Performance Analysis of Beamforming in Two Hop Amplify and Forward Relay Networks

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Abstract—The performance of beamforming in a two hop Amplify and Forward (AF) relay network is analyzed. This network consists of a single relay which is used to amplify and forward the signal from the source to the destination. The source and destination are both equipped with multiple antennas while the relay has a single antenna. In this paper, we derive closed form expressions for the outage probability and probability density function of the received SNR. We also present exact symbol error rate expressions for the two hop AF relay network and show that full spatial diversity order, which corresponds to the minimum number of antennas at the source and destination, can be achieved. Our analytical results are confirmed through comparison with Monte Carlo simulations.

I. INTRODUCTION

Multiple-Input Multiple-Output (MIMO) systems offer performance benefits over single antenna systems as they can provide higher data rates and the ability to combat fading by providing spatial diversity [1]. When channel state information (CSI) is available at the source and destination, beamforming schemes are implemented by maximum-ratio-transmission (MRT) [2] or maximum-ratio-combining (MRC) [3] at the transmitter and receiver respectively. Beamforming has been well studied in literature because it maximizes the received signal-to-noise ratio (SNR) [4].

Recently, relay transmission has also been shown to achieve spatial diversity through node cooperation [5, 6]. They also provide the benefit of extending coverage without requiring large transmitter powers. The two most common relaying protocols are Decode and Forward (DF) and Amplify and Forward (AF). AF relay protocols are simple schemes, which amplifies the signal transmitted from the source and forwards it to the destination [7–9]. AF techniques may use knowledge of the source to relay channel to assist in signal amplification [7]. We will refer to this as channel-assisted AF (CA-AF). AF techniques may also use knowledge of the statistics of the noise plus knowledge of the source to relay channel to assist in the amplification of the signal [9]. We will refer to this as channel-noise-assisted AF (CNA-AF). When only one relay is used to forward the data, this is referred to as a two hop network.

Two hop networks using CA-AF have been studied in Rayleigh fading environments when there are different transmit SNR at the source and relay [10, 11]. This has been extended to Nakagami-m fading environments using CA-AF in [10], when the transmit SNR at the source and relay are the same, and in [12] when the transmit SNR are different. However, the derived lower bounds for the bit error performance in [12] were shown to be quite loose, especially at high SNR. In addition, the transmitter, receiver and relay were constrained to single antennas, and only CA-AF systems were considered.

In this paper, we consider two hop AF relay networks where multiple antennas at the source and destination are used for beamforming in two hop networks using CA-AF and CNA-AF. We first derive closed form statistics for the receive SNR of the proposed system for both CNA-AF and CA-AF. We then present new exact expressions for the average symbol error rate (SER) for a variety of modulation formats using CA-AF. To gain insights, we then present the diversity order and array gain for both CNA-AF and CA-AF systems. We are able to prove that the proposed system achieves the maximum possible spatial diversity order, which is equal to the minimum number of antennas at the source and destination. The results are confirmed through comparison with Monte Carlo simulations.

The rest of the paper is organized as follows. In Section II, the two hop beamforming AF network system model is described. We then present new closed form expressions for the SNR probability density function (p.d.f.) and outage probability and SER in Section III. We then proceed by verifying our SINR distribution results and SER expressions in Section IV.

II. TWO HOP BEAMFORMING AF NETWORK SYSTEM MODEL

Consider a wireless communications system where the source with \( N_t \) antennas is communicating with the destination with \( N_r \) antennas through a relay with a single antenna. In this two-hop system, we assume that the source does not have a direct link to the destination. The received scalar signal at the relay can be written as

\[
y_R = h_1^\dagger w_t x + n_R
\]

where \( h_1 \) is the \( N_t \times 1 \) channel vector from the source to the relay with Rayleigh fading entries, \( w_t \) is the \( N_t \times 1 \) transmit weight vector, \( x \) is the transmitted scalar symbol with zero mean and unit variance, \( n_R \) is additive white gaussian noise (AWGN) satisfying \( E[|n_R|^2] = \sigma^2 \) and \( (\cdot)^\dagger \) denotes conjugate transpose. According to the principles of MRT [2], we choose

\[
w_t = \frac{h_1}{\| h_1 \|_F} \Rightarrow \| \cdot \|_F \text{ is the Frobenius norm.}
\]

This results in the following equivalent expression

\[
y_R = \sqrt{\lambda} h_1^\dagger x + n_R
\]
where \( \lambda(z) = |z|^2 \). The received scalar signal at the relay is then multiplied by a gain \( G \), and transmitted to the destination. The \( N_r \times 1 \) received vector signal at the destination can be written as

\[
y_D = h_2 G(\sqrt{\lambda(h_1)} x + n_R) + n_D
\]

where \( h_2 \) is the \( N_r \times 1 \) channel matrix with Rayleigh fading entries from the relay to the destination and \( n_D \) is the \( N_r \times 1 \) additive white gaussian noise (AGWN) vector satisfying \( E[n_D n_D^\dagger] = \mathbf{I}_{N_r} \sigma^2 \). We then multiply the received signal \( y_D \) by a \( 1 \times N_r \) receive weight vector \( \mathbf{w}_D^\dagger \) to obtain

\[
r_D = \mathbf{w}_D^\dagger y_D = \mathbf{w}_D^\dagger h_2 G(\sqrt{\lambda(h_1)} x + n_R) + \mathbf{w}_D^\dagger n_D.
\]

According to the principles of MRC [3], we choose \( n_D = \frac{h_2}{\|h_2\|_F} r \). This results in the following equivalent expression

\[
r_D = \sqrt{\lambda(h_1)} G(\sqrt{\lambda(h_1)} x + n_R) + \mathbf{w}_D^\dagger n_D.
\]

The resulting SNR at the destination after algebraic manipulation can be written as

\[
\gamma_{eq} = \frac{\lambda(h_1) \lambda(h_2)}{\lambda(h_1) + \sigma^2}.
\]

One choice for the gain \( G \) is given by [9]

\[
G^2_{CNA-AF} = \frac{1}{\lambda(h_1) + \sigma^2}.
\]

which has the effect of limiting the output power of the relay if \( \lambda(h_1) \) is low. We will refer to this as CNA-AF. Substituting (7) into (6) gives

\[
\gamma_{eq1} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 + 1}.
\]

where \( \gamma_i = \frac{\lambda(h_i)}{\sigma^2} \) for \( i = 1, 2 \). When the relay has no knowledge of the noise statistics, we can choose the gain \( G \) as [7]

\[
G^2_{CA-AF} = \frac{1}{\lambda(h_1)}.
\]

This has the effect of inverting the channel and we will refer to this scheme as CA-AF. Substituting (9) into (6) gives

\[
\gamma_{eq2} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}.
\]

A. Statistical characterization of the Received SNR

We first obtain closed form expressions for the outage probability and p.d.f. of \( \gamma_{eq1} \) and \( \gamma_{eq2} \) defined in (8) and (10) respectively. The outage probability is an important quality of service measure defined as the probability that the received SNR drops below an acceptable SNR threshold \( \gamma_{th} \). The c.d.f. of \( \gamma_{eq1} \) and \( \gamma_{eq2} \) are given by the following theorem.

**Theorem 1:** The c.d.f. of \( \gamma_{eq1} \) and \( \gamma_{eq2} \) are given by substituting \( c = 1 \) and \( c = 0 \) respectively, into

\[
F_{\gamma_{eq1}}(\gamma_{th}) = 1 - \frac{2e^{-\gamma_{th}}}{(m_2 - 1)\gamma_{th}^{m_2 - \frac{1}{2}}} \sqrt{\frac{\gamma_{th}}{\gamma_{th}}}
\]

\[
\times \sum_{p=0}^{m_1 - 1} \left(\frac{\gamma_{th}^2 + \gamma_{th}}{\gamma_1 \gamma_2}\right)^{\frac{1}{2}} \sum_{j=0}^{p} \left(\frac{\gamma_{2\gamma_{th}}}{\gamma_1 (\gamma_1 + c)}\right)^{\frac{1}{2}}
\]

\[
\times \sum_{q=0}^{m_2 - 1} \left(\frac{\gamma_{2\gamma_{th}}}{\gamma_1 \gamma_{th}}\right)^{\frac{1}{2}} K_{q+j-p+1} \left(\frac{\gamma_{2\gamma_{th}}}{\gamma_1 \gamma_{th}}\right).
\]

**Proof:** The proof follows by substituting the above expressions into (11).

**Corollary 1:** The p.d.f. of \( \gamma_{eq1} \) and \( \gamma_{eq2} \) are given by substituting \( c = 1 \) and \( c = 0 \) respectively, into

\[
f_{\gamma_{eq1}}(\gamma) = \frac{2e^{-\gamma} e^{-\gamma}}{N_r - 1} \frac{\gamma^{\frac{1}{2} - 2\gamma}}{\sqrt{\gamma + \gamma_2}} \left(\gamma_1 \gamma_2\right)^{\frac{1}{2}}
\]

\[
\times \sum_{j=0}^{p} \left(\frac{\gamma_2 \gamma_2}{\gamma_1 (\gamma_1 + 1)}\right)^{\frac{1}{2}} \sum_{q=0}^{N_r - 1} \left(\frac{\gamma_2 (\gamma + 1)}{\gamma_1 \gamma}\right)^{\frac{1}{2}}
\]

\[
\left(\frac{K_{1+j-p+q}}{\gamma_1 \gamma_2}\right) \left(\gamma_1 \gamma_2 \right)^{\frac{1}{2}}
\]

\[
\gamma((1 + j + N_r + q) \gamma_1 \gamma_2 - (\gamma_1 + \gamma_2) \gamma)
\]

\[
- \left(\gamma + 2\gamma\right) \left(\gamma(1 + \gamma)\right)^{\frac{1}{2}} K_{2+j-p+q} \left(2 \gamma(1 + \gamma)^{\frac{1}{2}}\right).
\]

**Proof:** The proof follows by differentiating (11) w.r.t. \( \gamma_{th} \), and substituting \( \gamma_{th} = \gamma \).

B. Symbol Error Rate Analysis

In this subsection, we derive closed form expressions for the average symbol rate of the two hop CA-AF scheme with various modulation formats. We consider this scheme instead of CNA-AF for mathematical tractability and also because it provides a tight upper bound for CA-AF in the high SNR regime. Our results apply for all general modulation formats that have a SER expression of the form

\[
P_s = E[a Q(\sqrt{2\gamma})]
\]

where \( Q(\cdot) \) is the Gaussian-Q function and \( a \) and \( b \) are modulation specific constants. Such modulation formats include...
BPSK \((a = 1, b = 1)\), BFSK with orthogonal signalling \((a = 1, b = 0.5)\) or minimum correlation \((a = 1, b = 0.715)\) and \(M\)-ary PAM \((a = 2(M - 1)/M, b = 3/(M^2 - 1))\) [3]. Our new results also provide the approximation SER to other modulation formats for which (13) is an approximation, e.g. \(M\)-PSK \((a = 2, b = \sin^2(\pi/M))\).

The SER of the two hop CA-AF scheme is given as

\[
P_s = \frac{a \sqrt{b}}{2} \left( \frac{1}{\sqrt{b}} - \frac{2}{\Gamma(N_r)} \sum_{k=0}^{N_t-1} \frac{1}{k!} \sum_{j=0}^{N_r+k-1} \left( \frac{N_r + k - 1}{j} \right) \Gamma(N_r + 2k - j - \frac{1}{2}) \Gamma(N_r + k + 1) \right)
\]

\[
4^{k-j-1} \gamma_2^{j+1-N_r-k} \Gamma(N_r + 2k - j - \frac{1}{2}) \Gamma(N_r + k + 1) \left( \frac{b + 1}{\gamma_1} + \frac{1}{\gamma_2} \right) \frac{2}{\sqrt{\gamma_1 \gamma_2}} \Gamma(N_r + 2k - j - \frac{1}{2}) \Gamma(N_r + k + 1)
\]

where \(\gamma_1 = N_r + 2k - j - \frac{1}{2}, \gamma_2 = k - j - \frac{1}{2}\) and \(2F_1(\cdot; \cdot; \cdot)\) is the Gauss’ hypergeometric function defined in [13].

To prove this, we begin by rewriting the SER expression given in (13) directly in terms of the c.d.f. of the output SNR as follows [14]

\[
P_s = \frac{a \sqrt{b}}{2\pi} \int_0^\infty e^{-bu} \sqrt{u} F_\gamma(u) du.
\]

We then proceed by substituting (11) with \(c = 0\) into (15), and solving the resultant integral using [13, Eq. 6.621.3] to obtain the desired result. Although (14) may appear complicated, it essentially involves finite summations of functions which can be efficiently evaluated.

C. High SNR SER Analysis

To gain further insights, we now consider the SER at high SNR. In the high SNR regime, the performance of CAN-AF approaches the performance of CA-AF. We thus only consider the SER of the CA-AF system. In the high SNR regime, the key factors governing system performance are the diversity order and array gain. We will now present closed form expressions for these factors.

We use a general SISO SER result from [15] in which the SER can be approximated in the high SNR regime by considering a first order expansion of the p.d.f. of \(\gamma_{\text{end}}\). We omit the proof of the first order expansion due to space limitations. The high SNR SER expression given in terms of the array gain \(G_a\) and diversity order \(G_d\) can be written as

\[
P_s^\infty = \frac{G_a}{\gamma_1^{G_d / 2}} G_d + \frac{1}{2}
\]

where

\[
G_a = \frac{a \sqrt{b}}{2\pi} \sqrt{G_d(2b)^{G_d}}
\]

Note that (16) contains the diversity order expressed in terms of \(\gamma_1\). We can obtain an equivalent expression in terms of \(\gamma_2\) by substituting \(\gamma_1 = \gamma_2\) and \(\gamma_2 = \gamma_1\) in (16), (17) and (18).

We can obtain an upper bound to the diversity order by applying the min-max cut from the source to the relay, and from the relay to the source. This gives the upper bound to be the minimum of the number of antennas at the transmitter and receiver, since there is only one antenna at the relay. We have thus proved that applying beamforming at the transmitter and receiver can achieve the maximum diversity order possible in a two hop relay system.

IV. NUMERICAL RESULTS

In this section, we confirm our analytical results through comparison with Monte Carlo simulations.

Fig. 1 shows the c.d.f. of the received SNR using CAN-AF for various antenna configurations. The ’Analytical’ curves from (11) with \(c = 1\) clearly agree with the Monte Carlo simulated curves. We see that the outage probability is significantly improved as the number of antenna increases.

Fig. 2 shows the p.d.f. of the received SNR using CAN-AF for various antenna configurations. The ’Analytical’ curves from (12) with \(c = 1\) clearly agree with the Monte Carlo simulated curves.

Fig. 3 shows the SER using CA-AF with BPSK modulation for various antenna configurations. The ’Analytical’ curves are from (14) with \(a = 1\) and \(b = 1\), and match the Monte Carlo simulated curves. The ’Analytical (High SNR)’ curves are from (16). These curves clearly converge to the exact SER in the high SNR regime, confirming that the maximum diversity order is possible. In addition, the diversity order is clearly the same for the two systems with antenna configuration \(N_t = 3, N_r = 3\) and \(N_t = 4, N_r = 3\). This confirms that the diversity order of the two hop beamforming system is determined by the minimum number of antennas at the source and the destination.

Fig. 4 and 5 shows the SER using CA-AF with 4-PAM and QPSK modulation respectively for various antenna configurations. The ’Analytical’ curves are from (14) with \(a = 1.5\)
and $b = 0.2$ for 4-PAM and $a = 2$ and $b = 0.5$ for QPSK modulation. Again, we see an exact match with the Monte Carlo simulated curves. The 'Analytical (High SNR)' curves are from (16).

V. CONCLUSION

We analyze the performance of beamforming in two hop AF relay networks in Rayleigh fading environments. We first derive new closed form expressions for the statistics of the SNR at the receiver. We then derive exact SER expressions for arbitrary SNR, and the array gain and diversity order at high SNR. Our results indicate that the diversity order is equal to the minimum number of antennas at the source and destination. Our results are confirmed through comparison with Monte Carlo simulations.
We start with the c.d.f. and p.d.f. of $\gamma_1$, given respectively by

$$F_{\gamma_1}(\gamma_1) = 1 - e^{-\frac{\gamma_1}{N_1}} \sum_{p=0}^{N_1-1} \frac{\gamma_1^p}{p!} \quad (21)$$

and

$$f_{\gamma_1}(\gamma_1) = \frac{\gamma_1^{N_1-1} e^{-\frac{\gamma_1}{N_1}}}{(N_1-1)!} \frac{1}{\gamma_1} \quad (22)$$

Note that (8) and (10) can be written in a general form given by

$$\gamma_{eq} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 + c} \quad (23)$$

for $c = 1$ and $c = 0$ respectively. We focus on deriving the c.d.f. of $\gamma_{eq}$.\hfill

Now the c.d.f. of (23) can be written as

$$\Pr(\gamma_{eq} < \gamma_{th}) = \Pr \left( \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 + c} < \gamma_{th} \right)$$

$$= \int_0^{\gamma_{th}} \Pr \left( \frac{\gamma y}{\gamma_1 + y + c} < \gamma_{th} \right) f_{\gamma_1}(y)dy$$

$$= \int_0^{\gamma_{th}} \Pr \left( \frac{\gamma y}{\gamma_1 + y + c} < \gamma_{th} \right) f_{\gamma_1}(y)dy$$

$$= \int_0^{\gamma_{th}} \left[ 1 - \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy \right] \frac{f_{\gamma_1}(y)}{\gamma_1} dy$$

$$= 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

$$= 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

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$$= 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

Using the identity $F_X(x) = 1 - F_X(x)$, we substitute the c.d.f. of $\gamma_1$ given in (21) and p.d.f. of $\gamma_2$ given in (22) with $\gamma_1 = \gamma_2$ and $N_1 = N_2$ to the last line of (24), which gives

$$F_{\gamma_{eq}}(\gamma_{th}) = 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

$$= 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

$$= 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

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$$= 1 - \int_0^{\gamma_{th}} \frac{f_{\gamma_1}(y)}{\gamma_1} \int_{\gamma_{th}}^{\gamma_{2th}} \frac{f_{\gamma_2}(y)}{\gamma_{2th}} dy$$

After some algebraic manipulation, we obtain

$$F_{\gamma_{eq}}(\gamma_{th}) = 1 - \frac{e^{-\gamma_{th}\left(\frac{1}{N_1} + \frac{1}{N_2}\right)}}{(N_1-1)!\gamma_1^N} \sum_{p=0}^{N_1-1} \frac{\gamma_{th}^p}{p!} \mathcal{I}_0$$

where

$$\mathcal{I}_0 = \int_0^{\gamma_{th}} e^{-\gamma_{th}(w+c)} \frac{1}{\gamma_{th}} \frac{N_1-1}{(N_1-1)!\gamma_1^N} \gamma_{th}^p w^{-p} dw \quad (27)$$

We then apply the binomial theorem in (27) to obtain

$$\mathcal{I}_0 = \sum_{j=0}^{p} \sum_{q=0}^{N_r-1} \frac{p^j}{q} (N_r-1-q) \mathcal{I}_1$$

where

$$\mathcal{I}_1 = \int_0^{\gamma_{th}} e^{-\gamma_{th}(w+c)} \frac{1}{\gamma_{th}} \frac{N_1-1}{(N_1-1)!\gamma_1^N} \gamma_{th}^p w^{-p} dw \quad (29)$$

We solve the integral in (29) using [16, Eq. 3.471.9] and proceed by substituting the resultant expression into (28) and then into (26), yielding the desired result in (11).