The effect of beach slope on the tsunami run-up induced by thrust fault earthquakes

Chao An, Yongen Cai*

*Department of Geophysics, Peking University, Beijing, 100871, China

Abstract

Analytical methods for studying tsunami run-up become untenable when the tsunami wave runs on beaches. In this study, we use a finite-element procedure that includes the interaction between solid and fluid based on the potential flow theory to simulate the dynamics of tsunami wave induced by a thrust fault earthquake in order to investigate the effect of different beach slopes on the tsunami run-up. The simulated run-up shows significantly differences from that predicted by the analytical solution. The maximum run-up shows a negative linear relationship with the square root of the cotangent of the beach slope, similar to the result of the analytical solution using a solitary wave as the incident wave, but with different slopes and amplitudes.

Keywords: Earthquake; Tsunami, Run-up; Beach effect

1. Introduction

The process of a tsunami consists of three phases: the generation of an oceanic wave by an impulsive event (i.e., earthquake), the propagation of the wave in the open ocean, and the evolution of the wave over a sloping beach (run-up). Since it is the run-up that makes the tsunami a devastating event, most studies have focused on the run-up phase of the process. Early research on tsunami run-up dates back to Kaplan (1955) [1], who generated a periodic wave that travels in water across a region of constant depth and studied its evolution on a sloping beach. He derived an empirical relationship between the maximum run-up and the ratio between the amplitude and the wavelength of the incident wave. Keller and Keller (1964) [2] presented analytical solutions for the run-up of monochromatic waves using linear shallow water equations and showed that the amplification factor was related to the ratio between the amplitude and the wavelength of the incident wave. Gardner et al. (1967) [3] developed the INS (inverse-scattering) method to solve nonlinear differential equations for solitons; they showed that any disturbance in the ocean would fission into a series of solitary waves at far enough distance; they then analyzed how solitary waves may evolve on a sloping beach. Synolakis (1987) [4] derived analytical solutions for the run-up of solitary waves using linear shallow water equations, showing that the maximum run-up is proportional to the square root of the cotangent of the sloping beach angle and the 5/4 power of the ratio of the wave height and amplitude.
water depth. Tadepalli and Synolakis (1994) [5] indicated that local tsunamis might not propagate enough distance to develop to solitary waves; thus they suggested that local tsunamis should be described by the run-up of N-waves. Tadepalli and Synolakis (1994) [5] also derived analytical solutions of the run-up of N-waves and showed that the leading-depression N-waves have larger maximum run-up than the leading-elevation N-waves. Geist (1999) [6] demonstrated how subduction zone events may generate N-waves and developed approximate formulae based on the asymptotic results of Tadepalli and Synolakis (1994) [5].

One of the basic assumptions in the analytical models based on the shallow water theory is that the amplitude of the wave is significantly less than the depth of the water column. This assumption becomes untenable when the waves run up on the beach. Numerical procedures have also been used to study tsunamis. For example, Ohmachi et al. (2001) [7] set up a two-dimensional model that includes an ocean basin with a thrust fault in a half space beneath the seafloor and simulated a tsunami by imposing the dynamic displacement of the seafloor resulting from a thrust event on the bottom of the seawater. They showed that the water wave height is remarkably larger than the static seabed displacement. However, they did not consider the run-up phase of the tsunami. Zahibo et al. (2006) [8] studied the nonlinear effects at tsunami propagation and run-ups and concluded that nonlinear effect may not be significant in the prediction of run-ups. None of these studies included the dynamic interaction between the tsunami and the seafloor except in Cai et al. (2009) [9].

In this paper, we use a two-dimensional model that includes the interaction between the tsunami and the seafloor to study the relationship between the maximum run-up and beach slopes. The result shows that the maximum run-up is negatively related to the beach slope, with a linear relationship between the maximum run-up and the square root of the cotangent of the beach slope, similar to the result of the analytical solution using a solitary wave as the incident wave, but with significantly different slopes and amplitudes.

2. Tsunami model with seawater and seafloor interaction

2.1. Basic equations

We express the continuity and motion equations, respectively, in terms of the velocity potential $\phi$

$$\frac{\partial \rho_w}{\partial t} + \nabla \cdot (\rho_w \nabla \phi) = 0$$

$$\frac{\partial}{\partial t} \nabla \phi + \frac{1}{2} \nabla \left( (\nabla \phi) \cdot (\nabla \phi) \right) + \frac{\nabla p}{\rho_w} + g = 0,$$

where $\nabla$ is defined by $\nabla = (\partial / \partial x, \partial / \partial z)$, $g = (0, g)$ is the acceleration of gravity, and the velocity vector $\mathbf{v} = \nabla \phi$. If we assume that seawater is weakly compressible and the changes in density and velocity are small, the relationship between density and pressure can be approximated as

$$\frac{\rho_w}{\rho_{w0}} = 1 + \frac{p}{\kappa},$$

where $p$ is the pressure of the fluid, $\rho_w$ is its density, $\rho_{w0}$ is its initial density, and $\kappa$ is the bulk modulus of the fluid. The continuity and motion equations can be simplified to

$$\frac{\rho_{w0}}{\kappa} \frac{\partial p}{\partial t} + \rho_{w0} \nabla^2 \phi = 0$$

(1)
\[
\frac{\partial \phi}{\partial t} + \frac{p}{\rho_w} - \Omega = 0,
\]

(2)

where \( \Omega \) denotes the gravity potential.

Insert Equation (2) into (1) we obtain

\[
- \rho_w \frac{\partial^2 \phi}{\partial t^2} + \kappa \nabla^2 \phi = -\rho_w \frac{\partial \Omega}{\partial t}
\]

(3)

In the solid part of the model, the equation of motion is

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{g} + \frac{1}{\rho_s} \nabla \cdot \mathbf{\sigma},
\]

(4)

where the displacement vector is defined as \( \mathbf{u} = (u_x, u_y) \), \( \rho_s \) denotes the density of solid and \( \mathbf{\sigma} \) the stress tensor is defined as

\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_z \end{bmatrix}.
\]

The constitutive equation is

\[
\mathbf{\sigma} = \lambda (\nabla \cdot \mathbf{u}) \mathbf{I} + 2\mu \mathbf{\varepsilon},
\]

where \( \lambda = \nu E / (1 + \nu) / (1 - 2\nu) \) and \( \mu = E / 2(1 + \nu) \) are Lame’s constants and \( \mathbf{\varepsilon} \) is the strain tensor defined as

\[
\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) & \frac{\partial u_z}{\partial z} \end{bmatrix}.
\]

2.2. Mechanic model

Since many large tsunamis are induced by thrust earthquakes in subduction zones, we set up a model to study tsunamis induced by thrust earthquakes, which considers the dynamic interaction between the tsunami and the seafloor (Fig. 1).

In Figure 1, \( V_2 \) represents the volume occupied by seawater with an undisturbed constant depth of 4km. \( V_3 \) represents the solid part of the model with a density of \( 2.7 \times 10^3 \) kg/m\(^3\), Young’s modulus of \( 8.3 \times 10^9 \) Pa (approximately the Young’s modulus of basalt) and Poisson’s ratio of 0.25. \( V_1 \) represents the volume occupied by the fault zone with a width of 300m, a maximum depth of 7km and a dip angle of 10°. We assume that, before the earthquake, the fault zone has the same material properties as those of the solid seafloor; in the simulation, an earthquake is ‘induced’ on the fault by artificially reducing the shear modulus of the fault zone – a procedure used successively in Hu et al. (2009) [10]. \( V_d \), a zone 10km wide on both the left and the right sides of the model and
5 km thick at the bottom, is set up to absorb the reflected waves. Material in this zone is set the same as in V3 but with a Rayleigh damping of 1.0 for both the P- and S-waves.

Fig. 1. Two-dimensional tsunami model that allows the dynamic interaction between seawater and seafloor. V1: fault zone; V2: seawater; V3: seafloor and continent; V4: absorbing boundary zone; S1 is a point in the hanging wall at the top of the fault; W2 is a point on the sea surface 150 km to the right of S1; W3 is the shoreline on the right side of the model; W4 is the shoreline on the left side of the model; W5 is a point on the sea surface 250 km to the left of S1.

Boundary conditions of the model are given as below. A horizontal displacement of 4.8 km is applied on the left side of the model (B1 in Fig. 1). This condition, together with gravity, is to simulate a maximum principal stress in the horizontal direction and a minimum principal stress in the vertical direction, a tectonic condition required by the thrust earthquake. The boundaries at the right side and at the bottom of the model are constrained in normal direction but free in the tangential direction; the top of the continent and the seawater (B3 and B5) are free. The normal velocity and stress on the interface between the seawater and the seafloor (B6) are continuous, i.e.,

\[ n \cdot v \big|_{B_6} = n \cdot u \big|_{B_6} \quad \text{and} \quad p \big|_{B_6} = n \cdot (\sigma \cdot n^\top) \big|_{B_6}. \]

The simulation consists of two steps: First, the horizontal displacement is applied. Second, the shear modulus in the fault zone V1 is reduced to induce an earthquake. This procedure has been applied successfully by Hu et al. (2009) [10] to study earthquake faulting and stress transfer in southern California. The amount of reduction in the shear modulus may be constrained by the observed shear displacement in the field. The time step is set to 1 s and the total time of simulation is 8000 s. The finite element program used is ADINA coded by ADINA R & D, Inc.

3. Results

3.1. Dynamic process of faulting and tsunami generation and propagation

Figures 2(a) and (b) show, respectively, the dislocation along the fault surface and the vertical displacement at the observation point S1 (Fig. 1). The dislocation at S1 is at its maximum of ~25 m and the average dislocation is ~17.5 m along the fault. Figure 2(b) shows that the fault oscillates approximately for 80 s before finally stabilizes. The maximum vertical displacement at S1 is about 4.1 m and stabilizes at ~3.7 m. The difference between the maximum and stabilized displacements (0.4 m) can remarkably affect the height of the tsunami in the model.

Figure 2(c) shows the snapshots of the displacements on the seafloor and sea surface. At \( t = 10 s \), the displacement on the sea surface follows closely that of the seafloor. The wave on the sea surface has a crest of 4 m and a trough of 2 m over a length of 50 km above the hanging wall. At \( t = 300 s \), the source wave has split into two tsunami waves propagating in opposite directions with different amplitudes and waveforms, and travel in velocities (192 m/s to the right and 194 m/s to the left, respectively) smaller than that predicted by the shallow water theory (198 m/s). Notice that the wave above the hanging wall has an extra crest in front of the biggest crest while the wave above the foot wall does not.

Figure 2(d) and (e) show the ground displacement at the two observation points W2 and W5 at distances 150 km to the right and 250 km to the left of S1, respectively. The first arrival is P waves, followed by long-period surface waves of large amplitude. The latter are inferred to be transferred waves from Rayleigh waves travelling along the
seafloor. These seismic waves arriving at W2 and W5 are much earlier than the tsunami waves, about 580 and 1150s earlier respectively. Also seen in Figure 2(d) is a small crest ahead of the biggest crest in the tsunami wave travelling to the right above the hanging wall, but there is no such crest in the wave travelling to the left above the foot wall.

Fig. 2. Earthquake source and tsunami generation and propagation
(a) Dislocation along the fault surface; (b) Vertical displacement of S1; (c) Vertical displacement of the seafloor and sea surface at different times; (d) Vertical displacements of W2, 150km to the right of S1; (e) Vertical displacements of W5, 250km to the left of S1.

Here we compare the water level calculated at the observation point W3 with the tidal gauge records of Iwanai station during the 1993 Hokkaido Nansei-oki earthquake tsunami (Fig. 3. The figure on the left side is taken from Ohmachi et al. (2001) [7]). Iwanai station is located above the hanging wall of the northmost fault (Satake et al. (1995) [11]), similar to W3 in our model. Since the fault geometry, water depth and distance from seashore in the model are different from those of the Hokkaido Nansei-oki earthquake, we cannot compare the calculated amplitude and the travel time of tsunami waves with those recorded at the Iwanai station. On the other hand, the calculated waveform is closely similar to the recorded waveform (see Fig. 3), supporting the viability of the calculation.

Fig. 3. Comparison between calculated water level and tidal gauge records

Fig. 4. Using analytical functions to fit the stable tsunami waves.
(a) Right side beach; (b) Left side beach
3.2. Run-ups

In this section we compare the tsunami run-ups calculated from our numerical model and those predicted by shallow water theory.

The simulated tsunami waves become stable at distance far from the source. For the comparison with the run-ups predicted by the shallow water theory, we use analytical functions to fit the stable tsunami waves and use these as the incident waves in the analytical solutions to predict run-ups. The incident tsunami waves in shallow water theory are monochromatic (Keller and Keller, 1964 [2]), solitary (Synolakis, 1987 [4]) and N-waves (Tadepalli and Synolakis, 1994 [5]). The method and formulae for fitting the stable tsunami waves with the analytical functions are provided in Appendix; the fitting results are shown in Figure 4.

Figure 5(a) and (b) show the run-ups and difference between the analytical predictions and the calculated run-ups at the observation point W3 on the right side beach. The relative differences are about -6% at the slope of 40° and 19% at 10° for the monochromatic waves, -7% at 40° and 30% at 10° for solitary waves, and -2% at 20° and -18% at 40° for N-waves. Figure 5(c) and (d) show the run-ups and difference between the analytical predictions and the calculated run-ups at the observation point W4 on the left side beach. The relative differences are -30% at 20° and -17% at 10° for monochromatic waves, and -0.2% at 20° and 30% at 10° for solitary waves.

Thus at a fixed slope angle different incident waves predict different run-ups. It implies that we can use different incident waves to predict run-ups for different sloping beaches. For slopes between 10° and 30° above the hanging wall, the relative difference predicted by N-waves is about 8% on average.

The model also predicts that the maximum run-up decreases with the increase of the beach slope (Fig. 5(a) and (c)). A similar conclusion was drawn by Madsen (2008) [12] that the impact on flat beaches with slopes of the order 1/100 is huge compared to impact on steeper beaches.

![Fig. 5. Comparison between predicted and calculated run-ups.](image-url)
3.3. Relationship between run-up and beach slope

Figure 6 shows that a linear relationship exists between the maximum run-up and the square root of the cotangent of the beach slope $\sqrt{\cot \beta}$ for both the right side and the left side beaches - similar to the results of analytical solutions using solitary or N-waves as the incident waves. We can see from Figure 6 that the line fitted for the right side beach has a greater slope than that for the left side beach. This may be attributed to the fact that the wave traveling to the right has a smaller wavelength (approximately 55km) than the one traveling to the left (approximately 85km). This explanation has been suggested by the results from analytical solutions of solitary or N-waves.

![Figure 6. Relationship between maximum run-up and $\sqrt{\cot \beta}$](image)

4. Conclusions

We used a numerical model that includes the mechanical interaction between the seawater and seafloor to simulate the tsunami induced by a thrust fault earthquake. The results from this simulation lead to the following conclusions:

1) The source wave of a tsunami induced by a thrust fault earthquake is located above the hanging wall and then it splits into two tsunami waves - one above the hanging wall and the other above the foot wall - with different amplitudes, waveforms and velocities. The velocities are smaller than those predicted by the shallow water theory.

2) The maximum run-up decreases with increasing beach slope, with a linear relationship between the maximum run-up and $\sqrt{\cot \beta}$ - consistent with the analytical solution of solitary waves and N-waves. However, the slope of our relation is less than that of the solitary and N-wave solutions.

3) The run-up predicted by our numerical model agrees with that predicted from the N-wave solution for slopes between 10° and 30° above the hanging wall.
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References


Appendix

A1. Analytical run-up of monochromatic, solitary and N-waves using linear shallow water wave equations

In this section, we use \( d \) denoting the undisturbed constant water depth and \( \beta \) denoting the angle of sloping beach. For an incident wave of monochromatic waves with the form \( z = H \left( 2\pi \left( \frac{x - x_0}{\lambda} + \pi/2 \right) \right) \), the run-up was derived by Keller and Keller, cited by Synolakis (1987)[4]. The maximum run-up is

\[
R_{\infty} = \frac{2}{J_0 \left( \frac{4\pi d \cot \beta}{\lambda} \right) - i J_1 \left( \frac{4\pi d \cot \beta}{\lambda} \right)} H, 
\]

where \( H \) and \( \lambda \) are the amplitude and wavelength of the incident wave and \( J_0 \) and \( J_1 \) are zeroth and first-order Bessel functions of the first kind.

For an incident wave of solitary waves with the form \( z = H \frac{\sech^2 \left( \gamma \left( x - x_0 \right) \right)}{\gamma^2} \), the run-up was derived by Synolakis (1987)[4]. The maximum run-up is \( R_{\infty} = 2.831 \frac{\sqrt{\cot \beta}}{H/d} \) for \( \gamma = \sqrt{3H/4d^2} \). For an arbitrary \( \gamma \), using the same method we derive the maximum run-up as

\[
R_{\infty} = 3.043 \frac{\sqrt{d \cot \beta}}{H}. 
\]
For an incident wave of symmetrical N-waves, the run-up is given by Tadepalli and Synolakis (1994)\cite{5}. Using the same method, for an incident wave of unsymmetrical N-waves with the form

\[ z = -H \operatorname{sech}^2(\gamma(x-x_1)) \tanh(\gamma(x-x_2)), \]

we derive the run-up as

\[ R(t) = \sqrt{8\pi E \gamma \cot \beta} \csch^2(\nu) \left[ \sum_{n=1}^{\infty} (-1)^{n+1} \left( -\frac{1}{n^2} + \frac{1}{n^2} e^{2\nu L} + \sinh(2\nu) n^2 e^{2\nu L} \right) \right], \]

where \( L = |x_1 - x_2| \), \( \chi = \exp\left( -2\gamma(L + x_1 - ct + d \cot \beta) \right) \), \( \nu = \gamma L \). For certain incident waves, we can determine \( L \) and \( \nu \) and then get the maximum run-up by calculating the maximum value of \( R(t) \).

A2. Wave fit

At \( t=500s \), the tsunami wave travelling to the right is still over the region of constant water depth and is about to hit the right side beach (see Fig. a1, the center of the small crest in front is 40km from the foot of the beach).

Cut part of the wave and fit it using monochromatic, solitary and N-waves (Fig. a2). The fitting result is:

- Monochromatic wave: \( z(x) = 1.86 \sin \left( \frac{2\pi}{111.6km} (x-106.4km) + \frac{\pi}{2} \right) \) m;
- Solitary wave: \( z(x) = 2.23 \operatorname{sech}^2 \left( 6.65 \times 10^{-2} \text{km}^{-1} (x-107.9km) \right) \) m;
- N-wave: \( z(x) = 2.20 \operatorname{sech}^2 \left( 6.66 \times 10^{-2} \text{km}^{-1} (x-121.1km) \right) \tanh \left( 6.66 \times 10^{-2} \text{km}^{-1} (x-127.8km) \right) \) m, with \( \nu = \gamma L = 1.0455 \) and the run-up is \( R_N = 2.699 \sqrt{\gamma d \cot \beta \ H} \).

Notice that the run-up is not related to the sign of \( H \).

For the left side beach, cut part of the wave traveling to the left at \( t=4500s \) (when the center of the main crest traveling to the left is about 120km to the foot of the left side beach). Use the same method used for the right side beach to fit the wave (Fig. a3) and predict the run-ups. Now the wave is in far field and N-wave fitting is not used.
Fig. a2. Fit the tsunami wave traveling to the right using monochromatic, solitary and N-waves.

Fig. a3. Fit the tsunami wave traveling to the left using monochromatic and solitary waves.