A problem on unique representation bases

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Abstract

In this paper we construct a unique representation basis whose growth is more than $x^{1/2-\varepsilon}$ for infinitely many positive integers $x$, which solves a problem posed by Nathanson in [M.B. Nathanson, Unique representation bases for the integers, Acta Arith. 108 (2003) 1–8].

MSC: 11B13; 11B34; 11B05

Let $A$ be a set of integers, and let

$$r_A(n) = \#\{(a, b) : a, b \in A, a \leq b, a + b = n\}.$$ 

A set $A$ of integers is called an additive basis for the integers if $r_A(n) \geq 1$ for all $n \in \mathbb{Z}$, and a unique representation basis if $r_A(n) = 1$ for all $n \in \mathbb{Z}$. A set $B$ of integers is called a Sidon set if $r_B(n) \leq 1$ for all $n \in \mathbb{Z}$. Thus a unique representation basis is a Sidon set that is a basis for the integers. Recently, Nathanson [2] proved that a unique representation basis for the integers can be arbitrarily sparse. An interesting problem is to find a dense unique representation basis. Nathanson [2] constructed a unique representation basis $A$ whose growth is logarithmic in the sense that the number of elements $a \in A$ with $|a| \leq x$ is bounded above and below by constant multiples of $\log x$. Nathanson [3] studied related problems with given representation functions. Let

$$A(y, x) = \#\{a \in A : y \leq a \leq x\}.$$ 

Nathanson [2] asked the following problem:

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Does there exist a number \( \theta < 1/2 \) such that \( A(-x, x) \leq x^\theta \) for every unique representation basis \( A \) and for all sufficiently large \( x \)?

In this note, we show that the answer to the problem is negative.

**Theorem.** For any \( \varepsilon > 0 \), there exists a unique representation basis \( A \) for the integers such that for infinitely many positive integers \( x \), we have

\[
A(-x, x) \geq x^{1/2-\varepsilon}.
\]

For a set \( A \) and any integer \( c \), define

\[
A + c = \{a + c : a \in A\}.
\]

**Lemma 1.** Let \( A \) be a nonempty finite set of integers with \( r_A(n) \leq 1 \) for all \( n \in \mathbb{Z} \) and \( 0 \not\in A \).

If \( m \) is an integer with \( r_A(m) = 0 \), then there exists a finite set \( B \) of integers such that \( A \subseteq B \), \( r_B(n) \leq 1 \) for all \( n \in \mathbb{Z} \), \( r_B(m) = 1 \) and \( 0 \not\in B \).

**Proof.** Let \( b = \max\{|a| : a \in A\} \). Take \( c = 4b + |m| \) and

\[
B = A \cup \{-c, c + m\}.
\]

It is easy to verify that the four sets

\[
2A, \quad A - c, \quad A + c + m, \quad \{m, -2c, 2c + 2m\}
\]

are disjoint each other. Hence \( r_B(n) \leq 1 \) for all \( n \in \mathbb{Z} \) and \( 0 \not\in B \) by \( c \neq 0 \) and \( c + m \neq 0 \). This completes the proof of **Lemma 1**.

**Lemma 2.** Let \( A \) be a nonempty finite set of integers with \( r_A(n) \leq 1 \) for all \( n \in \mathbb{Z} \) and \( 0 \not\in A \).

Then, for any \( \varepsilon > 0 \) and \( M > 0 \), there exists an integer \( x > M \) and a finite set \( B \) of integers with \( 0 \not\in B \), \( A \subseteq B \), \( r_B(n) \leq 1 \) for all \( n \in \mathbb{Z} \) and \( B(-x, x) \geq x^{1/2-\varepsilon} \).

**Proof.** It is well known that for any positive integer \( m \) there exists a Sidon set \( S \subseteq [1, m] \) with \( |S| \geq \sqrt{m} + o(\sqrt{m}) \) (see [1,4]). Thus there exists an integer \( x > M + (25T)^{1/(2\varepsilon)} \) and a set \( D \) of positive integers with \( r_D(n) \leq 1 \) for all positive integers \( n \) and \( D(1, x/(5T)) \geq \frac{1}{2} \sqrt{x/(5T)} \), where \( T = \max\{|a| : a \in A\} \). Let \( B = A \cup \{5Tb : b \in D\} \). Then \( 0 \not\in B \) and

\[
B(-x, x) \geq D(1, x/(5T)) \geq \frac{1}{2} \sqrt{x/(5T)} \geq x^{1/2-\varepsilon}.
\]

It is easy to verify that \( r_B(n) \leq 1 \) for all \( n \in \mathbb{Z} \). This completes the proof of **Lemma 2**.

**Proof of the Theorem.** We shall use induction to construct an ascending sequence \( A_1 \subseteq A_2 \subseteq \cdots \) of finite sets of integers and a sequence \( \{x_i\}_{i=1}^\infty \) of positive integers with \( x_{i+1} > x_i \) for all \( i \) such that for any positive integer \( k \), we have

(i) \( r_{A_k}(n) \leq 1 \) for all \( n \in \mathbb{Z} \);
(ii) \( r_{A_{2k}}(n) = 1 \) for all \( n \in \mathbb{Z} \) with \( |n| \leq k \);
(iii) \( A_{2k-1}(-x_k, x_k) \geq x_k^{1/2-\varepsilon} \);
(iv) \( 0 \not\in A_k \).

Let \( A_1 = \{-1, 1\} \) and \( x_1 = 1 \). Suppose that we have \( A_1, A_2, \ldots, A_{2l-1} \) and positive integers \( x_1 < x_2 < \cdots < x_l \). Let \( m \) be an integer with minimum absolute value and \( r_{A_{2l-1}}(m) = 0 \). If \( l = 1 \), then \( |m| = 1 = l \). If \( l > 1 \), then by \( A_{2l-2} \subseteq A_{2l-1} \) we have \( r_{A_{2l-2}}(m) = 0 \). By the
inductive hypothesis and (ii) we have \( m \geq l \). By Lemma 1 there exists a finite set \( B \) of integers such that \( A_{2l-1} \subseteq B \), \( r_B(n) \leq 1 \) for all \( n \in \mathbb{Z} \), \( r_B(m) = 1 \) and \( 0 \notin B \). If \( r_B(-m) = 0 \), then there exists a finite set \( B' \) of integers such that \( B \subseteq B' \), \( r_{B'}(n) \leq 1 \) for all \( n \in \mathbb{Z} \), \( r_{B'}(m) = 1 \) and \( 0 \notin B' \). Now, let \( A_{2l} = B \) if \( r_B(-m) \neq 0 \), and let \( A_{2l} = B' \) if \( r_B(-m) = 0 \). Then \( A_{2l} \) satisfies (i), (ii), (iv) and \( A_{2l-1} \subseteq A_{2l} \). By Lemma 2 there exists a finite set \( A_{2l+1} \) of integers and an integer \( x_{l+1} > x_l \) such that (i), (iii) and (iv) hold, and \( A_{2l} \subseteq A_{2l+1} \). Let

\[
A = \bigcup_{k=1}^{\infty} A_k.
\]

By (ii), we have that \( r_A(n) = 1 \) for all \( n \in \mathbb{Z} \). So \( A \) is a unique representation basis for the integers. By (iii), we have

\[
A(-x_k, x_k) \geq x_k^{1/2 - \varepsilon}.
\]

This completes the proof. \( \square \)

Finally, we pose the following open problems:

(1) Does there exist a real number \( c > 0 \) and a unique representation basis \( A \) such that

\[
A(-x, x) \geq c\sqrt{x}
\]

for infinitely many positive integers \( x \)?

(2) Does there exist a real number \( c > 0 \) and a unique representation basis \( A \) such that

\[
A(-x, x) \geq c\sqrt{x}
\]

for all real numbers \( x \geq 1 \)?

(3) Does there exist a real number \( \theta < \frac{1}{2} \) such that for any unique representation basis \( A \) there are infinitely many positive integers \( x \) with \( A(-x, x) < x^{\theta} \)?

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References