On Sensor Network Localization Using SDP Relaxation

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Abstract—Sensor network localization attempts to determine the locations of a group of sensors given the distances between some of them. The Semidefinite Programming (SDP) relaxation of this problem is widely used to determine the locations of the sensors \cite{1}. In this paper, we analyze and determine a number of conditions that guarantee that the SDP relaxation is exact, i.e. gives the correct solution. Our main contribution is twofold. We present the first non-asymptotic bound on the connectivity range requirement of the sensors in order to ensure the network is uniquely localizable. And second, we introduce a new class of graphs that can always be localized by an SDP relaxation. Specifically, we show that adding a simple objective function to the SDP will ensure that the solution is correct when applied to a triangulation graph. Since triangulation graphs are very sparse, with only \( m \) edges, this is informationally efficient, requiring an almost minimal amount of distance information. We also analyze a number of objective functions for the SDP relaxation to determine good heuristics to solve the localization problem for a general graph.

I. INTRODUCTION

Graph Realization is a commonly studied topic which attempts to map the nodes in a graph \( G(V, E) \) to locations in the Euclidean space, based on the non-negative weights of the edges, i.e. the weight of each edge becomes the Euclidean distance between each pair of incident points. There are a number of applications of the graph realization problem \cite{2,3,4,5}. In this paper, we focus on the application to Sensor Network Localization (SNL).

A sensor network consists of a collection of sensors whose locations are unknown, and anchors whose locations are known. A common property of a sensor network is that each sensor detects others within a given connectivity (or radio) range and determines the distance from itself to these other sensors. Given this set of sensors and distances, the goal is to determine the exact location of each of the sensors. The problem becomes a graph realization problem by forming the weighted, undirected graph \( G(V, E) \); the node set \( V \) represents the sensors, and each non-negative weighted edge in \( E \) represents a known distance between two sensors.

The SNL problem has received a lot of attention recently because of the formulation of its relaxation as a Semidefinite Program (SDP) \cite{1,7,8}. This formulation can find the exact locations of the sensors, given that the graph has certain properties. In this paper, we present a number of conditions that guarantee unique localizability of the SDP formulation of the SNL problem, i.e. conditions that ensure the SDP will give the correct solution.

A. Background

We are given a graph \( G(V, E) \) in dimension \( d \), where the vertices consist of anchors \( \{a_1, \ldots, a_m\} \) (where \( m \geq d + 1 \)) whose locations are known and sensors \( \{x_1, \ldots, x_n\} \), whose locations are unknown. The edge set also consists of two distinct sets, \( E = N_x \cup N_a \), where \( N_x = \{d_{ij}^2 : (i, j) \in E\} \) is the set of edges between sensors, and \( N_a = \{d_{ij}^2 : (i, j) \in E\} \) is the set of edges between an anchor and a sensor. The problem of finding the locations of the sensors (i.e., the remaining vertices whose locations are unknown), can be formulated as finding the points \( x_1, x_2, \ldots, x_n \in \mathbb{R}^d \), such that

\[
\begin{align*}
\|x_i - x_j\|^2 &= d_{ij}^2, \quad \forall (i, j) \in E \\
\|a_k - x_j\|^2 &= d_{kj}^2, \quad \forall (k, j) \in E
\end{align*}
\]

(1)

SDP Formulation

The following SDP relaxation is a well known method for solving the Sensor Network Localization problem \cite{1,9}.

\[
\begin{align*}
\max \quad & 0 \\
\text{s.t.} \quad & Z_{(1:d,1:d)} = I_d \\
& (0; e_i - e_j)^T Z d_{ij}^2, \quad \forall (i, j) \in N_x \\
& (a_k - e_j)^T Z d_{kj}^2, \quad \forall (k, j) \in N_a \\
& Z \succeq 0
\end{align*}
\]

(2)

where \( e_i \) is the \( i \)th column of the identity matrix in \( \mathbb{R}^{n \times n} \). The solution to this formulation is a matrix
The constraint $Z \succeq 0$ implies that $Y \succeq X^T X$. If $Y = X^T X$, then the above formulation finds a matrix $Z$ such that the columns of its submatrix $X = [x_1 \ x_2 \ \cdots \ x_n]$ satisfy (1).

**Definition 1:** A sensor network is **uniquely localizable** if there is a unique $X \in \mathbb{R}^{d \times n}$ that satisfies (1), and there is no $\tilde{X} \in \mathbb{R}^{h \times n}$ for $h(>d)$, that satisfies (1) and $\tilde{X} \neq (X; 0)$. In other words, there is no non-trivial extension of $X \in \mathbb{R}^{d \times n}$ into dimension $h(>d)$ that also satisfies (1). When the SNL problem is uniquely localizable, the maximum rank solution of (2) has rank $d$, and its solution matrix $Z$ satisfies $Y = X^T X$ [7]. The dual of the SDP relaxation,

$$\begin{align*}
\min \quad & I_d \cdot V + \sum_{(i,j) \in N_x} y_{ij} d_{ij}^2 + \sum_{(k,j) \in N_u} w_{kj} d_{kj}^2 \\
\text{s.t.} \quad & \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} + \sum_{(i,j) \in N_x} y_{ij} \begin{pmatrix} 0; e_i - e_j \end{pmatrix} \begin{pmatrix} 0; e_i - e_j \end{pmatrix}^T \\
& + \sum_{(k,j) \in N_u} w_{kj} \begin{pmatrix} a_k; -e_j \end{pmatrix} \begin{pmatrix} a_k; -e_j \end{pmatrix}^T \succeq 0
\end{align*}$$

(3)

is useful, in that the solution to the dual tells us key properties about the primal. We define the $(d + n) \times (d + n)$ dual slack matrix to be

$$U = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} + \sum_{(i,j) \in N_x} y_{ij} \begin{pmatrix} 0; e_i - e_j \end{pmatrix} \begin{pmatrix} 0; e_i - e_j \end{pmatrix}^T + \sum_{(k,j) \in N_u} w_{kj} \begin{pmatrix} a_k; -e_j \end{pmatrix} \begin{pmatrix} a_k; -e_j \end{pmatrix}^T$$

The dual slack matrix $U$ is optimal if and only if it is feasible and the complementarity condition holds, i.e., $ZU = 0$. If complementarity holds, then $\text{rank}(Z) + \text{rank}(U) \leq (d + n)$, and since $\text{rank}(Z) \geq d$, this means that $\text{rank}(U) \leq n$. Thus, if an optimal dual slack matrix has rank $n$, then every solution to (2) has rank $d$ [7].

**Definition 2:** A sensor network is **strongly localizable** if there is an optimal dual slack matrix with rank $n$. A graph realization is **globally rigid** if for $X^1 \in \mathbb{R}^{d \times n}$ and $X^2 \in \mathbb{R}^{d \times n}$, $\|x^1_i - x^2_j\| = \|x^1_i - x^2_j\|$ for all $(i,j) \in E$ implies $X^1 = X^2$ [7], [10].

### B. Our Contributions

In this paper, we present a number of conditions that guarantee unique localizability of the SDP formulation of the problem, i.e., conditions that ensure the SDP will give the correct solution so that the sensor network can be localized in polynomial time. Our result is twofold.

1. A very popular graph in the context of sensor network localization is the unit-disk graph. It has been observed that, when the disk radius (or radio range) increases, more sensors in the network would be correctly localized. There is an asymptotic analysis to explain this phenomenon [11]. Our second result is to present the first non-asymptotic bound on the connectivity range requirement of the nodes in order to ensure the network is uniquely localizable. This may have a practical impact by providing guidance on communication power ranges that ensure the network is localizable.

2. The basic SDP localization model [2] is an SDP feasibility problem. An open question has been to determine whether adding certain objective functions to the basic model improves localizability of the problem; that is, the SDP feasible region contains high rank solutions, but the SDP optimal solution is guaranteed to be unique and low rank. We give an affirmative answer to a generic class of graph. Our result may also have an influence on Compressed Sensing, which also uses an objective function to produce the sparsest solution. Based on this idea, we produce numerical results by comparing several SDP objective functions to illustrate their effectiveness.

Moreover, although our theoretical analyses are based on exact distance measurements, similar extensions of our model– established in earlier SDP work –would be applicable to noisy distance data.

### C. Paper Organization

The organization of this paper is as follows. First, Section III gives an upper bound on the connectivity radius in a sensor network that guarantees unique localizability. In Section IV we show that given a triangulation (i.e., planar, chordal and convex) graph, if the sum of the distances between nodes that do not have an edge between them is maximized, then the graph will be strongly localizable. We use this idea, and test a number of heuristic objective functions on a large number of random sensor networks to determine how well each works in practice. Our results for these heuristics are presented in Section V.

II. **Bounding Connectivity Radius**

In this section, we consider unit disk graph model [12]–[14]: i.e., two sensor nodes (or a sensor node and an anchor node) are connected (their mutual distance is known) if and only if their mutual distance is less than $r$. We also assume $n$ nodes are uniformly distributed in the unit square. Then, we find an upper bound for $r$.
that grants unique localizability with high probability. First, we form an upper bound on the connectivity range to ensure the graph is a \((d+1)\)-lateration graph.

**Definition 3:** For some \(d, n \geq 1\), the graph \(G(V, E)\) is a \((d+1)\)-lateration graph if there exists a permutation of the vertices, \(\{\pi(1), \pi(2), \ldots, \pi(n)\}\), such that the edges of the sub-graph \(\pi(1), \ldots, \pi(d+1)\) form a complete graph, and each successive vertex \(\pi(j)\) for \(j \geq d+2\) is connected to \(d+1\) vertices in the set \(\{\pi(1), \ldots, \pi(j-1)\}\). This permutation of the vertices, \(\pi\), is called the \((d+1)\)-lateration ordering.

It is shown in [15] that if a graph \(G\) has a \((d+1)\)-lateration ordering, then it is universally rigid. Zhu et al. [15] provide a rigorous proof, however the intuition behind the proof is that a complete graph on \(d+1\) vertices is uniquely defined, and any point connected to \(d+1\) uniquely defined points is also uniquely defined.

Define \(r(p)\) to be the connectivity range of the sensors so the graph is uniquely localizable with probability at least \(p\). Thus, to find a bound on \(r(p)\), we first find a connectivity range for which the graph \(G(V, E)\) will be a \((d+1)\)-lateration graph with at least probability \(p\).

### A. Ensuring a Clique in the Graph

First, we find a bound for the range \(r\) such that there is at least one clique in the graph. We approach the problem by splitting the total area into a number of sub-hypercubes in dimension \(d\); this will allow us to analyze sub-sections of the area and their containing nodes, as opposed to analyzing nodes individually. Let \(\mathcal{H}\) be the \(d\)-dimensional area in which each point \(x_i \in \mathbb{R}^d\) lies, without loss of generality, we assume \(\mathcal{H} = [0, 1]^d\).

This generality is clear if the area is a hypercube in \(d\) dimensions, however, we show in Lemma 1 that this generality can be extended to areas of different forms as well.

**Lemma 1:** Assume there is at least one node in each sub-square. If at each section of the area \(\mathcal{H}\), the minimum distance between boundaries is at least \(3\ell\), for \(\ell\) the edge length of each sub-square in the grid, then the analysis of \(\mathcal{H}\) can be generalized to assume it is a unit hypercube in \(\mathbb{R}^d\).

**Proof:** See appendix

The realistic implications of Lemma 1 are that if the area becomes too thin in sections, a \((d+1)\)-lateration ordering may not be possible. This is very reasonable, since it implies that a \((d+1)\)-lateration may not be possible in an area similar to the one in Figure 3 since localization may not spread across a narrow area.

We split \(\mathcal{H}\) into \(M\) equal sub-hypercubes in \(\mathbb{R}^d\), say \(h_1, h_2, \ldots, h_M \subseteq \mathcal{H}\), where each sub-hypercube, \(h_i\), will have a volume of \(1/M\), and the length of each of its edges will be \(\ell := 1/\sqrt{M}\) (assume \(M = b^d\) where \(b\) is an integer number).

**Theorem 1:** If \(r \geq \sqrt{\pi} / \sqrt{M}\), and \(M \leq n^{d-1}/d\), (or equivalently \(r \geq \sqrt{\pi n/d}\)), there is at least one clique of \(d+1\) vertices in the graph \(G(V, E)\).

**Proof:** Emitted; based on the pigeon-hole principle.

We can initialize the \((d+1)\)-lateration algorithm by choosing \(r\) according to this lower bound, and choosing the nodes of the \((d+1)\)-clique as the first nodes in the \(d\)-lateration ordering. However, we should refine the bound for \(r\) to make sure there is also an ordering \((\pi)\) that \((d+1)\)-lateration is possible with a high probability. This is done in the following sections.

### B. Binomial Distribution Model

By assuming that the nodes are uniformly distributed throughout the area \(\mathcal{H}\), we can say that the nodes are binomially distributed throughout each sub-hypercube of \(\mathcal{H}\). However, since the total number of nodes, \(n\), is constant, the number of nodes in each sub-hypercube is dependent on the number in the other sub-hypercubes. Thus, we make the slight modification to the distribution assumption by assuming that the number of nodes in each sub-hypercube is independent. Hence, the number of points in each sub-hypercube will be independently binomially distributed: if \(Y_i\) is the number of points in the sub-hypercube \(h_i\), then \(Y_i \sim B(n, 1/M)\).

Using this binomial distribution model, let \(S_n = \sum_{i=1}^{M} Y_i\) denote the total number of nodes in the hypercube \(\mathcal{H}\). Then, by properties of the binomial distribution, \(S_n \sim B(n, 1/M)\).

**Lemma 2:**

\[
\begin{align*}
E[S_n] & = M \cdot E[Y_1] = M \left( \frac{n}{M} \right) = n \\
\text{Var}(S_n) & = M \cdot \text{Var}(Y_1) = M \cdot \left[ \frac{n}{M} \left( 1 - \frac{1}{M} \right) \right] \\
& = n \left( 1 - \frac{1}{M} \right)
\end{align*}
\]

Thus, \(S_n \rightarrow 1\) almost surely. This shows that the assumption of binomially distributed sensors throughout each sub-hypercube is statistically equivalent to assuming a uniform distribution on all the nodes throughout the larger area \(\mathcal{H}\).

### C. Connectivity Bound

We now form further constraints on the connectivity range \(r\) to ensure that \(G\) is a \((d+1)\)-lateration graph. We assume that the points are binomially distributed in each sub-hypercube, with parameters \(n\) and \(1/M\). First, \(r\) must satisfy Theorem 1 since this ensures a \((d+1)\)-clique in the graph \(G\). These \(d+1\) vertices in the clique represents the first \(d+1\) vertices in the permutation \(\pi\) of the \((d+1)\)-lateration graph (Definition 3).
We construct an improved bound on the probability of localizability through an ordering of the hypercubes, \( h_i \in \mathcal{H} \), and hence an ordering on the points. We prove the following lemmas for the case where \( d = 2 \), and we refer to the sub-hypercubes as squares. However, we note that the same analysis can be applied to hypercubes in other dimensions.

**Lemma 2:** Assume that each sub-square in \( \mathcal{H} \in \mathbb{R}^2 \) has at least one point, and \( r \geq 2\sqrt{d}\ell \). If there are three successive squares in the same row, then the locations of points in all sub-squares in those three successive columns are unique. Similarly, if the locations of points in three successive sub-squares in the same column are known, then the locations of all points in sub-squares in these three successive rows are known.

**Proof:** Using a coordinate representation, where each sub-square is given a pair \((i, j)\) that represents its horizontal and vertical location in the larger hypercube, three successive squares in the same row correspond to three sub-squares at positions \((i, j - 1), (i, j), (i, j + 1)\). Let \( \mathcal{S} \) represent this set of three sub-squares. We want to show that if the squares in \( \mathcal{S} \) have points whose locations are known, then the unique locations of all points in sub-squares in the three corresponding columns \((j - 1, j)\) and \((j, j + 1)\) can also be uniquely determined. A similar result holds for three successive sub-squares in the same column. We will prove the case for three successive sub-squares in the same row, in the set \( \mathcal{S} \), and the second case is proved similarly.

Let \( \mathcal{S} \) represent the set of three sub-squares in a row, with localizable points. If none of the sub-squares in \( \mathcal{S} \) are on a boundary \((i, j > 1)\), then the point(s) in the sub-square in the middle of the three columns and one row up, i.e. at position \((i + 1, j)\), are within distance \( r \) of all the points in sub-squares in the set \( \mathcal{S} \). Thus, since each of the three squares in \( \mathcal{S} \) contains at least one point, the point(s) in this sub-square, at \((i + 1, j)\), are uniquely localizable. Moreover, the points in the sub-square one row up from the squares in \( \mathcal{S} \), and in corresponding left-most column, i.e. at \((i + 1, j - 1)\), are also localizable, since they will be connected to all points in two sub-squares of \( \mathcal{S} \), and all points in the sub-square at \((i + 1, j)\).

Similarly, the points in the sub-square one row up from the squares in \( \mathcal{S} \), and in the corresponding right-most column, i.e., at \((i + 1, j + 1)\) are also localizable.

Thus, the three sub-squares one row up from those in \( \mathcal{S} \), and in the same columns, are also localizable. This pattern extends to both increasing and decreasing row indices, so that all successive sub-squares in the three successive columns corresponding to \( \mathcal{S} \), i.e. columns \( j - 1, j \) and \( j + 1 \), are all localizable.

**Lemma 3** states that if there is a sequence of three squares in a row or column with localizable points, then the localizability extends to all squares in the corresponding columns or rows. For example consider the scenario Figure 1. Assume the locations of all nodes in sub-squares \((1, 2), (2, 2)\) and \((3, 2)\) are known. Then, by Lemma 2, the locations of all nodes in the first, second and third rows are known.

**Lemma 3:** Assume there is at least one point in each sub-square and \( r \geq 2\sqrt{d}\ell \), where \( \ell \) is the edge length of each sub-square. Then the associated graph is a \((d + 1)\)-lateration graph if either

a) There is a clique in a non-corner grid
b) There is a clique in a corner point and one of its neighbor squares has at least two nodes

**Proof:** We will show that if the conditions of Lemma 3 are satisfied, then there exists an \((d + 1)\)-lateration ordering on the nodes in the graph. First, note that since \( r \geq 2\sqrt{d}\ell \), all nodes in a sub-square are connected to all nodes in neighboring sub-squares.

**Part (a):** We will use the case of Figure 1 however it is easy to extend this proof to a general graph. Figure 1 shows a clique in a non-corner sub-square, i.e., in square \((2, 1)\). Let the points in this clique be the initial \( d + 1 \) points in the \((d + 1)\)-lateration ordering. All nodes in sub-squares \((1, 1), (1, 2), (2, 2), (3, 1)\) and \((3, 2)\) are connected to this clique, and thus they are localizable by \((d + 1)\)-lateration; let the points in these squares be next in the \((d + 1)\)-lateration ordering.

By Lemma 2, all nodes in the first three rows of sub-squares are localizable. Since all sub-squares have at least one point, and there are at least three columns in the first three rows, the same argument applies, and there is a \((d + 1)\)-lateration ordering on the nodes. Therefore, if the conditions of Lemma 3 hold, the associated graph is a \((d + 1)\)-lateration graph.

**Part (b):** Now consider Figure 1 where there is a clique in a corner sub-square, and there are at least 4
two nodes in sub-square $s_1$, the first $d + 1$ nodes in the $(d + 1)$-lateration ordering. All nodes in sub-squares $(2, 1), (2, 2)$, and $(1, 2)$ are connected to the clique and hence localizable; these nodes will be next in the $(d + 1)$-lateration ordering. Next, let nodes in $(3, 1)$ and $(3, 2)$ be the succeeding nodes in the ordering. Using a similar argument as before from Lemma 2, we can construct a $(d + 1)$-lateration on the nodes in the graph, and all nodes are localizable.

Therefore, if the conditions of Lemma 3 hold, the associated graph is a $(d + 1)$-lateration graph.

The above lemma provides sufficient, but not necessary conditions on a network for $(d + 1)$-lateration, which will imply unique localizability. Moreover, they are strict conditions for a sensor network, since the distribution of sensors in a network may not always ensure that there is one sensor in each sub-square. Thus, we extend these conditions to a more general case, and allow for the possibility of empty sub-squares. Clearly, too many empty sub-squares will result in a graph that is not uniquely localizable; also, if empty sub-squares exist, there must be other restricting conditions to ensure the graph is not too sparse to ensure localizability. Thus, we establish new properties of the graph that allow for empty sub-squares.

Definition 4: Two neighboring sub-squares are called adjacent neighbors if they do not share any edges; otherwise, they are called simple neighbors. An empty sub-square is called saturated if all its simple neighbors have at least two nodes and one of the simple neighbors has at least 3 nodes.

Lemma 4: Assume every empty sub-square is saturated and $r \geq 2\sqrt{\ell}$, where $\ell$ is edge length of each sub-square. Then the associated graph is a $(d + 1)$-lateration graph if

a) There is a clique in a non-corner sub-square

b) There is a clique in a corner sub-square, one of its neighbors has at least two nodes, and at most one of its neighbors is empty

Proof: Part (a): Figure 2 shows an example of a non-corner clique, and saturated empty squares. Consider the saturated empty sub-square at $(2, 3)$. Similar to Lemma 3, letting all nodes in $(1, 1), (1, 2), (2, 2), (3, 1)$, and $(3, 2)$ succeed the clique’s nodes in the $(d + 1)$-lateration ordering ensures these nodes are localizable. Next, include the nodes in $(1, 3)$ and $(3, 3)$ in the ordering, and finally the nodes in $(2, 4)$. This ensures a $(d + 1)$-lateration ordering among these nodes.

Now, consider two empty sub-squares that are adjacent neighbors, i.e., sub-squares $(3, 7)$ and $(2, 8)$ in Figure 2. Assume the nodes in sub-squares $(1, 6), (2, 6)$ and $(3, 6)$ are already in the $(d + 1)$-lateration ordering, and hence their locations are known. Include the nodes in $(1, 7), (2, 7)$ and $(1, 8)$ sequentially in the ordering, so that these nodes are localized. There are 3 nodes in $(2, 9)$, thus if we start localization from this clique, nodes in $(1, 9)$ and $(3, 9)$, are localizable and finally nodes in $(3, 8)$ are localizable (with respect to the new starting point). Therefore, the original $(d + 1)$-lateration and new $(d + 1)$-lateration share more than $(d + 1)$ nodes, and the whole graph is uniquely localizable.

Similar to the proof of Lemma 3, we can use the result of Lemma 2 to show that a $(d + 1)$-lateration ordering exists on all points in the graph when the conditions of Lemma 3 hold.

Part (b): Figure 2 shows an example of the condition in Lemma 4. Similar to the proof in Part (a), it is easy to observe a $(d + 1)$-lateration ordering here.

Therefore, if the conditions of Lemma 4 hold, then the associated graph is a $(d + 1)$-lateration graph, and hence is $d$-uniquely localizable.

Next, we find a lower bound for the probability that a unit disk graph can be localizable using 3-lateration. Define

$$C = \{\text{There are only corner cliques}\}$$

and

$$\hat{C} = \{\text{There is a non-corner clique}\}.$$

The probability of a graph with uniformly distributed sensors being universally rigid is

$$P\{\text{universally rigid}\} = P\{\text{universally rigid} | \hat{C}\}P(\hat{C}) \quad + \quad P\{\text{universally rigid} | C\}P(C) \quad \geq \quad P\{\text{universally rigid} | \hat{C}\}P(\hat{C})$$

Let $\ell = \frac{\sqrt{2}}{\sqrt{n}}$ be the edge length of each grid, and $r = \frac{\sqrt{2}}{\sqrt{n}}$ be the connectivity range, so that $M = \frac{1}{\sqrt{n}}$ is the total number of grids. The distribution of nodes in each sub-square is binomial $B(n, \frac{1}{\sqrt{n}})$, and there are a total of $(M - 4)$ non-corner sub-squares, thus

$$P(\hat{C}) = 1 - \left(\sum_{i=0}^{2} \binom{n}{i} \left(\frac{1}{\sqrt{n}}\right)^i \left(1 - \frac{1}{\sqrt{n}}\right)^{n-i}\right)^{M-4}$$
Let \( k \) be the number of empty sub-squares. By Lemma 3 \( \mathbb{P}\{\text{universally rigid}|k = 0, \hat{C}\} = 1 \), and if \( p_0 \) is the probability that a specific sub-square is empty, we have \( \mathbb{P}\{k = i\} = \left(\frac{1}{M}\right)^i (1 - p_0)^{n-j} \).

We also use Lemma 4 to get the bound
\[
\mathbb{P}\{\text{universally rigid}|k = i, \hat{C}\} \geq \mathbb{P}\{\text{empty sub-squares are saturated}|k = i, \hat{C}\}
\]

Lemma 4 requires empty squares not have empty simple neighbors; thus we first find the probability that a square does not have empty simple neighbors. Assume there are \( k \) empty squares. Because of independence assumption, these empty grids are uniformly distributed. Given the first empty square in the grid, the probability that the next empty square is not a simple neighbor of the first empty square is \((1 - \frac{4q}{M-q-1})\), the probability that the 3rd empty square is not a simple neighbor of the first two empty squares is greater than \((1 - 2 \cdot \frac{4q}{M-q-1})\), and so on. For an empty grid to be saturated, all its simple neighbors must have at least two nodes, and one of them must have at least three nodes. This happens with probability
\[
\hat{\rho} = \mathbb{P}\{\text{Simple neighbors have at least two nodes}\} - \mathbb{P}\{\text{Simple neighbors have exactly two nodes}\}
\]
\[
= \left[1 - \sum_{j=0}^{1} \left(\frac{n}{j}\right) \left(\frac{1}{M}\right)^j (1 - \frac{1}{M})^{n-j}\right]^4
\]
\[
- \left(\binom{n}{2} \left(\frac{1}{M}\right)^2 (1 - \frac{1}{M})^{n-2}\right)^4
\]  
(6)

Thus,
\[
\mathbb{P}\{\text{empty sub-squares are saturated}|k = i, \hat{C}\} \geq \prod_{q=0}^{i} (1 - \frac{4q}{M-q-1})\hat{\rho}
\]  
(7)

Note that the right hand side of the above equation is positive if \( q \leq (M-1)/5 \). Thus, we only consider grids with less than \((M-1)/5\) empty squares. Finally,

\[
\mathbb{P}\{\text{universally rigid}\} \geq \mathbb{P}\{\text{universally rigid}|\hat{C}\} \mathbb{P}\{\hat{C}\}
\]
\[
= \sum_{i=0}^{M} \mathbb{P}\{\text{universally rigid}|k = i, \hat{C}\} \mathbb{P}\{k = i\} \mathbb{P}\{\hat{C}\}
\]
\[
\geq \mathbb{P}\{\hat{C}\} \left(\mathbb{P}\{k = 0\} + \sum_{i=1}^{M} \mathbb{P}\{k = i\}\right)
\]
\[
\times \mathbb{P}\{\text{empty sub-squares are saturated}|k = i, \hat{C}\}
\]
\[
\geq \mathbb{P}\{\hat{C}\} \left(\mathbb{P}\{k = 0\} + \sum_{i=1}^{\left[(M-1)/5\right]} \mathbb{P}\{k = i\}\right)
\]
\[
\times \prod_{q=0}^{i} (1 - \frac{4q}{M-q-1})\hat{\rho}
\]  
(8)

For different values of \( n \) (the number of nodes), we can find values of \( \alpha \) such that the right hand side of Equation 8 is at least 0.99. Figures 3a and 3b show \( \alpha \) and \( r \) versus the number of nodes \( n \) such that the right hand side of Equation 8 is at least 0.99.

Aspnes et al. [11] prove that for \( d = 2 \), if \( r > \frac{2\sqrt{\log n}}{\sqrt{n}} \), then graph is asymptotically localizable. Their proof is asymptotic and can be extended to form of \( r > \frac{g(n)}{\sqrt{n}} \) for any strictly increasing function \( g(\cdot) \). Our result is not an asymptotic bound and is true for any value of \( n \).

### III. UNIQUE LOCALIZATION OF TRIANGULATION GRAPH

In this section, we show that adding an objective function of maximizing the sum of certain distances in a triangulation graph (in \( R^2 \)) will result in the SDP relaxation giving the correct location of nodes.

**Definition 5:** Consider a set of points \( \mathcal{P} = \{p_1, p_2 \ldots p_n\} \) in \( R^2 \). A triangulation, \( \mathcal{T}_P \), of the points \( \mathcal{P} \) is a subdivision of the convex hull of \( \mathcal{P} \) into simplices (triangles) such that the edges of two simplices do not intersect, or share a common face.

Triangulation graphs and their properties have been studied in the literature [16]–[18]. Bruck et al. [16] show that embedding on a unit disk graph with local angle information (angles between nodes) is NP-hard, while the same problem on a triangulation graph is not. Araujo et al. [17] introduced an algorithm to construct a triangulation graph from a unit disk graph with \( O(n \log n) \) bit communication between nodes.

**Definition 6:** For a triangulation \( \mathcal{T}_P \), we define a graph \( G_{\mathcal{T}_P}(V, E) \) such that \( V = \mathcal{P} \) and \( (p_i, p_j) \in E \) if and only if \( (p_i, p_j) \) is an edge of a simplex in \( \mathcal{T}_P \).

We formally decompose a triangulation \( \mathcal{T}_P \) into an initial clique \( (K_3) \) and a set of actions \( A = \{a_1, a_2 \ldots a_m\} \), where an action \( a_i \) consists of adding a node and connecting it to two adjacent nodes, or two connected external nodes. A node is called external if it is not strictly inside the convex hull of a cycle in the graph.

**Lemma 5:** A triangulation can be constructed recursively by taking a simplex (triangle), adding a node to the exterior of the graph and connecting it to two adjacent nodes such that the new edges do not cross any existing edges or by adding an edge between two external nodes.
Definition 7: In a triangulation graph, adjacent triangles are two triangles which share a common edge. A virtual edge between two vertices \(i\) and \(j\) exists when \(i\) and \(j\) belong to adjacent triangles, but \((i, j) \notin E\). The set of virtual edges between sensors is denoted by \(E_v\), and between sensors and anchors is denoted by \(E_a\).

Theorem 2: If the objective function of the SDP formulation maximizes the sum of the lengths of virtual edges in a generic triangulation graph, the solution has rank \(d\) (i.e., it is exact).

Proof: The theorem is clearly true for \(n = 3\) (for \(n\) the number of nodes), hence the base case holds. We assume it is true for any triangulation graph with \(n\) nodes and we want to show that it is also true for graphs with \(n + 1\) nodes. First, we look at the SDP relaxation and its dual problem.

Primal SDP with objective

\[
\begin{align*}
\text{max} & \quad \sum_{(k,j) \in E_v} (a_k; -e_j)(a_k; -e_j)^T \cdot Z \\
& \quad + \sum_{(i,j) \in E_a} (e_i; e_j)(e_i; e_j)^T \cdot Z \\
\text{s.t.} & \quad Z(1:d, :d) = I_d \\
& \quad (e_i; e_j)(e_i; e_j)^T \cdot Z = d_{ij}^2, \forall (i,j) \in E \\
& \quad (a_k; -e_j)(a_k; -e_j)^T \cdot Z = d_{k,j}^2, \forall (k,j) \in \bar{E} \\
& \quad Z \succeq 0
\end{align*}
\]

Dual of SDP with objective

\[
\begin{align*}
\text{minimize} & \quad I \cdot V + \sum_{(i,j) \in E} y_{i,j}d_{i,j}^2 \\
& \quad + \sum_{(i,j) \in E} w_{k,j}d_{k,j}^2 \\
\text{subject to} & \quad U = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \\
& \quad + \sum_{(i,j) \in E} (e_i; e_j)(e_i; e_j)^T \\
& \quad + \sum_{(i,j) \in E} w_{k,j}(a_k; -e_j)(a_k; -e_j)^T \\
& \quad - \sum_{(i,j) \in E_v} (e_i; e_j)(e_i; e_j)^T \\
& \quad - \sum_{(i,j) \in E_v} (a_k; -e_j)(a_k; -e_j)^T \\
& \quad U \succeq 0
\end{align*}
\]

Let \(X^n\) be the correct location of nodes, where superscript \(n\) represents the number of nodes. Then \(Z^n = \left(\begin{array}{cccc} I_d \\ X^n & (X^n)^T & X^n \end{array}\right)\) is a feasible solution to the primal SDP. If we find a feasible solution to the dual \(U^n\), such that \(U^n Z^n = 0\), then \(Z^n\) is optimal. Moreover, if we find \(U^n\) with rank \(n\), and \(U^n Z^n = 0\), then all optimal solutions of the primal have rank at most equal to \(d\). Let \(U^n = \left(\begin{array}{ccc} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{array}\right)\), where \(U_{11} \in \mathbb{R}^{d \times d}\) and \(U_{22} \in \mathbb{R}^{n \times n}\). By the induction assumption, since \(U^n\) is optimal and has rank \(n\), we know \(U_{22} \succeq 0\) and \(U^n Z^n = 0\).

We decompose the triangulation graph into an initial simplex \(K_3\), and actions \(A = \{a_1, a_2, \ldots, a_m\}\). Without loss of generality, we assume the nodes in the first triangle are anchor nodes and let the last node added to the graph be \(x_{n+1}\).

The complementarity condition \(U^n Z^n = 0\) means the elements of \(U^n\) represent a stress on each edge such that the total force at all non-anchor nodes is zero (assuming, without loss of generality, a stress of \(-1\) on all virtual edges). For example, assume in Figure 35 we have a dual slack matrix \(U^n\) on nodes 1-7, and we add node 8. Nodes \((2, 4, 6, 8)\) make a clique (when including the virtual edge between 4 and 8). Thus, the sub-graph \((2, 4, 6, 8)\) is universally rigid and its stress matrix is positive semi-definite with rank 1 \([19]\).

In the general case, take \(x_{n+1}\), the last node added to the graph. Consider the new triangle created by adding \(x_{n+1}\), its adjacent triangle and the virtual edge, which
construct a 4-clique. Let $\Omega_0$ be the corresponding PSD stress matrix (with rank at least 1). We assume that $\Omega_0$ is normalized such that the stress associated with the virtual edge is $-1$. Let $i$ and $j$ be the nodes to which $x_{n+1}$ is connected, and let $k$ be the adjacent node of the virtual edge. Define $\Omega_0$ to be the equilibrium matrix of the edges and virtual edge added with $x_{n+1}$.

$$
\Omega_0 = \begin{pmatrix}
-1+y_{ik}+y_{kj} & -y_{ik} & -y_{ik} & 1 \\
-y_{ik} & -y_{ik} & -y_{ik} & -y_{ik} \\
-y_{ik} & -y_{ik} & -y_{ik} & -y_{ik} \\
-1+y_{ik}+y_{kj}+y_{i,n+1} & -y_{ik} & -y_{ik} & -y_{ik} \\
\end{pmatrix}
$$

Note that it is easy to show that $(-1+y_{i,n+1}+y_{j,n+1}) > 0$. Now, consider the updated stress matrix

$$
U_{22}^{n+1} = [U_{22}^{n} 0] + \Omega
$$

where $\Omega \in \mathbb{R}^{(n+1) \times (n+1)}$ is the stress matrix of the new edges, i.e., $\Omega([i,j,k,n+1],[i,j,k,n+1]) = \Omega_0$.

Since $U_{22}^{n+1}$ is an equilibrium matrix for the updated graph, the complementarity condition must hold, i.e. $U_{22}^{n+1} Z^{n+1} = 0$. Moreover, since $\Omega \succeq 0$ and $U_{22}^{n} \succeq 0$, it must hold that $U_{22}^{n+1} \succeq 0$. It is just left to show that $U_{22}^{n+1} \succ 0$, which will imply that $\text{rank}(U_{22}^{n+1}) = n$.

Let’s assume it’s not true, i.e. there is a vector $z$ such that $z^T U_{22}^{n+1} z = 0$, then

$$
z^T [U_{22}^{n} 0] z + z^T \Omega z = 0
$$

which only holds if $z^T [U_{22}^{n} 0] z = 0$ and $z^T \Omega z = 0$.

Since $U_{22}^{n} \succ 0$, this means that the first $n$ elements of $z$ are zero, i.e., $z_{(1:n)} = 0$. Thus,

$$
z^T \Omega z = z_{(n+1)}^2 \Omega_{n+1} = z_{(n+1)}^2 (-1+y_{i,n+1}+y_{j,n+1}) = 0
$$

which implies $z_{(n+1)} = 0$ since $(-1+y_{i,n+1}+y_{j,n+1}) > 0$. Therefore, $z^T U_{22}^{n+1} z = 0$ if and only if $z = 0$, implying $U_{22}^{n+1} \succ 0$ and $\text{rank}(U_{22}^{n+1}) = n$. ■

IV. HEURISTIC OBJECTIVE FUNCTION

The standard SDP formulation of the sensor network localization problem does not have an objective function. We tested a number of heuristic objective functions on a large number of random sensor networks to determine the best objective function. The following heuristics are used:

1. (LSM) Initially maximize the sum of all non-edges, and do the following until either (i) the total error in edge lengths is sufficiently small, or (ii) the maximum number of iterations has been reached: (a) Randomly choose an edge and minimize its length in the objective function (while maximizing all others) (b) If this results in a smaller total error in edge lengths, then minimize this edge length in the objective function; otherwise, maximize it.

2. (IET) For each non-edge, run two SDP localizations: (a) Maximize all edge lengths, (b) Maximize all edge lengths, except the length of the given edge, which is minimized. If the second method results in a small error, then minimize the length of this non-edge in the objective function; otherwise, maximize it.

3. (WEM) Run the SDP localization with no objective function to find a set of locations, $\hat{X}$. Then, find the edge $(i, j)$ with maximum error. Run the SDP localizations two more times, with the objective functions of maximizing and minimizing the length of this edge from either $i$ or $j$ to an anchor; choose to maximize (minimize) this distance in the objective function if maximizing (minimizing) the distance from $i$ or $j$ to an anchor resulted in less error.

4. (MAX) Maximize the sum of all the non-edge lengths

5. (ZERO) No objective function

The plots in Figures 3a and 3b show the results of each heuristic method. As we can see, LSM gave by far the best results, however took much longer than (MAX) and (ZERO).

V. CONCLUSION

In this paper, we find a non-asymptotic bound for the connectivity range of unit disk graphs such that the resulting graph is uniquely localizable. We also prove that for an appropriate objective function in triangulation graphs, the SDP relaxation guarantees exact localization.

REFERENCES


Fig. 3: An example of an area that may not have a $(d+1)$-lateration.
section, under the above assumptions in Lemmas 3 and 4, the ordering will continue across the narrow mid-section to the lower section; this follows almost directly from Lemma 2.

Therefore, if the minimum distance between boundaries of $\mathcal{H}$ is at least $3\ell$, the analyses of a network in $\mathcal{H}$ can be generalized to that of a unit hypercube in $\mathbb{R}^d$. 