]
where \( c = 2 + 3c_0, \ 0 \leq \gamma \leq 1 \). We show the conclusion as follows.
If \( 0 \leq x \leq h \), then \( x \leq x^\gamma h^{1-\gamma} \) and
\[
\omega(s_1 - f, x) \leq \omega(s_1, x) + \omega(f, x) \leq (2 + 3c_0) \omega(f, x) \leq c \omega(f, x^\gamma h^{1-\gamma})
\]
by (3.2); if $x > h$, then $h < x^\gamma h^{1-\gamma}$ and

$$
\omega(s_1 - f, x) \leq 2||s_1 - f||
\leq 2\omega(f, h)
\leq c\omega(f, x^\gamma h^{1-\gamma})
$$

by (3.2). Therefore (3.3) follows.

Thus, we have obtained

**Theorem 3.1.** For the linear interpolating spline $s_1$ of $f \in C(\Gamma)$, there hold estimates (3.1) and (3.3).

For $f \in H^\mu(\Gamma)$, we let $v = \gamma \mu$, then (3.3) becomes

$$
\omega(s_1 - f, x) \leq cM_\mu(f)x^v h^{\mu-v}.
$$

(3.3')

Furthermore, we have

**Corollary 3.2.** If $f \in H^\mu(\Gamma)$, then

$$
||s_1(f) - f||_{H^v} \leq cM_\mu(f)h^{\mu-v}
$$

(3.4)

for $0 \leq v \leq \mu$.

Now we discuss the cubic interpolating splines. Here we require that

$$
\frac{\max_j |\Delta t_j|}{\min_j |\Delta t_j|} \leq c_1 < \infty, \quad \max_{1 \leq j \leq N-1} \frac{|\Delta t_j| + |\Delta t_{j+1}|}{|\Delta t_j + \Delta t_{j+1}|} \leq c_2 < 2,
$$

(3.5)

and the boundary conditions are given by

$$
s_3'(t_0) = 0, \quad s_1'(t_N) = 0, \quad \text{if } f \in C(\Gamma),
$$

$$
s_3'(t_0) = f'(t_0), \quad s_1'(t_N) = f'(t_N), \quad \text{if } f \in C^1(\Gamma),
$$

$$
s_3''(t_0) = f''(t_0), \quad s_1''(t_N) = f''(t_N), \quad \text{if } f \in C^2(\Gamma).
$$

For $f \in C^m$, $m = 0, 1 \text{ or } 2$, we have

$$
||s_3^{(r)}(t) - f^{(r)}(t)|| \leq c\omega(f^{(m)}, h)h^{m-r}, \quad r = 0, 1, \ldots, m
$$

(3.6)
and
\[ \omega(s_2^{(m)}, x) \leq \omega(f^{(m)}, x), \quad x \geq 0. \] (3.7)

Because of the similarity, we only give their proof under \( m = 2 \).

The cubic spline has the following expression:
\[
s_3(t) = \frac{M_{j-1}}{6 \Delta t_j} (t_j - t)^3 + \frac{M_j}{6 \Delta t_j} (t - t_{j-1})^3 \\
+ \left( \frac{f_{j-1} - M_{j-1} \Delta t_j}{6} \right) (t_j - t) + \left( \frac{f_j - M_j \Delta t_j}{6} \right) (t - t_{j-1}),
\]
\[ t \in \Gamma_j, \quad j = 1, 2, \ldots, N, \] (3.8)

where \( M_j = s_3''(t_j), j = 0, 1, \ldots, N \) which are determined by the equation
\[ AM = D \] (3.9)

with \( M = [M_0, M_1, \ldots, M_N]^T \). Here the matrices \( A \) and \( D \) are given by
\[
A = \begin{bmatrix}
1 & 0 & & \\
\mu_1 & 2 & \lambda_1 & \\
& \ddots & \ddots & \ddots \\
& & \mu_{N-1} & 2 & \lambda_{N-1} \\
& & & 0 & 1
\end{bmatrix}
\]

and
\[
D = [f_0'', 6f[t_0, t_1, t_2], \ldots, 6f[t_{N-2}, t_{N-1}, t_N], f_N'']^T,
\]
respectively, where \( \mu_j = \frac{\Delta t_j}{\Delta t_j + \Delta t_{j+1}} \), \( \lambda_j = 1 - \mu_j \) and \( f[t_{j-1}, t, t_{j+1}] \) is the second divided difference of \( f \) at the points \( t_{j-1}, t, t_{j+1} \).

Let \( C^{N+1} \) be a \( N + 1 \) dimension complex space with maximum norm \( \| \cdot \| \) and \( x = (x_0, x_1, \ldots, x_N)^T \in C^{N+1} \). Suppose \( k \) such that \( \|x| = |x_k| \). We have
\[
\|Ax\| \geq \begin{cases} 
|x_k| & \text{if } k = 0, N, \\
|\mu_k x_{k-1} + 2x_k + \lambda_k x_{k+1}| & \text{if } 1 \leq k \leq N - 1 \\
(2 - c_2)\|x\|.
\end{cases}
\]

It means
\[
\|A^{-1}\| \leq 1/(2 - c_2). \] (3.10)
Let $f'' = [f_0'', f_1'', \ldots, f_N'']^T$. From

$$6f[t_{j-1}, t_j, t_{j+1}] - (\mu_j f''_{j-1} + 2f''_j + \lambda_j f''_{j+1})$$

$$= 6(f[t_{j-1}, t_j, t_{j+1}] - \frac{1}{2}f''_j) + \mu_j (f''_j - f''_{j-1}) + \lambda_j (f''_j - f''_{j+1})$$

and [1]

$$|f[t_{j-1}, t_j, t_{j+1}] - \frac{1}{2}f''_j| \leq c_\omega(f'', |\Delta t_j|),$$

we have

$$||D - A f'|| \leq c_\omega(f'', h).$$

Then by (3.9), (3.10) and (3.12),

$$||M - f'|| = ||A^{-1}(D - A f')|| \leq \frac{c}{2 - c_2} c_\omega(f'', h)$$

or

$$|M_j - f''_j| \leq c_\omega(f'', h)$$

and also

$$|M_k - f''(t)| \leq c_\omega(f'', h), \quad k = j - 1, \quad j, t \in \Gamma_j$$

for $j = 1, 2, \ldots, N$. From (3.8), the second derivative of the cubic spline is

$$s''_2(t) = \frac{t_j - t}{\Delta t_j} M_{j-1} + \frac{t - t_{j-1}}{\Delta t_j} M_j, \quad t \in \Gamma_j, \quad j = 1, 2, \ldots, N$$

and from (3.14) and (3.15), we have

$$|s''_2(t) - f''(t)| = \left| \frac{t_j - t}{\Delta t_j} (M_{j-1} - f''(t)) + \frac{t - t_{j-1}}{\Delta t_j} (M_j - f''(t)) \right|$$

$$\leq \frac{|t_j - t| + |t - t_{j-1}|}{|\Delta t_j|} \max_{k = j-1} \{|M_k - f''(t)|\}$$

$$\leq c_\omega(f'', h), \quad t \in \Gamma_j, \quad j = 1, 2, \ldots, N.$$
and
\[ s_3(t) - f(t) = \int_{t_{j-1}}^{t} [s'_3(\tau) - f'(\tau)] \, d\tau, \quad t \in \Gamma_j, \]
where \( j = 1, 2, \ldots, N. \)

Let \( t, t' \in \Gamma. \) If \( |t - t'| > h, \) then, by (3.6),
\[
|s''_3(t) - s''_3(t')| \leq |s''_3(t) - f''(t)| + |f''(t) - f''(t')| + |f''(t') - s''_3(t')|
\leq c' \omega(f'', h) + \omega(f'', |t - t'|)
\leq c \omega(f'', |t - t'|),
\]
and if \( 0 < |t - t'| \leq h, \) from
\[
|s'''_3(t)| = \frac{|M_j - M_{j-1}|}{|\Delta t_j|}
\leq \frac{1}{|\Delta t_j|} \left( |M_j - f''_j| + |f''_j - f''_{j-1}| + |f''_{j-1} - M_{j-1}| \right)
\leq c \frac{\omega(f'', h)}{|\Delta t_j|}
\leq c \frac{\omega(f'', h)}{h}, \quad t \in \Gamma_j, \quad j = 1, 2, \ldots, N,
\]
where we have used (3.13) and (3.5), it follows
\[
|s''_3(t) - s''_3(t')| \leq \int_{t'}^{t} s'''_3(\tau) \, d\tau
\leq c \frac{|t - t'|}{h} \omega(f'', h)
\leq c \frac{|t - t'|}{h} \left( 1 + \frac{h}{|t - t'|} \right) \omega(f'', |t - t'|)
\leq c \omega(f'', |t - t'|). \quad (3.19)
\]
Hence (3.7) is valid for \( x > 0. \) Of course, (3.7) is valid if \( x = 0, \) and so (3.7) is true. Similar to (3.3), we also have
\[
\omega(s^{(m)}_3 - f^{(m)}, x) \leq c \omega(f^{(m)}(x^\gamma h^{1-\gamma}), \quad x \geq 0, \quad (3.20)
\]
where \( 0 \leq \gamma \leq 1. \) Thus we have proved

**Theorem 3.3.** *For the cubic interpolating spline \( s_3 \) of \( f \in C^m(\Gamma), \) there hold the estimates (3.6) and (3.20) where \( m = 0, 1, \) or 2.*
For $f \in C^{m,\mu}(\Gamma)$, (3.20) becomes

$$\omega(s_3^{(m)} - f^{(m)}, x) \leq c M_{\mu}(f^{(m)}) h^{\mu - v}, \quad 0 \leq v \leq \mu$$  \hspace{1cm} (3.20')

and thus we have

**Corollary 3.4.** If $f \in C^{m,\mu}(\Gamma)$, then

$$\|s_3(f) - f\|_{C^{m,v}} \leq c M_{\mu}(f^{(m)}) h^{\mu - v},$$ \hspace{1cm} (3.21)

where $0 \leq v \leq \mu$.

### 4. COMPLEX SPLINE APPROXIMATION

In this section, we discuss the complex spline approximation for singular integral operators. For convenience, we assume that the operator $A_w$ is bounded on $C^{m,\mu}(\Gamma)$ and let $s(\phi)$ denote the linear or cubic interpolating spline of $\phi \in C^{m,\mu}(\Gamma)$. Here the splines are those defined in the previous section and $m = 0, 1, \text{ or } 2$. In the case of linear splines, $m$ automatically equals 0.

It is effective to use complex splines for the numerical evaluation of Cauchy-type singular integrals. The following theorem is about their error estimates.

**Theorem 4.1.** If $f \in C^{m,\mu}(\Gamma)$, then

$$\|A_w s(f) - A_w(f)\|_{C^k} \leq c M_{\mu}(f^{(m)}) h^{m+\mu-k-1}, \quad 0 \leq k \leq m - 1$$ \hspace{1cm} (4.1)

and

$$\|A_w s(f) - A_w(f)\|_{C^{m,v}} \leq c \|f\|_{C^{m,\mu}} h^{\mu - v}, \quad 0 \leq v \leq \mu.$$ \hspace{1cm} (4.2)

**Proof.** Using Theorem 2.4 and the results in Section 3, we have

$$\|A_w s(f) - A_w(f)\|_{C^k} \leq c \|s(f) - (f)\|_{C^{k+1}}$$

$$\leq c M_{\mu}(f^{(m)}) h^{m+\mu-k-1}, \quad 0 \leq k \leq m - 1$$ \hspace{1cm} (4.3)

and

$$\|A_w s(f) - A_w(f)\|_{C^{m,v}} \leq c \|s(f) - f\|_{C^{m,v}}$$

$$\leq c \|f\|_{C^{m,\mu}} h^{\mu - v}, \quad 0 \leq v \leq \mu.$$ \hspace{1cm} (4.4)

if $f \in C^{m,\mu}(\Gamma)$. The proof is complete.
Now we consider the operator approximation by complex splines. Let

\[ A_{w,A} = \tilde{s}A_w s. \] (4.5)

The spline operators \( \tilde{s} \) and \( s \) may be different. For convenience, we assume \( \tilde{s} = s \) here.

**Theorem 4.2.** If \( \varphi \in C^{m,\mu}(\Gamma) \), then

\[
\| A_{w,A}(\varphi) - A_w(\varphi) \|_{C^{m,\nu}} \leq c\| \varphi \|_{C^{m,\mu}} h^{\mu - \nu}, \quad 0 \leq \nu \leq \mu. \] (4.6)

**Proof.** By Corollary 3.2 or 3.4 and Theorem 2.4, we have

\[
\| A_{w,A}(\varphi) - A_w s(\varphi) \|_{C^{m,\nu}} \leq cM( (A_w s(\varphi))^{(m)}) h^{\mu - \nu}
\]
\[
\leq c\| s(\varphi) \|_{C^{m,\mu}} h^{\mu - \nu}
\]
\[
\leq c\| \varphi \|_{C^{m,\mu}} h^{\mu - \nu} \] (4.7)

and from Theorem 4.1, it follows that

\[
\| A_{w,A}(\varphi) - A_w(\varphi) \|_{C^{m,\nu}}
\]
\[
\leq \| A_{w,A}(\varphi) - A_w s(\varphi) \|_{C^{m,\nu}} + \| A_w s(\varphi) - A_w(\varphi) \|_{C^{m,\nu}}
\]
\[
\leq c\| \varphi \|_{C^{m,\mu}} h^{\mu - \nu}, \quad 0 \leq \nu \leq \mu. \] (4.8)

The theorem has been proved. \( \blacksquare \)

5. APPLICATIONS AND REMARKS

As a direct application of the results obtained in the previous section, we give a scheme for the numerical solution of singular integral equations over open arcs. We know that the Cauchy-type singular integral equation

\[
a(t)\varphi(t) - \frac{b(t)}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau + \lambda \int_{\Gamma} k(t, \tau) \varphi(\tau) d\tau = f(t), \quad t \in \Gamma \] (5.1)

can be regularized as a Fredholm integral equation

\[
(I + \lambda A_w K)y = f^*, \] (5.2)

where \( Ky(t) = \int_{\Gamma} w_1(\tau)k(t, \tau)y(\tau) d\tau \), \( y = \varphi/w_1 \), \( w_1 = [(a^2 - b^2)w]^{-1} \), \( f^* = A_w f + bN_{k-1} \) and \( N_{k-1} \) is a given polynomial with degree \( k - 1 \) \( (N_{k-1} = 0 \) if \( \kappa \leq 0 \) \) for details, see [12]). Here all given functions are Hölder continuous,
\( \lambda \) is a constant, and undetermined function \( \varphi \) is restricted in \( h_0 \), which is a function class whose functions are integrable on \( \Gamma \) and are Hölder continuous on \( \Gamma \) except at the endpoints. Using \( A_{w, \lambda} \) to replace \( A_w \) in (5.2), we have \((I + \lambda A_{w, \lambda} K)y_A = f^*\), and then letting \( u_A = y_A - f^* \), \( g_A = -\lambda A_{w, \lambda} Kf^* \), we get

\[
(I + \lambda A_{w, \lambda} K)u_A = g_A. \tag{5.3}
\]

If \( \lambda \) is not an eigenvalue of (5.2) and \( h \) is small enough, (5.3) has unique solution and the solution is a spline (cf. [9]). Thus, we can obtain the approximate solution of (5.1) by collocation method and the error estimate is easily obtained from Theorem 4.2. The problem will be discussed in detail in another paper.

For \( \mu = 1 \), the constants \( c \) and some other applications, we present some remarks.

**Remark 1.** The assumption \( \mu < 1 \) is only for convenience. Examining the proof of Lemma 2.2, we see that if \( \mu = 1 \) (2.10) is still true but (2.11) is not.

**Remark 2.** The constants \( c \) in theorems of Section 2 depend on \( a, b, m, \mu \) and \( \Gamma \). In Section 3, constants \( c \) depend on \( \Gamma \) and also depend on \( c_1 \) if the splines are cubic. Thus, in Theorem 4.1 and 4.2 the constants are related to \( a, b, m, \mu, \Gamma \) and \( c_1 \), but do not depend on \( v \).

**Remark 3.** Consider the following Cauchy-type integral:

\[
Tf(t) = \int_{\Gamma} \frac{f(\tau)}{\tau - t} d\tau, \tag{5.4}
\]

which is approximated by

\[
T_A f(t) = \int_{\Gamma} \frac{s(f)(\tau)}{\tau - t} d\tau. \tag{5.5}
\]

If \( g(\alpha) = g(\beta) = 0 \), it is easy to prove

\[
|Tg(t)| \leq c \int_0^1 \frac{\omega(g, y)}{y} dy, \quad t \in \Gamma. \tag{5.6}
\]

Thus, by (3.3) or (3.20) we have

\[
|(T - T_A)f(t)| \leq c \int_0^1 \frac{\omega(f - s(f), y)}{y} dy \leq \frac{c}{1 - \gamma} \int_0^\beta \frac{\omega(f, y)}{y} dy,
\]

\[ t \in \Gamma \tag{5.7} \]
where $0 \leqslant \gamma < 1$. This means that if $f$ satisfies Dini condition then $T_A f$ is uniformly convergent to $T f$ on $\Gamma$. We can also obtain

$$\left| (T - T_A) f (t) \right| \leqslant \frac{c}{1 - \gamma} \omega (f^{(m)}; h) h^{m - 1 + \gamma}, \quad t \in \Gamma$$  \hspace{1cm} (5.8)

if $f \in C^m$, $m = 1$ or 2.

**Remark 4.** If function $w$ has integrable singularities at the endpoints of $\Gamma$, such as $w_1$ mentioned above, we can make a transformation with $w$ multiplied by a simple polynomial $s(t) = \frac{b}{\sqrt{1 - t^2}}$, $t \in (-1, 1)$, such that the operator becomes a smooth operator divided by $s(t)$ (cf. [16]). For example, the operator $T_1 f(t) = \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau)}{1 - \tau^2} d\tau, \quad t \in (-1, 1)$ (5.9)
can be transformed into

$$T_1 f(t) = \frac{1}{\sigma(t)} \left( \frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1 - \tau^2}} f(\tau) d\tau + \frac{1}{\pi} \int_{-1}^1 \frac{\tau + t}{\sqrt{1 - \tau^2}} f(\tau) d\tau \right),$$  \hspace{1cm} (5.10)

where $\sigma(t) = 1 - t^2$.

6. **PROOF OF LEMMA 2.2**

**Proof.** Equations (2.10) and (2.11) are equivalent to

$$\left| S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2) \right| \leqslant c M_{\mu}(f^{(m)}) \delta^\mu (1 + |\ln \delta|)$$  \hspace{1cm} (6.1)

or

$$\left| S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2) \right| \leqslant c \left( |u| + \int_0^1 \frac{\omega(u,y)}{y} dy \right) M_{\mu}(f^{(m)}) \delta^\mu, \quad (6.2)$$

if $u$ satisfies Dini condition and $u(\alpha) = u(\beta) = 0$, where $t_1, t_2 \in \Gamma$ and $|t_2 - t_1| = \delta$. Now we prove (6.1) and (6.2).

For convenience, suppose $0 < \delta < 1$ and $\alpha < t_1 < t_2 < \beta$.

(i) If $|t_1 - \alpha| > \delta$ and $|\beta - t_2| \leqslant \delta$, let

$$\left| S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2) \right| = m! \left| \int_{\alpha t_1}^{\alpha t_2} + \int_{t_1 \beta}^{t_2 \beta} u(\tau) \left[ \frac{f(\tau) - T_m(f)(\tau, t_2)}{(\tau - t_2)^{m+1}} - \frac{f(\tau) - T_m(f)(\tau, t_1)}{(\tau - t_1)^{m+1}} \right] d\tau \right|$$

$$= |I_1 + I_2|.$$
From (2.5) and \( |t_1 \beta| = |t_1 t_2| + |t_2 \beta| \leq 2c_0 \delta \), it follows that
\[
|I_2| = m! \left| \int_{t_1 \beta} u(\tau) \left[ \frac{f(\tau) - T_m(f)(\tau, t_2)}{(\tau - t_2)^{m+1}} - \frac{f(\tau) - T_m(f)(\tau, t_1)}{(\tau - t_1)^{m+1}} \right] d\tau \right|
\leq c |u| ||M_\mu(f^{(m)})|| \int_{t_1 \beta} (|\tau - t_2|^{m+1} + |\tau - t_1|^{m+1}) |d\tau|
\leq c |u| ||M_\mu(f^{(m)})|| \delta^m. \tag{6.3}
\]

Define the function
\[
h(\tau) = T_m(f)(\tau, t_1) - T_m(f)(\tau, t_2), \quad \tau \in \Gamma. \tag{6.4}
\]

Then
\[
h(\tau) = h(t_2) + h'(t_2)(\tau - t_2) + \cdots + \frac{h^{(m)}(t_2)}{m!}(\tau - t_2)^m \tag{6.5}
\]
because \( h(\tau) \) is an \( m \)-degree polynomial of \( \tau \). Noting that
\[
h^{(k)}(t_2) = T_{m-k}(f^{(k)})(t_2, t_1) - f^{(k)}(t_2) \tag{6.6}
\]
and (2.5), we have
\[
\frac{m!}{k!} |h^{(k)}(t_2)| \leq c \binom{m}{k} M_\mu(f^{(m)}) \delta^{m+k} \tag{6.7}
\]
for \( k = 1, 2, \ldots, m \), therefore
\[
i_1 \equiv m! \left| \int_{t_1 \beta} u(\tau) \frac{T_m(f)(\tau, t_2) - T_m(f)(\tau, t_1)}{(\tau - t_2)^{m+1}} d\tau \right|
\leq m! \sum_{k=0}^{m} \frac{1}{k!} \int_{t_1 \beta} \frac{|u(\tau)|}{|\tau - t_2|^{m-k+1}} |d\tau|
\leq c M_\mu(f^{(m)}) \sum_{k=0}^{m} \binom{m}{k} \delta^{m-k} \int_{t_1 \beta} \frac{|u(\tau)|}{|\tau - t_2|^{m-k+1}} |d\tau|
\leq c M_\mu(f^{(m)}) ||u|| \delta^m \left( 1 + \ln \frac{1}{\delta} \right). \tag{6.8}
\]
If \( u \) satisfies Dini condition and \( u(\beta) = 0 \),
\[
\int_{\tau_1}^{\tau_2} \frac{|u(\tau)|}{|\tau - \tau_2|} d\tau = \int_{\tau_1}^{\tau_2} \frac{|u(\tau) - u(\beta)|}{|\tau - \tau_2|} d\tau \leq \int_{\tau_1}^{\tau_2} \omega(u, |\tau - \beta|) d\tau.
\]

For \( \tau \in (\tau_1, \tau_2) \),
\[
c_0|\tau - \tau_2| \geq |\tau_1| = |t_1| \geq |t_2 - t_1| = \delta,
\]
but \( |\beta - \tau| \leq \delta \), thus \( |\beta - \tau| \leq c_0|\tau - \tau_2| \) and
\[
|\tau - \beta| \leq |\tau - \tau_2| + |t_2 - \beta| \leq (1 + c_0)|\tau - \tau_2|.
\]
So we have
\[
\int_{\tau_1}^{\tau_2} \frac{|u(\tau)|}{|\tau - \tau_2|} d\tau \leq \int_{\tau_1}^{\tau_2} \frac{\omega(u, (1 + c_0)|\tau - \tau_2|)}{|\tau - \tau_2|} d\tau \leq c \int_0^1 \frac{\omega(u, y)}{y} dy \quad (6.11)
\]
and
\[
i_1 \leq cM_\mu(f^{(m)})(||u|| + \int_0^1 \frac{\omega(u, y)}{y} dy) \delta^\mu. \quad (6.12)
\]
By (2.5),
\[
i_2 = m! \left| \int_{\tau_1}^{\tau_2} u(\tau)[f(\tau) - T_m(f)(\tau, t_1)] \left[ \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right] d\tau \right|
\leq c||u||M_\mu(f^{(m)}) \int_{\tau_1}^{\tau_2} \frac{|\tau - t_1|^{m+\mu}|(\tau - t_1)^{m+1} - (\tau - t_2)^{m+1}|}{|\tau - t_1|^{m+1}|\tau - t_2|^{m+1}} d\tau
\leq c||u||M_\mu(f^{(m)}) |t_1 - t_2| \sum_{k=0}^m \int_{\tau_1}^{\tau_2} \frac{|\tau - t_1|^{k+\mu - 1}}{|\tau - t_2|^{k+1}}|d\tau|. \quad (6.13)
\]
Let \( l_1 = |\tau_1|, \ l_2 = |\tau_2| \) and \( s \) be the parameter of arc length. Using integration by parts and \( l_1 < l_2 \),
\[
\int_{\tau_1}^{\tau_2} \left| \frac{|\tau - t_1|^{k+\mu - 1}}{|\tau - t_2|^{k+1}} \right| |d\tau| \leq c \int_0^{l_1} \frac{|\tau t_1|^{k+\mu - 1}}{|\tau t_2|^{k+1}} |ds| = c \int_0^{l_1} \frac{(l_1 - s)^{k+\mu - 1}}{(l_2 - s)^{k+1}} |ds|
\leq c \left[ \frac{k + \mu}{k + \mu + 1} \right] \int_0^{l_1} \frac{(l_1 - s)^{k+\mu}}{(l_2 - s)^{k+2}} |ds| - \frac{1}{k + \mu l_2^{k+1}}
\]
\[
\leq c \frac{k+1}{k+\mu} \int_0^{l_1} \frac{(l_2-s)^{k+\mu}}{(l_2-s)^{k+2}} \, ds
\]

\[
\leq c \frac{k+1}{k+\mu} \frac{1}{1-\mu} \left[ \frac{1}{(l_2-l_1)^{1-\mu}} - \frac{1}{l_2^{1-\mu}} \right],
\]

but \( l_2 - l_1 = |t_1 t_2| > \delta \), then

\[
\int_{x t_1} \left| \frac{\tau - t_1}{\tau - t_2} \right|^{k+\mu-1} d\tau \leq c \frac{k+1}{k+\mu} \frac{1}{1-\mu} \frac{1}{(l_2-l_1)^{1-\mu}} \leq c \delta^{\mu-1}
\]  

for \( k = 0, 1, \ldots, m \), so we obtain

\[
i_2 \leq c M_\mu(f^{(m)}) \delta^\mu.
\]

Thus from

\[
I_1 = m! \int_{x t_1} u(\tau) \left[ \frac{f(\tau) - T_m(f)(\tau, t_1)}{(\tau - t_2)^{m+1}} - \frac{f(\tau) - T_m(f)(\tau, t_1)}{(\tau - t_1)^{m+1}} \right] d\tau
\]

\[
= m! \int_{x t_1} u(\tau) \left\{ \frac{T_m(f)(\tau, t_1) - T_m(f)(\tau, t_2)}{(\tau - t_2)^{m+1}} + \left[ f(\tau) - T_m(f)(\tau, t_1) \right]\left( \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right) \right\} d\tau,
\]

\[
|I_1| \leq i_1 + i_2
\]

\[
\leq \begin{cases} 
  c M_\mu(f^{(m)}) ||u|| \delta^\mu (1 + |\ln \delta|) & \text{if } u(\beta) \neq 0, \\
  c M_\mu(f^{(m)}) \left( ||u|| + \int_0^1 \frac{\omega(u, y)}{y} \, dy \right) \delta^\mu & \text{if } u(\beta) = 0 
\end{cases}
\]

and together with (6.3), we obtain

\[
|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)| \leq |I_1| + |I_2|
\]

\[
\leq \begin{cases} 
  c M_\mu(f^{(m)}) ||u|| \delta^\mu (1 + |\ln \delta|) & \text{if } u(\beta) \neq 0, \\
  c M_\mu(f^{(m)}) \left( ||u|| + \int_0^1 \frac{\omega(u, y)}{y} \, dy \right) \delta^\mu & \text{if } u(\beta) = 0 
\end{cases}
\]

and there follows (6.1) and (6.2).
(ii) If \(|t_1 - x| \leq \delta\) and \(|\beta - t_2| > \delta\), similarly, we also have (6.18) but \(x\) in place of \(\beta\). Hence, (6.1) and (6.2) are valid.

(iii) If \(|t_1 - x| > \delta\) and \(|\beta - t_2| > \delta\), let

\[
|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)|
= m! \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} u(\tau) \left( \frac{f(\tau) - T_m(f)(\tau, t_1)}{(\tau - t_1)^{m+1}} - \frac{f(\tau) - T_m(f)(\tau, t_2)}{(\tau - t_2)^{m+1}} \right) d\tau \right|
= |I_1 + I_2 + I_3|.
\]  

(6.19)

Similar to the proof of (6.3),

\[
|I_2| \leq c M_\mu(f^{(m)}) \delta^\nu.
\]  

(6.20)

We rewrite \(I_1 + I_3\) as \(i_1 + i_2\) where

\[
i_1 = m! \int_{\mathbb{R}} u(\tau) \frac{T_m(f)(\tau, t_1) - T_m(f)(\tau, t_2)}{(\tau - t_2)^{m+1}} d\tau
\]

+ \(m! \int_{\mathbb{R}} u(\tau) \frac{T_m(f)(\tau, t_1) - T_m(f)(\tau, t_2)}{(\tau - t_1)^{m+1}} d\tau,
\]

(6.21)

\[
i_2 = m! \int_{\mathbb{R}} u(\tau) \left[ f(\tau) - T_m(f)(\tau, t_1) \right] \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} d\tau
\]

+ \(m! \int_{\mathbb{R}} u(\tau) \left[ f(\tau) - T_m(f)(\tau, t_2) \right] \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} d\tau.
\]

(6.22)

Using (6.4) and (6.5), we also rewrite \(i_1\) as

\[
m! \sum_{k=0}^{m-1} \left[ \int_{\mathbb{R}} \frac{u(\tau)}{(\tau - t_2)^{m-k+1}} d\tau + \int_{\mathbb{R}} \frac{u(\tau)}{(\tau - t_1)^{m-k+1}} d\tau \right]
\]

+ \(\int_{\mathbb{R}} \frac{u(\tau)}{\tau - t_2} d\tau + \int_{\mathbb{R}} \frac{u(\tau)}{\tau - t_1} d\tau \) \[f^{(m)}(t_1) - f^{(m)}(t_2)]

\[= i_{11} + i_{12}.
\]

(6.23)
Similar to the proof of (6.8) and (6.15), we have

\[ |i_{11}| \leq c |u| |M_\mu(f^{(m)})\delta^\mu, \quad (6.24) \]

\[ |i_{12}| \leq c |u| |M_\mu(f^{(m)})\delta^\mu \left(1 + \ln \frac{1}{\delta}\right) \quad (6.25) \]

and

\[ |i_2| \leq c |u| |M_\mu(f^{(m)})\delta^\mu. \quad (6.26) \]

If \( u \) satisfies Dini condition and \( u(a) = u(\beta) = 0 \), we have

\[
\left| \int_{\tau_1}^{\tau} \frac{u(\tau)}{\tau - t_2} d\tau + \int_{\tau_2}^{\tau} \frac{u(\tau)}{\tau - t_1} d\tau \right|
\]

\[ = \left| \int_{\tau_1}^{\tau} \frac{u(\tau) - u(t_2)}{\tau - t_2} d\tau + \int_{\tau_2}^{\tau} \frac{u(\tau) - u(t_1)}{\tau - t_1} d\tau \right|
\]

\[ + |u(t_2)| \int_{\tau_1}^{\tau} \frac{1}{\tau - t_2} d\tau + |u(t_1)| \int_{\tau_2}^{\tau} \frac{1}{\tau - t_1} d\tau
\]

\[ \leq \int_{\tau_1}^{\tau} \frac{\omega(u, |\tau - t_2|)}{|\tau - t_2|} d\tau + \int_{\tau_2}^{\tau} \frac{\omega(u, |\tau - t_1|)}{|\tau - t_1|} d\tau
\]

\[ + |u(t_2)| \left| \ln \frac{|t_1 - t_2|}{|a - t_2|} + i\theta_1 \right| + |u(t_1)| \left| \ln \frac{|\beta - t_1|}{|t_1 - t_2|} + i\theta_2 \right|, \quad (6.27) \]

where \( \theta_1 \) is the angle between \( t_1 - t_2 \) and \( a - t_2 \) and \( \theta_2 \) is the angle between \( \beta - t_1 \) and \( t_2 - t_1 \), and

\[ |u(t_2)| \left| \ln \frac{|t_1 - t_2|}{|a - t_2|} + i\theta_1 \right| + |u(t_1)| \left| \ln \frac{|\beta - t_1|}{|t_1 - t_2|} + i\theta_2 \right|
\]

\[ = |[u(t_2) - u(t_1)] \ln |t_2 - t_1| - [u(t_2) - u(a)] \ln |a - t_2|
\]

\[ + [u(t_1) - u(\beta)] \ln |\beta - t_1| + i[\theta_1 u(t_2) + \theta_2 u(t_1)]
\]

\[ \leq c \left( |u| + \sup_{0 < x \leq 1} \omega(u, x) \ln \frac{1}{x} \right), \quad (6.28) \]
therefore
\[
\left| \int_{\tau_1}^{\tau_2} \frac{u(\tau)}{\tau - \tau_1} d\tau + \int_{\tau_1}^{\beta} \frac{u(\tau)}{\tau - \tau_1} d\tau \right| \leq c \left( ||u|| + \int_0^1 \frac{\omega(u, y)}{y} dy \right), \quad (6.29)
\]
so that
\[
|i_{12}| \leq c M_\mu(f^{(m)}) \left( ||u|| + \int_0^1 \frac{\omega(u, y)}{y} dy \right) \delta^\mu. \quad (6.30)
\]
Now we obtain
\[
|i_1| \leq |i_{11}| + |i_{12}|
\]
\[
\leq \begin{cases} 
  c M_\mu(f^{(m)}) \left( ||u|| + \int_0^1 \frac{\omega(u, y)}{y} dy \right) \delta^\mu & \text{if } u(x) = u(\beta) = 0, \\
  c M_\mu(f^{(m)}) ||u|| \delta^\mu \left( 1 + \ln \frac{1}{\delta} \right) & \text{otherwise}
\end{cases} \quad (6.31)
\]
and together with (6.20) and (6.26), (6.1) and (6.2) are valid in this case.

(iv) If $|t_1 - x| \leq \delta$ and $|\beta - t_2| \leq \delta$, then $|\Gamma| = |zt_1| + |t_1 t_2| + |t_2 \beta| \leq 3c_0 \delta$. From (2.5),
\[
|S_{u,m}(f)(t_1) - S_{u,m}(f)(t_2)|
\]
\[
\leq c ||u|| M_\mu(f^{(m)}) \int_\Gamma (|\tau - t_1|^{-1+\mu} + |\tau - t_2|^{-1+\mu}) d\tau
\]
\[
\leq c ||u|| M_\mu(f^{(m)}) |\Gamma|^\mu
\]
\[
\leq 3c_0 c ||u|| M_\mu(f^{(m)}) \delta^\mu \quad (6.32)
\]
and then (6.1) and (6.2) are valid.

Now we have proved that (6.1) and (6.2) are true in each case. The proof is complete.

ACKNOWLEDGMENTS

The author is grateful to the editor and the reviewers for their valuable suggestions. The author also thanks Professor Wei Lin, Dr. John H Easton and Dr. Xiao Bin Liu for their generous support and assistance.

REFERENCES