On solving univariate sparse polynomials in logarithmic time

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Abstract

Let \( f \) be a degree \( D \) univariate polynomial with real coefficients and exactly \( m \) monomial terms. We show that in the special case \( m = 3 \) we can approximate within \( \varepsilon \) all the roots of \( f \) in the interval \([0, R]\) using just \( O(\log(D)\log(D \log \frac{R}{\varepsilon})) \) arithmetic operations. In particular, we can count the number of roots in any bounded interval using just \( O(\log^2 D) \) arithmetic operations. Our speed-ups are significant and near-optimal: The asymptotically sharpest previous complexity upper bounds for both problems were super-linear in \( D \), while our algorithm has complexity close to the respective complexity lower bounds. We also discuss conditions under which our algorithms can be extended to general \( m \), and a connection to a real analogue of Smale’s 17th Problem.

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1. Introduction

Real-solving—the study of solving systems of polynomial equations over the real numbers—occupies a curious position within computational algebraic geometry. From the point of view of computational complexity, classical algebraic geometry has left real-solving almost completely untouched. For example, rigorous lower bounds for the arithmetic complexity of finding approximations to the roots of polynomial systems did not appear until the work of Renegar in the late 1980s [Ren87,Ren89], and finding optimal bounds continues to be the subject of much active research [BMST97,BCSS98,MP99,Roj99b,GLS01,MPR03].

As for counting real roots, there is a beautiful result that one can bound their number independently of the degrees of the underlying polynomials: Askold Khovanski proved an explicit upper bound singly exponential in the number of variables and the total number of monomial terms [Kho91]. While this fewnomial bound is far from optimal in higher dimensions [LRW03], it is significantly smaller than the number of complex roots when the underlying polynomials have sufficiently high degree. (Sparse polynomials are sometimes also known as lacunary polynomials and, over \( \mathbb{R} \), are a special case of fewnomials—a more general class of analytic functions of parametrized complexity [Kho91].)

We are then naturally lead to suspect that a similar improvement is possible for the harder problem of approximating the real roots. So can one solve sparse polynomial systems over the real numbers significantly faster than via the usual algorithms based on complex algebraic geometry? The existence of general speed-ups of this nature is still an open problem, even in the univariate case: For example, until the present paper, it was still unknown whether the real roots of a univariate trinomial of degree \( D \) could be approximated within a number of arithmetic operations sub-linear in \( D \) [MP99,Roj99b,GLS01,MPR03].

We now answer this last question affirmatively as follows: Let \( I \subseteq \mathbb{C} \) be any subset, \( f \in \mathbb{C}[x_1] \) a degree \( D \) polynomial, and suppose \( \zeta_1, \ldots, \zeta_m \) are all the distinct roots of \( f \) in the region \( I \). By \( \varepsilon \)-approximating the roots of \( f \) in \( I \) we will mean finding complex numbers \( z_1, \ldots, z_m \in I \) such that for any \( i \in \{1, \ldots, m\} \), we have \( |\zeta_i - z_i| < \varepsilon \) for some \( j \in \{1, \ldots, m\} \). (So \( \varepsilon \)-approximating in \( I \) implies a correct root count in \( I \) as well). When \( I \subseteq \mathbb{R} \) and \( f \in \mathbb{R}[x_1] \), we will further stipulate that the \( z_i \) all lie in \( \mathbb{R} \).

**Theorem 1.** Let \( R, \varepsilon > 0 \) and suppose \( f \in \mathbb{R}[x_1]\backslash\{0\} \) has degree \( D \) and at most 3 monomial terms. Then we can \( \varepsilon \)-approximate all the roots of \( f \) in the closed interval \([0, R]\) using just \( O(\log(D)\log(D \log \frac{R}{\varepsilon})) \) arithmetic operations. In particular, we can count exactly the number of roots of \( f \) in any bounded interval using just \( O(\log^2 D) \) arithmetic operations.

As is standard, we count arithmetic operations as field operations in the field over \( \mathbb{Q} \) generated by the coefficients of \( f \). Note also that our underlying algorithm handles degenerate roots (i.e., roots of multiplicity > 1) with no difficulty.
Remark 1. Throughout this paper, all $O$-constants and $\Omega$-constants are absolute and effectively computable.

Remark 2. Our speed-ups are significant: The current asymptotically sharpest (sequential worst-case) arithmetic complexity upper bound for $\varepsilon$-approximating the roots of a general degree $D$ univariate polynomial in the open disc $\{z \mid |z| \leq R\}$ is $O(D \log^5(D) \log \log \frac{R}{\varepsilon})$ [BP94,NR96]. The analogous upper bound for counting real roots in an interval is $O(D \log^2(D) \log \log D)$, via the technique of Sylvester–Habicht sequences [Roy96,LM01]. In particular, no sharper upper bounds were known before for the problems considered in Theorem 1 above.

Remark 3. Our speed-ups are also near-optimal: For any fixed $D,R \geq 2$, our $\varepsilon$-approximation algorithm matches (up to an asymptotically constant multiple) the $\Omega(\log \log \frac{1}{\varepsilon})$ arithmetic complexity lower bound known for $\varepsilon$-approximating the roots of $x^2 - N$ where $1 \leq N \leq 2$ [BMST97]. As for counting the roots, there do not appear to be any explicit lower bounds known for the arithmetic complexity of counting the real roots of $m$-nomials. However, one should note that the arithmetic complexity of just evaluating a degree $D$ monomial is $\Omega(\log D)$ in the worst case [dMS96,Mor97].

Our algorithms are based on an earlier hybrid algorithm of Y. Ye which combines bisection and Newton iteration, a new observation on the Sturm sequences of trinomials (Theorem 3 of Section 1.2), and some analytic estimates (Theorem 5 of Section 2). We in fact give a more general algorithm that applies to certain univariate $m$-nomials and implies a deeper complexity result in Theorem 6 of Section 2. However, for $m \geq 4$, there are two main obstructions to showing that our more general algorithm has complexity sub-linear in $D$. We refer to these obstructions as Problems A and B, and describe the first now.

Problem A. Is there an absolute constant $\kappa$ such that one can count exactly the number of real roots (in any input interval) of arbitrary $m$-nomials of degree $D$ using, say, just $O(\log^{\kappa m} D)$ arithmetic operations?

One may note that we have hedged our bets in Problem A by letting the complexity bound increase exponentially in $m$. This is motivated by the NP-hardness of the multivariate analogue of Problem A.

Proposition 1. Suppose $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is an $m$-nomial and we measure the size of $f$ as the total number of bits necessary to write the binary expansions of all the coefficients and exponents of $f$. Then it is NP-hard (in the classical Turing sense, using the preceding notion of size) to decide whether $f$ has a real root or not.

The proof follows immediately from a standard reduction of the special case of $n$-variate polynomials of degree 6 to 3-SAT (in conjunctive normal form) [GJ79]. So NP-hardness starts at $m \geq \Omega(n^6)$, if not earlier. Via a more intricate argument, one
can show that NP-hardness starts at $m \geq 6n + 6$, if not earlier [RS04]. To obtain the coarser lower bound, one respectively substitutes 0, 1, $1 - x$, $x + y - xy$, and a new equation, for each False, True, $\neg x$, $x \lor y$, and conjunction in a conjunctive normal form. By adding equations of the form $x(1 - x) = 0$ to force the variables to be 0-1, one can thus change any 3-SAT instance into a real system of equations of degree $\leq 3$. Summing the squares of the equations, one then obtains a single degree 6 polynomial whose feasibility is equivalent to the existence of a satisfying assignment for one’s original 3-SAT instance.

So root counting is subtle for $n$-variate $m$-nomials, and this persists even for fixed $n$: We present some examples in Section 1.2 revealing that the classical approach of Sturm sequences will most likely need to be abandoned for $n = 1$ and $m > 3$. Nevertheless, speed-ups for $m$ fixed still appear possible and are of considerable practical interest for large $D$.

Noting that $\varepsilon$-approximation extends naturally to polynomial systems (by $\varepsilon$-approximating each coordinate separately), a consequence of our univariate trinomial algorithm is the following result which may be of use for solving general pairs of bivariate trinomials.

**Theorem 2.** Suppose $f \in \mathbb{R}[x, y]$ has exactly 3 monomial terms, $D$ is the degree of $f$, and $\delta > 0$ is any constant. Then, using just $O(\log(D) \log(\log(D)))$ arithmetic operations and $O(\log^{2+\delta} D)$ bit operations, we can $\varepsilon$-approximate all isolated inflection points, vertical tangents, and singular points of the curve $\{(x, y) \in [0, R]^2 | f(x, y) = 0\}$.

The asymptotically sharpest (sequential worst-case) arithmetic complexity upper bound for $\varepsilon$-approximating the roots of a system of $n$ polynomial equations in $n$ variables are at best polynomial in $\log \log R$ and a quantity sometimes attaining $D^n$ (see, e.g., [MP99,Roj99b,GLS01,MPR03]). The importance of Theorem 2 lies in that it is a first step toward a practical two-dimensional analogue of bisection. The latter in turn is a first step toward generalizing our algorithms here to higher dimensions.

We leave the bit complexity of our algorithms for a future paper. In particular, unless stated otherwise, our underlying model of computation here is the BSS model (with inequality) over $\mathbb{R}$ [BSS89,BCSS98]. Those unfamiliar with this model can simply think of such a machine as one’s favorite lap-top computer, augmented with unlimited memory, a flawless operating system, and additional registers that allow arithmetic and inequality checking with real numbers as well as bits. (Strictly speaking, a BSS machine is different—but still polynomial-time equivalent to—this simplified model.) However, let us touch upon a deeper question: The connection of our work to Smale’s notion of *approximate roots* (see, e.g., [Sma86,BCSS98] and Section 2 below).

**Definition 1.** Suppose $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$, $F := (f_1, \ldots, f_n)$, and the total number of distinct exponent vectors in the monomial term expansions of $f_1, \ldots, f_n$ is $m$. We then call $F$ a *real m-sparse n x n polynomial system*. 
Definition 2. Let $B \subseteq \mathbb{C}^n$ be any open ball and $F : B \to \mathbb{C}^n$ an analytic function. Also, given any $z_0 \in B$, let us define the sequence $(z_i)_{i=0}^\infty$ by $z_i := z_{i-1} - \frac{1}{i} F(z_{i-1})$ for all $i \in \mathbb{N}$. Then, if there is a root $\zeta \in B$ of $F$ such that $|z_{i+1} - \zeta| \leq 8^2 |z_0 - \zeta|$ for all $i$ (i.e., if Newton iteration for $F$, starting at $z_0$, converges quadratically to a root of $F$), we call $z_0$ an approximate root of $F$ with associated root $\zeta$.

Real Analogue of Smale’s 17th Problem. For fixed $n$, can all real roots of an $m$-sparse $n \times n$ polynomial system be found approximately, on the average, in polynomial time with a uniform algorithm? More precisely, let $F$ be an $m$-sparse $n \times n$ polynomial system with maximal exponent $D$ and coefficients that are, say, independent standard real Gaussian random variables. Is there a uniform algorithm that finds a set of approximate roots close to all the real roots of $F$, with average-case arithmetic complexity $O((m \log D)^k)$ for some constant $k$ depending only on $n$?

The polynomial system $((x_1 - 1)(x_1 - 2), \ldots, (x_n - 1)(x_n - 2))$ clearly shows that fixing $n$ is necessary in our real analogue above. The appellation uniform merely emphasizes that there be a single algorithm, with all steps explicit and constructive, which works for all inputs [BCSS98].

The original statement of Smale’s 17th Problem [Sma98] (see also [Sma00, p. 287]) differs from our analogue above as follows: (1) $n$ is allowed to vary (so one seeks an absolute constant $v$), (2) one averages over choices of complex coefficients, and (3) one instead asks for a single complex approximate root. Smale also left the underlying probability distribution unspecified. We observe that a positive answer to the following variant of Problem A would be quite useful in the direction of our real analogue:

Stochastic Version of Problem A: Suppose $f$ is a univariate $m$-nomial of degree $\leq D$ with coefficients that are real standard Gaussian random variables. Is there a uniform algorithm that counts exactly the number of roots of $f$ (in any input interval), with average-case arithmetic complexity $O((m \log D)^k)$ for some absolute constant $k$?

Smale’s original 17th Problem remains unsolved, although a partial affirmative answer (an algorithm containing a non-constructive step which may be called many times) was found by Shub and Smale in the mid-1990s [SS94, BCSS98].

1.1. Earlier work on approximating roots

When can we solve a polynomial system in time polylogarithmic in the degree of the underlying complex algebraic set? An affirmative answer is trivial for the special case of a single polynomial with $\leq 1$ monomial term, and was known at least since the mid-1970s for the (univariate) binomial case, e.g., [Bre76; Ye94, Section 4]. On the other hand, little seems to be known about the case of 3 or more monomial terms: To the best of the authors’ knowledge, the only result close in spirit to Theorem 1 is a result of Daniel Richardson [Ric93] implying that the arithmetic complexity of counting the number of real roots of $c_1 + c_2 x^d + c_3 x^D$ (with $0 < d < D$) is polynomial in $d$ and $\log D$. 
Not much else seems to be known about the intrinsic complexity of real solving, or even real root counting, for univariate polynomials with 4 or more monomial terms. One known speed-up over the usual univariate algorithms over \( \mathbb{C} \) is [Roj00a, Main Theorem 1.2] which gives an arithmetic complexity bound of \( O(m \log(D) \log(D \log \frac{K}{\varepsilon})) \) for \( \varepsilon \)-approximating the roots of

\[
c_1 x^{a_1} + \cdots + c_k x^{a_k} - c_{k+1} x^{a_{k+1}} - \cdots - c_m x^{a_m},
\]
when \( 0 \leq a_1 \leq \cdots \leq a_m = D \) are integers and all the \( c_i \) are positive real numbers.

It is also not difficult via the results above to construct various systems of multivariate trinomials which admit super-fast solving in the sense of Theorem 1 as well. For example, if one has \( f_1 \in \mathbb{R}[x_1], f_2 \in \mathbb{R}[x_1, x_2], \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n] \), and all the \( f_i \) are trinomials of degree \( \leq D \), then to solve \( F := (f_1, \ldots, f_n) \) one can simply solve \( f_1 \) first and then recursively solve the resulting smaller system. Letting \( N_F \) denote the number of roots in the nonnegative orthant of such an \( F \), it is easily checked that \( N_F \leq 2^n \) (see, e.g., [LRW03, Theorem 3, Part (c)]). We can then easily derive, via Theorem 1, an arithmetic complexity upper bound of \( O(N_F^{\log 2}(D) \log \log \frac{K}{\varepsilon}) \) for the more general problem where we instead \( \varepsilon \)-approximate all the roots of \( F \) in the orthant-wedge

\[
W_R^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq R^2, \quad x_i \geq 0 \text{ for all } i \}.
\]

Extending these results to general trinomial systems, not to mention general sparse systems, remains an open problem. Nevertheless, for a general binomial system \( F \) with exactly \( D < \infty \) roots in \( \mathbb{C}^n \) (counting multiplicities), one can \( \varepsilon \)-approximate all its roots in \( W_R^n \) using just \( O((\log D)(B^3 \log^2(n) + \log \log \frac{K}{\varepsilon})) \) arithmetic operations and \( O(n^3 B^{2+\delta} \log^{2+\delta} n) \) bit operations, where \( B \) is the total number of bits needed to write down the exponents of \( F \) and \( \delta > 0 \) is an arbitrary constant [Roj00a, Main Theorem 1.3]. (The bound in [Roj00a] is written in a slightly different manner and we have here taken the liberty of improving the bit complexity portion by employing a recent result of van der Kallen on computing the Hermite normal form of an integral matrix [vdK00]).

1.2. A new algebraic observation and earlier work on counting roots

As for merely counting the real roots, there are still surprising gaps in our knowledge. For instance, the main general algebraic techniques for real root counting—Sturm–Habicht sequences [Stu35,Hab48,Roy96,LM01] and Hermite’s Method [Her56,PRS93,Roy96]—originated in the 19th century and are still hard to improve upon.

**Definition 3.** For any sequence of real numbers \( s := (s_1, \ldots, s_k) \), the *number of sign alternations of \( s \)*, \( N_s \), is the number of pairs \((j, j')\) with \( 1 \leq j < j' \leq k \), \( s_j s_{j'} < 0 \), and \( s_i = 0 \) for all \( i \in \{j + 1, \ldots, j' - 1\} \). Also, for any polynomial \( f \in \mathbb{R}[x_1] \), define the sequence \((p_i)_{i=0}^{\infty}\) where \( p_0 := f \), \( p_1 := f' \), \( p_{i+2} := q_i p_{i+1} - p_i \), \( q_i \) is the quotient of \( \frac{p_{i+1}}{p_i} \), and \( p_K \) is
the last element of the sequence not identically equal to 0. We then call \((p_0, \ldots, p_K)\) the Sturm sequence of \(f\).

**Lemma 1** (Sturm [Stu35], Roy [Roy96]). Following the notation of Definition 3, let \(a, b \in \mathbb{R}\) with \(a \leq b\), \(A := (p_0(a), \ldots, p_K(a))\), and \(B := (p_0(b), \ldots, p_K(b))\). Then the number of distinct roots of \(f\) in the open interval \((a, b)\) is exactly \(N_A - N_B\). In particular, \([a_1 < \cdots < a_m\) and \(c_1, \ldots, c_m \in \mathbb{R}] \Rightarrow c_1x^{d_1} + \cdots + c_mx^{d_m}\) has no more than \(N(c_1, \ldots, c_m) \leq m - 1\) positive roots.

**Remark 4.** The latter part of the above lemma is Descartes’ Rule of Signs which dates back to the famous philosopher’s 1637 book La Geometrie (see also [SL54, p. 160]).

The following observation on the Sturm sequence of a trinomial, which we use in proving Theorem 1, may be of independent interest.

**Theorem 3.** Following the notation above, suppose \(f\) has at most 3 monomial terms. Then \(K \leq 3 \lceil \log_2 D \rceil + 2\) and \((p_1, \ldots, p_K)\) consists solely of binomials, monomials, and/or constants. In particular, the entire Sturm sequence of \(f\) can be evaluated at any real number using just \(O(\log^2 D)\) arithmetic operations.

Extending our last theorem to polynomials with more monomials appears unlikely: First, note that the quotient of the division of two binomials can be quite non-sparse, e.g., \(\frac{x^{d_1} - 1}{x - 1} = x^{d_1} + \cdots + x + 1\). So the expansion of any intermediate quotients must be avoided. Furthermore, the tetranomial case already gives some indication that Sturm sequences may in fact have to be completely abandoned: the following example shows that, for all \(D \geq 2\), the fourth element of the Sturm sequence of a degree 2\(D\) tetranomial can already be a \((D + 1)\)-nomial.

**Example 1.** Consider the tetranomial \(p_0(x) := x^{2D} + x^{D+1} + x^D + 1\) and its derivative \(p_1 := p_0' = 2Dx^{2D-1} + (D + 1)x^D + Dx^{D-1}\). The resulting Sturm sequence then continues with \(p_2 = \frac{D-1}{2D} x^{D+1} - \frac{1}{2} x^D - 1\) and from here it is easy to see (by a writing a simple recursion for the resulting long division) that the quotient \(q_3\) of \(\frac{p_1}{p_2}\) is a polynomial of degree \(D-2\) with exactly \(D - 1\) monomial terms. Thus, \(p_3 := q_3p_2 - p_1\) has degree \(D\) and at least \(D + 1\) monomial terms.

We are also willing to conjecture that the maximal length of the Sturm sequence of a degree 2\(D\) tetranomial is \(\Omega(D)\). (Maple experiments have verified this up to \(D = 150\).) Nevertheless, while the behavior of tetranomials is thus more complicated, this example need not rule out a more clever method to circumvent these difficulties.

### 1.3. A connection to discriminants

If one insists on relying on Sturm sequences then one is naturally lead to the univariate sparse discriminant. Briefly, given an \(n\)-variate \(m\)-nomial \(f\) with
indeterminate coefficient vector $C$ and exponents contained in $A \subset \mathbb{Z}^n$, its *sparse discriminant* (or *A-discriminant*), $\Delta_A(f)$, is the unique (up to sign) irreducible polynomial in $\mathbb{Z}[C]\setminus\{0\}$ of lowest degree which vanishes whenever $C$ is specialized so that $f$ has a root in $(\mathbb{C}\setminus\{0\})^n$ in common with $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ [GKZ94]. It is not hard to see that if (a) $n = 1$, (b) the coefficients of $f$ are all constants, and (c) $A$ is exactly the set of exponent vectors of $f$, then we have $\Delta_A(f) = 0 \iff K = 1$ in the Sturm sequence $(p_0, \ldots, p_K)$ of $f$. So the following is clear.

**Proposition 2.** Suppose $f \in \mathbb{R}[x]$ has set of exponent vectors $A$ and one can compute the signs of the Sturm sequence of $f$ evaluated at any $r \in \mathbb{R}$ using just $T(A)$ arithmetic operations. Then one can decide the vanishing of $\Delta_A(f)$ using just $T(A)$ arithmetic operations.

It would be quite enlightening to understand the converse of Proposition 2 and its possible obstructions.

Perhaps more than coincidentally—recalling Theorem 1—$\Delta_A(f)$ can be computed in polynomial time when $f$ is a trinomial: up to sign, the formula is simply

$$\Delta_{\{a_2, a_3\}}(c_1 + c_2 x^{a_2} + c_3 x^{a_3}) = a_2^n c_3^2 a_3^{-1} - (a_2^n - 1) a_3^{-1} (a_2 - a_3)^{a_3 - a_2} c_2 c_3^{a_1}$$

when $a_2$ and $a_3$ are relative prime and $0 < a_2 < a_3$ (see [GKZ94, p. 406] and Remark 7 of Section 3 below). Whether polynomial time complexity persists for general monomials is already non-trivial for the tetranomial case since the underlying discriminants become much more complicated. For example, up to sign, $\Delta_{\{0,6,10,31\}}(c_1 + c_2 x^6 + c_3 x^{10} + c_4 x^{31})$ is the following homogeneous 21-nomial of degree 35:

$-21^{10} 3^{21} 5^{10} 7^{21} c_1^4 c_3^{31} - 2^{13} 3^{24} 5^{5} 7^{21} c_1^2 c_2^5 c_3^{28} - 2^{14} 3^{27} 7^{21} c_2^1 c_3^{25}$

$+ 2^8 3^{15} 5^{6} 7^{16} \cdot 19 \cdot 31^{6} 11731 c_1^7 c_2^3 c_3^{22} c_4^2$

$- 2^8 3^{13} \cdot 5 \cdot 7^{13} \cdot 19 \cdot 31^4 \cdot 4931 \cdot 11924843839 c_1^5 c_2^8 c_3^{20} c_4^2$

$- 2^6 3^{13} 5^{6} 7^{13} 31^{13} c_1^{12} c_2 c_3^{18} c_4^4 - 2^8 3^{16} 5^{12} 7^{13} \cdot 17 \cdot 31^2 \cdot 3629537 c_1^3 c_2^{13} c_3^{17} c_4^2$

$- 2^6 3^9 \cdot 5 \cdot 7^{11} \cdot 29 \cdot 31^{11} \cdot 6361 \cdot 7477163 c_1^{10} c_2^6 c_3^{15} c_4^4$

$- 2^10 3^{19} 5^{22} 7^{13} c_1 c_2^{18} c_3^{14} c_4^2 - 2^6 3^{8} 5^{12} 7^{31} c_1^{10} c_2^{11} c_3^{12} c_4^2$

$- 2^{2} 7^{8} \cdot 5 \cdot 7^{31} 18 \cdot 3327253 c_1^{15} c_2^{10} c_3^{16} c_4^9 - 2^6 3^{4} 5^{22} 7^{4} 31^{7} 160730667473 c_1^6 c_2^{16} c_3^{9} c_4^4$

$+ 2^3 3^{5} 5^{12} 7^{4} \cdot 11 \cdot 29 \cdot 31^{16} \cdot 233 \cdot 1559 c_1^{13} c_2^9 c_3^{7} c_4^6$

$- 2^6 3^{32} 31^5 \cdot 248552699041 c_1^4 c_2^{21} c_3^{6} c_4^4 - 2^2 3^{4} 5^{7} \cdot 11 \cdot 29 \cdot 31^{25} c_1^{20} c_2^2 c_3^5 c_4^8$

$- 2^5 \cdot 35^{22} 31^{14} \cdot 113 \cdot 317 \cdot 461 c_1^{11} c_2^{14} c_3^{4} c_4^6$
whose largest coefficient has 47 digits. So we pose the following problem:

**Univariate Discriminant Complexity Problem.** Given any finite subset $A \subseteq \mathbb{Z}$, define $s(A)$ to be the total number of bits needed to write the binary expansions of all the points of $A$. Can one decide the vanishing of $A$-discriminants (for, say, specializations of the coefficients in $\mathbb{C}$) using a number of arithmetic operations polynomial in $s(A)$?

This highlights an embarrassing gap in what is known about the complexity of discriminant computation: it is known that deciding the vanishing of sparse discriminants for bivariate polynomials can be done in polynomial time on a Turing machine (resp. BSS machine over $\mathbb{C}$) only if $\text{NP} \subseteq \text{P}$ (resp. $\text{NP} \subseteq \text{BPP}$) [Roj00a, Main Theorem 1.4], but no such hardness result is known for the univariate case. The preceding inclusions (see [Pap95] for a beautiful introduction to complexity theory) are currently considered quite unlikely. On the positive side, it is known that $A$-discriminants (for arbitrary finite $A \subseteq \mathbb{Z}^n$) can be evaluated using a number of arithmetic operations polynomial in $s(A)$ when $A$ has $n + 1$ or fewer points [GKZ94, Proposition 1.8, pp. 274–275]. Also, the vanishing of $A$-discriminants can be decided within the complexity class $\text{P} \cap \text{NP}$ under a number-theoretic hypothesis strictly weaker than the Generalized Riemann Hypothesis [Roj04a].

2. **Speed-ups through a variant of $\mathcal{A}$-theory**

Checking whether a given point is an approximate root of a given polynomial can be done quite efficiently, thanks to the seminal work of Smale [Sma86]. Let us now formalize a refined version of this fact.

**Definition 4.** For any analytic function $f : \mathbb{C} \to \mathbb{C}$, let $\gamma(f, x) := \sup_{k \geq 2} \frac{|f^{(k)}(x)|}{k! |f(x)|^{k-1}}$.

**Remark 5.** It is worth noting that $1/\gamma(f, x_0)$ is a lower bound for the radius of convergence of the Taylor series of $f$ about $x_0$, so $\gamma(f, x_0)$ is finite whenever $f$ is nonsingular at $x_0$ [BCSS98, Proposition 6, p. 167].

Recall that a function $g : U \to \mathbb{R}$, defined on some connected domain $U \subseteq \mathbb{R}$ is convex iff $g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y)$ for all $x, y \in U$ and $\lambda \in [0, 1]$.

**Theorem 4** (Ye [Ye94, Theorem 2]). Suppose $f$ is convex and analytic on the open interval $(0, R)$. Also let $r \in (0, R)$ be any root of $f$ and suppose that there is an $\varepsilon > 0$ such that...
that $x_\gamma(f, x) \leq 0$, for all $x$ in the closed interval $\left[\frac{-r}{1+\frac{1}{2z}}, \frac{-r}{1-\frac{1}{2z}}\right]$. Then for all $z \in (0, R)$:

(1) If $f$ is monotonically decreasing in $[z, r]$ and $r \in [z, (1 + \frac{1}{2z})z]$ then $z$ is an approximate root of $f$.
(2) If $f$ is monotonically increasing in $[r, z]$ and $r \in [(1 - \frac{1}{2z})z, z]$ then $z$ is an approximate root of $f$.

Proposition 3 [Ye94, Proposition 1] Suppose $f_1, f_2 : U \to \mathbb{R}$ satisfy the hypotheses of Theorem 4, with $\bar{z}_1$ and $\bar{z}_2$, respectively, playing the role of $\bar{z}$, and that $f_1$ and $f_2$ are either both increasing or both decreasing. Then $f_1 + f_2$ also satisfies the hypotheses of Theorem 4, with $\bar{z} = \max\{\bar{z}_1, \bar{z}_2\}$.

Proposition 4. For any $D \in \mathbb{R}\setminus\{0\}$ and $c \in \mathbb{R}$, the function $f : [0, \infty) \to \mathbb{R}$ defined by $x \mapsto x^D - c$ satisfies $x_\gamma(f, x) = \left|\frac{D-1}{2}\right|$ for all $x > 0$.

Y. Ye observed the special case $D \in \mathbb{Z}\setminus\{0\}$ earlier in [Ye94, Example 1]. The proof their applies to arbitrary real $D$ as well.

An additional technical result we will need is an analytic estimate which globalizes Theorem 4. But first let us observe an oscillation property we will need for our main algorithm.

Definition 5. For any $D, m \in \mathbb{N}$ define $\mathcal{F}(D, m)$ to be the family of real $t$-nomials $f$ of degree $\leq D$ with $t \leq m$. For any $f \in \mathcal{F}(D, m) \setminus \mathbb{R}$, let $u_0(f) < \cdots < u_{N(f)}(f)$ be the ordered sequence comprised of 0, the roots of $f'$ in $(0, \infty)$, and $\infty$. We then say that $f$ is dampened if for all $i \in \{0, \ldots, N(f) - 1\}$, $f'''$ has at most one root in $(u_i(f), u_{i+1}(f))$.

We of course have $\mathcal{F}(D_1, m_1) \subseteq \mathcal{F}(D_2, m_2)$ whenever $D_1 \leq D_2$ and $m_1 \leq m_2$. Rolle’s Theorem applied to $f'$ tells us that $(u_t(f'), u_{i+1}(f'))$ (for $i \in \{1, \ldots, N(f') - 1\}$) always contains at least one root of $f''$. It is then easy to show via Descartes’ Rule and a routine calculation that $m$-nomials are always dampened when $m \leq 4$.

The invariant defined below generalizes the local quantity maximized in Theorem 4 above, and thus helps enforce the accelerated convergence of Newton’s method in certain cases of interest.

Definition 6. For any $D, m \geq 2$, any dampened $f \in \mathcal{F}(D, m)$, and $i \in \{1, \ldots, N(f) - 2\}$, define $v_i(f)$ to be the unique root of $f''$ in $(u_i(f), u_{i+1}(f))$, $v_{-1}(f) = 0$, and $v_{N(f)}(f) = \infty$. Also define $v_0(f)$ to be $u_0(f)$ or the unique root of $f''$ in $(u_0(f), u_1(f))$, according as $f'''$ is non-vanishing in $(u_0(f), u_1(f))$ or not. Also define $v_{N(f)-1}(f)$ to be $\infty$ or the unique root of $f''$ in $(u_{N(f)}(f), u_{N(f)}(f))$, according as $f''$ is non-vanishing in $(u_{N(f)}(f), u_{N(f)}(f))$ or not. Finally,
define \( \tilde{z}(D, m) \) to be
\[
\sup_{f \in \mathcal{F}(D, m) \text{ dampened } f(0) \neq 0 \text{ and } f'(0) = 0} \sup_{i \in \{0, \ldots, N(f) - 1\}} \sup_{x \in \{v_{i-1}(f), v_i(f)\} \cup \{u_i(f)\}} |x - u_i(f)| \gamma(f, x).
\]

While \( \tilde{z}(D, m) \) may be hard to compute exactly, we can at least bound it above and below explicitly in some cases of interest.

**Theorem 5.** For any \( D, m \geq 2 \) we have \( \tilde{z}(D, m) \leq \tilde{z}(D, m + 1) \). In particular,
\[
\frac{D - 1}{2} = \tilde{z}(D, 2) \leq \tilde{z}(D, 3) \leq (D - 1)(D - 2)/2.
\]

One application of our \( \tilde{z} \)-invariant is the following generalization of Theorem 1.

**Definition 7.** For any \( m \)-nomial \( f \), let \( \delta(f) \) be the smallest exponent appearing in \( f \). Let us then define the operators \( S : \mathcal{F}(D, m) \to \mathcal{F}(D, m) \), \( L_1 : \mathcal{F}(D, m) \to \mathcal{F}(D - 1, m - 1) \), and \( L_2 : \mathcal{F}(D, m) \to \mathcal{F}(D - 2, m - 1) \) by \( S(f) := x^{-\delta(f)}f \), \( L_1(f) := \frac{d}{dx}S(f) \), and \( L_2(f) := \frac{d}{dx}S(f) \).

**Theorem 6.** Following the notation of Definition 6, suppose \( f(x) := c_1x^{d_1} + c_2x^{d_2} + \cdots + c_mx^{d_m} \) is a real \( m \)-nomial such that

(a) \( m \geq 2 \) and \( 0 \leq a_1 < \cdots < a_m \).

(b) \( a_{i+1} = a_i + 1 \) and \( a_{j+1} = a_j + 1 \) for some \( i \neq j \Rightarrow a_m - a_1 = 2 \).

(c) All the polynomials \( \{S \circ L_{e_1} \circ \cdots \circ L_{e_k}(f) \mid k \in \{0, \ldots, m - 1\}, \ e_i \in \{1, 2\}\} \) are dampened.

Then one can find \( \epsilon \)-approximations to all the roots of \( f \) in \([0, R]\) using just
\[
O\left(2^m \left\{ \log(D)\log\left(\tilde{z}(D, m)\log\frac{R}{\epsilon}\right) + m^2 \mathcal{X}(D, m) \right\} \right)
\]
arithmetic operations, where \( \mathcal{X}(D, m) \) is any upper bound on the worst-case arithmetic complexity of counting the roots of an arbitrary \( f \in \mathcal{F}(D, m) \) in an arbitrary interval. In particular, if \( f \in \mathcal{F}(D, 4) \) then the polynomials
\[
\{S(f), S(L_{e_1}(f)), S(L_{e_1}(L_{e_2}(f))), S(L_{e_1}(L_{e_2}(L_{e_3}(f)))) \mid e_i \in \{1, 2\}\}
\]
are all dampened, for any \( D \in \mathbb{N} \).

It is easily checked (recalling our observations after Definition 5) that all binomials, all trinomials, and all tetranomials save those of the form \( c_1x^d + c_2x^{d+1} + c_3x^{D+d} + c_4x^{D+d+1} \) satisfy the hypotheses of our complexity bound above. Theorem 6 thus opens up the possibility of super-fast \( m \)-nomial solving when \( m > 3 \) is fixed. In
particular, good upper bounds on $\mathcal{K}(D, m)$ and $\tilde{\sigma}(D, m)$ appear to be a fundamental first step. Our aforementioned Problem A addresses $\mathcal{K}(D, m)$ so consider the following new problem.

**Problem B.** Is there an absolute constant $\kappa'$ such that $\tilde{\sigma}(D, m) = O(\log^{\kappa' m} D)$?

An interesting related problem, in the spirit of [MR04], is whether one can bound from below the probability that a random $m$-nomial of degree $D$ (with $m \geq 4$ fixed) is dampened.

We state our underlying algorithms in the next section. We then prove Theorems 1, 6, and 3 in Section 4. We conclude with the proofs of Theorems 2 and 5 in Sections 5 and 6, respectively.

### 3. The algorithm and subroutines

The central algorithm we use to prove Theorems 1 and 6, \texttt{MNOMIALSOLVE}, is detailed below. Succinctly, the key idea behind \texttt{MNOMIALSOLVE} is to subdivide the input interval into sub-intervals on which $\pm f$ is convex and monotonic, along with some additional sub-intervals on which $f$ is less well-behaved (Fig. 1).

Then, using special properties of sparse polynomials, a suitable combination of bisection and Newton iteration (detailed in the subroutine \texttt{HYBRID}) yields $\varepsilon$-approximations to all the roots in $(0, R)$. (Checking whether 0 and/or $R$ are roots can be done simply by evaluating there.) Each approximation is guaranteed to correspond to its own unique root, and the roots in intervals on which $f$ behaves badly are certified via a subroutine called \texttt{FASTERCOUNT}. The aforementioned subroutines are described shortly after our main algorithm.

**Algorithm \texttt{MNOMIALSOLVE}**

\textit{Input} Real numbers $R$ and $\varepsilon$ with $0 < \varepsilon < R$, a real $m$-nomial $f(x) := c_1x^{m_1} + \cdots + c_mx^{m_m}$ satisfying the hypotheses of Theorem 6, and an upper bound $\tilde{\sigma}^*$ on $\tilde{\sigma}(D, m)$.

![Fig. 1. $[-R, R]$ can be expressed as a union of 6 intervals on which $\pm f$ is convex and monotonic, and 5 additional intervals of width $<2\varepsilon$ containing the roots of $f'f''$.](image-url)
**Output** A (possibly empty) multiset $Z \subseteq \mathbb{R}$, such that every root of $f$ in $[0, R]$ is $\varepsilon$-approximated by a unique $z_i \in Z$, and $\#Z$ is exactly the number of roots of $f$ in $[0, R]$.

**DESCRIPTION**

**Step 0** Set $Z := \emptyset$.

**Step 1** If $m = 0$ then set $Z := \mathbb{R}$ and STOP.

**Step 2** If $m = 1$ and $a_1 > 0$ then set $Z := 0$ and STOP.

**Step 3** If $m = 1$ and $a_1 = 0$ then STOP.

**Step 4** If $a_1 > 0$ then set $Z := 0$, $f := f/x^{a_1}$, $a_i := a_i - a_1$ for all $i \geq 2$, and then $a_1 := 0$.

**Step 5** If $a_m = 1$ then set $Z := -c_1/c_2$ or $0$, according as $c_1 c_2 \leq 0$ or not, then STOP.

**Step 6** If $a_m = 2$ then make the change of variables $x := x - \varepsilon a_2$.

**Step 7** If $f(R) = 0$ then append $R$ to $Z$.

**Step 8** If $m = 2$ and $f(0)f(R) \geq 0$, then STOP.

**Step 9** If $m = 2$ and $f(0)f(R) < 0$, find an $\varepsilon$-approximation $z$ to the unique root of $f$ in $(0, R)$ via subroutine HYBRID (with input $(\varepsilon, R, \pm f, z^*)$, where the sign is chosen so that $f$ is convex on $(0, R)$), set $Z := Z \cup z$, and STOP.

**Step 10** Otherwise, if $a_2 > 1$, find via algorithm MONOMIALSOLVE an ordered sequence $u_0 < \cdots < u_{k_1}$ comprised of $0$, a set of $\varepsilon$-approximations (of the correct cardinality) in $(0, R)$ for the roots of $f''$ in $(0, R)$, and $R$. Then GOTO Step 12.

**Step 11** If $a_2 = 1$ then redefine $f(x) := x^a f(1/x)$ and $\varepsilon := \min 1, \varepsilon a_2^3, 1/\varepsilon, 1/2\varepsilon^2$ and make the change of variables $x := 1/x$. GOTO Step 10.

**Step 12** Define $v_{k_1} := R$.

**Step 13** Using algorithm FASTERCOUNT, decide if $f''$ has a root in $[u_0 + \varepsilon, u_1 - \varepsilon]$. If so, let $L := u_1 - u_0 - 2\varepsilon$. Then, evaluating $f''$ at $u_0 + \varepsilon$ and $u_1 - \varepsilon$, and using algorithm HYBRID with input $(\varepsilon, L, \pm f, z^*)$ (the sign chosen so that $\pm f$ is convex on $(u_0 + \varepsilon, u_1 - \varepsilon)$), define $v_0$ to be the $\varepsilon$-approximation found for the unique root of $f''$ in $[u_0 + \varepsilon, u_1 - \varepsilon]$. Otherwise, define $v_0$ to be $u_1$.

**Step 14** Using algorithm FASTERCOUNT, decide if $f''$ has a root in $[u_{k_1 - 1} + \varepsilon, u_{k_1} - \varepsilon]$. If so, let $L := u_{k_1} - u_{k_1 - 1} - 2\varepsilon$, $g(x) := f(x + u_{k_1 - 1})$. Then, evaluating $f''$ at $u_{k_1 - 1} + \varepsilon$ and $u_{k_1} - \varepsilon$, and using algorithm HYBRID with input $(\varepsilon, L, \pm g, z^*)$ (the sign chosen so that $\pm g$ is convex on $(u_{k_1 - 1} + \varepsilon, u_{k_1} - \varepsilon)$), define $v_{k_1 - 1}$ to be the $\varepsilon$-approximation found for the unique root of $f''$ in $[u_{k_1 - 1} + \varepsilon, u_{k_1} - \varepsilon]$. Otherwise, define $v_{k_1 - 1}$ to be $R$.

**Step 15** For all $i \in 1, \ldots, k_1 - 1$, count via algorithm FASTERCOUNT the number $m_i$ of roots of $f$ in $(u_i - \varepsilon, u_i + \varepsilon) \cap (0, R)$ closest to $u_i$ and no other root of $f''$ and, append $m_i$ copies of $u_i$ to $Z$.

**Step 16** For all $i \in 0, \ldots, k_1 - 1$, count via algorithm FASTERCOUNT the number $n_i$ of roots of $f$ in $(v_i - \varepsilon, v_i + \varepsilon) \cap (0, R)$ closest to $v_i$ and no other root of $f''$, and append $n_i$ copies of $v_i$ to $Z$. 
Step 17 FOR $i \in \{0, \ldots, k_1 - 1\}$ DO

Step 17(a) Evaluate $f$ at $u_i + \varepsilon$ and $v_i - \varepsilon$. If $f(u_i + \varepsilon) = 0$ (resp. $f(v_i - \varepsilon) = 0$) then append $u_i + \varepsilon$ (resp. $v_i - \varepsilon$) to $Z$ and GOTO Step 17(c).

Step 17(b) If $f(u_i + \varepsilon)f(v_i - \varepsilon) < 0$ then let $L_i := v_i - u_i - 2\varepsilon$ and $g_i(x) := f(x + u_i + \varepsilon)$. Then, using subroutine HYBRID with input $(\varepsilon, L_i, \pm g_i, \hat{a}^\varepsilon)$ (the sign chosen so that $g_i$ is convex on $(0, v_i - u_i - 2\varepsilon)$), find an $\varepsilon$-approximation in $(u_i + \varepsilon, v_i - \varepsilon)$ to the unique root of $f$ in $(u_i + \varepsilon, v_i - \varepsilon)$, and append this approximation to $Z$.

Step 17(c) Evaluate $f$ at $v_i + \varepsilon$ and $u_{i+1} - \varepsilon$. If $f(v_i + \varepsilon) = 0$ (resp. $f(u_{i+1} - \varepsilon) = 0$) then append $v_i + \varepsilon$ (resp. $u_{i+1} - \varepsilon$) to $Z$ and GOTO Step 18.

Step 17(d) If $f(v_i + \varepsilon)f(u_{i+1} - \varepsilon) < 0$ then let $L_i := u_{i+1} - v_i - 2\varepsilon$ and $g_i(x) := f(u_{i+1} - \varepsilon - x)$. Then, using subroutine HYBRID with input $(\varepsilon, L_i, \pm g_i, \hat{a}^\varepsilon)$ (the sign chosen so that $g_i$ is convex on $(0, u_{i+1} - v_i - 2\varepsilon)$), find an $\varepsilon$-approximation in $(v_i + \varepsilon, u_{i+1} - \varepsilon)$ to the unique root of $f$ in $(v_i + \varepsilon, u_{i+1} - \varepsilon)$, and append this approximation to $Z$.

Step 18 END FOR

Step 19 STOP

Remark 6. In reality, one should replace the use of subroutine FASTERCOUNT above by the best algorithm attaining the complexity bound of $K(D, m)$ alluded to in Theorem 6, whenever $m \geq 4$. We have omitted this detail above simply to keep the algorithm definite.

**SUBROUTINE HYBRID (Compare [P.277|Ye94])**

*Input* $\varepsilon, R \in \mathbb{R}$ (with $0 < \varepsilon < R$), a monotonic analytic function $\phi : (0, R) \to \mathbb{R}$ (with $\phi(\varepsilon)\phi(R) < 0$ and $\phi$ convex on $(0, R)$), and a positive upper bound $\hat{a}^\varepsilon$ on $z\gamma(\phi, z)$ valid for all $z \in (0, R)$.

*Output* An $\varepsilon$-approximation of the unique root of $\phi$ in $(0, R)$, using no more than $O(\log(z\log(\hat{a}^\varepsilon)))$ evaluations of $\phi$ and $\phi'$, and $O(\log(z\log(\hat{a}^\varepsilon)))$ additional arithmetic operations.

**DESCRIPTION**

**Step 0** Define $c_0$ to be $1 + \frac{1}{8\varepsilon}$ or $\frac{1}{1 - \frac{1}{8\varepsilon}}$, according as $\phi$ is decreasing or increasing on $(0, R)$. Then define $c_i := c_{i-1}^2$ for all $i > 0$, and compute $c_1, \ldots, c_M$ where $M$ is the first positive integer with $c_M = c_0^M \geq R$.

Finally, set $\hat{x} := \varepsilon$ and $\hat{k} := M$.

**Step 1** If $[(\phi$ is decreasing on $(0, R) and \phi(c_{\hat{k}-1}\hat{x}) > 0)$ or $(\phi$ is increasing on $(0, R) and \phi(c_{\hat{k}-1}\hat{x}) < 0)]$ then set $\hat{x} := c_{\hat{k}-1}\hat{x}$ and $\hat{k} := \hat{k} - 1$ and GOTO Step 1.

**Step 2** Otherwise, if $k > 0$ then set $\hat{k} := \hat{k} - 1$ and GOTO Step 1.
Step 3  Perform \( \log_2(3 + \log_2 \frac{B}{e}) \) iterations of Newton’s method (with \( \hat{x} \) as the starting point), then OUTPUT the very last iterate.

**Remark 7.** A well-known trick we will use frequently and implicitly is the computation of \( x^d \), where \( x \in \mathbb{R} \) and \( d \in \mathbb{Z} \), within \( 2 \lceil \log_2(|d| + 1) \rceil \) multiplications using \( \leq \lceil \log_2(|d| + 1) \rceil \) intermediate real numbers. Briefly, assuming the binary expansion of \( d \) is \((a_k \cdots a_0)_2\) (so \( a_0 \) is the 1’s bit), we simply compute \( x^{2^i} \), via recursive squaring, for all \( i \) with \( a_i \) nonzero, and then multiply the resulting numbers together.

For instance, \( 363 = 2^8 + 2^6 + 2^5 + 2^3 + 2 + 1 = (10110101)_2 \) and thus

\[
\begin{align*}
363^2 & = (\ldots (x^2) \ldots)^2 \cdot (\ldots (x^2) \ldots)^2 \cdot (\ldots (x^2) \ldots)^2 \cdot (x^2)^2 \cdot x,
\end{align*}
\]

which only requires 17 multiplications and enough memory for 9 intermediate real numbers—much faster than the naive 362 multiplications.

**SUBROUTINE FASTERCOUNT**

*Input* A polynomial \( p \in \mathbb{R}[x] \) with exactly \( m \geq 3 \) monomial terms, and \( a, b \in \mathbb{R} \) with \( a < b \).

*Output* The number of roots of \( p \) in the open interval \((a, b)\).

**DESCRIPTION**

**Step 0**  If \( m > 3 \) then use Sturm-Habicht sequences (as in, say, [LM01]) to count the number of roots in \((a, b)\) and STOP.

**Step 1**  Otherwise, let \( p_0 := p \), \( p_1 := p' \), let \( -p_2 \) be the remainder of \( p_0/p_1 \), and set \( i = 2 \).

**Step 2**  If \( p_i \in \mathbb{R} \) then set \( K := i \) and GOTO Step 5.

**Step 3**  Write \( p_i(x) := u_i x^{a_i} - u_0 x^{a_0} \), \( p_{i-1}(x) := v_1 x^{b_i} - v_0 x^{b_0} \) (where \( a_1 > a_0 \) and \( b_1 > b_0 \)), and let \( c_j := \left\lfloor \frac{b_j - a_1}{a_j - a_0} \right\rfloor \) for \( j = 0, 1 \).

**Step 4**  Replace \( i \) by \( i + 1 \), define \( p_i(x) := -v_1 \left( \frac{u_i}{u_0} \right)^{c_1} x^{b_i - c_1(a_1 - a_0)} + v_0 \left( \frac{u_i}{u_0} \right)^{c_0} x^{b_0 - c_0(a_1 - a_0)} \), and GOTO Step 2.

**Step 5**  Using recursive squaring (cf. Remark 7), evaluate and RETURN \( N_A - N_B \) (cf. Definition 3) where \( A \) (resp. \( B \)) is \((p_0(a), \ldots, p_K(a))\) (resp. \((p_0(b), \ldots, p_K(b))\)).

4. Correctness and complexity: proving Theorems 1, 6, and 3

**Proof of Theorem 1.** For open intervals, the second assertion follows immediately from Lemma 1 and Theorem 3. The case of closed intervals follows almost identically, save for an additional evaluation of \( f \) at the end-points of the interval which increases the complexity bound by (a negligible) \( O(\log D) \).
The first assertion follows immediately from Theorems 6 and 5, and the second assertion of Theorem 1 which we’ve just proved. □

Proof of Theorem 6. Henceforth, for convenience, we will say “time” in place of “arithmetic complexity.” Clearly, it suffices to show that MNOMIALSOLVE is correct and satisfies the stated complexity bound when we set $\bar{x} = \bar{x}(D, m)$. Toward this end, let us first observe that $L_1(f)$ has exactly $m - 1$ monomial terms and $L_2(f)$ has $\leq m - 1$ monomial terms. Descartes’ Rule then implies that $L_1(f)$ and $L_2(f)$ each have no more than $m - 2$ positive roots. Note also that $f$, $f'$, and $f''$ can all be evaluated using just $O(m \log D)$ arithmetic operations, thanks to Remark 7. These observations will be used implicitly throughout our proof.

The correctness of MNOMIALSOLVE is then straightforward for $m \leq 2$. Also, for $m = 2$ and $D \geq 3$, the complexity of MNOMIALSOLVE is clearly the same (asymptotically) as that of subroutine HYBRID with $\bar{x} = \frac{D - 1}{2}$, thanks to Proposition 4. So we can assume $m \geq 3$. Furthermore, by Step 4 of MNOMIALSOLVE, we can clearly assume that $a_1 = 0$.

In the special case where $D = 2$ and $m = 3$, note that Step 6 is but an implementation of completing the square. So, provided one shifts one’s $\varepsilon$-approximations by $\frac{a_2}{2\varepsilon}$ while running the remainder of MNOMIALSOLVE, we can clearly attain the stated complexity bound for $D = 2$. So we can assume additionally that $D \geq 3$ and now focus on Steps 10–18.

In the special case $a_2 = 1$, note that the change of variables $x \mapsto 1/x$ maps the interval $(r - \delta, r + \delta)$ to $\frac{1}{r^2 - \delta^2}(r - \delta, r + \delta)$ for $r > \delta > 0$. An elementary calculation then reveals that $\delta \leq \max\{1, \varepsilon^3/2\} \Rightarrow$ for any $\delta$-approximation $y$ of $x \geq e, 1/y$ is an $\varepsilon$-approximation of $1/x$. Note also that $x^{a_1}f(1/x) = c_3 + c_2x^{a_1-a_2} + c_1x^{a_3}$ (since $a_1 = 0$ after Step 4), and $a_3 - a_2 > 1$ since $a_2 = 1$ and $a_3 = D \geq 3$. So, provided one takes a reciprocal after running MNOMIALSOLVE, we see that the worst-case time complexity of any instance with $a_2 = 1$ is asymptotically no worse than that of the instances with $a_2 > 1$. So we can assume additionally that $a_2 > 1$.

Now, by assumption, $S(f)$ is dampened, as are all the inputs going into recursive calls of MNOMIALSOLVE. Note also that by our dampening assumption (and Rolle’s Theorem), the intervals from Steps 17(b) and 17(d) indeed contain exactly one root of $f''$. So our algorithm is well-defined and indeed finds $\varepsilon$-approximations to all the roots of $f$ in $[0, R]$, assuming subroutines FASTERCOUNT and HYBRID are correct. The correctness of the latter two subroutines is proved below, so let us now concentrate on proving our main complexity bound.

So let $\mathcal{C}(m)$ denote the time needed for MNOMIALSOLVE to execute completely for an input consisting of an $f \in \mathcal{F}(D, m)$ (satisfying the dampening assumptions of our current theorem), an interval length of $\leq R$, and a precision $\varepsilon$. Clearly, assuming subroutine HYBRID runs in time $O(m \log D) \log(\bar{x}' \log \frac{R}{\varepsilon})$ (which is covered in a separate proof below), we then have that $\mathcal{C}(m)$ must satisfy the following recurrence relation:

$$\mathcal{C}(m) \leq 2\mathcal{C}(m - 1) + O\left(m(m \log D) \log(\bar{x}' \log \frac{R}{\varepsilon})\right) + (2m - 4)\mathcal{C}(D, m),$$
where the first term corresponds to finding the \( \varepsilon \)-approximations of the roots of \( f' \) and \( f'' \) (Steps 10, 13, and 14), the last term corresponds to the application of algorithm \( \text{FASTERCOUNT} \) to the intervals about these roots (Steps 13–16), and the \( O \) term corresponds to the application of algorithm \( \text{HYBRID} \) to the intervals between these roots (Steps 17–18). (Note that we have implicitly used the fact that \( k(D, m) \) and \( \varkappa(D, m) \) are non-decreasing functions of \( D \) and \( m \).) In particular, for \( m \geq 4 \), the last term of our recurrence is justified by Remark 6.

Noting that we can regard \( R \) and \( \varepsilon \) as constants, it is then clear that to find the asymptotics of our recurrence, it suffices to find the asymptotics of the recurrence

\[
c_m \leq 2c_{m-1} + m^2 A + (2m - 4)B(m).
\]

A simple Maple calculation (\( \text{rsolve}(c(m) = 2^c(m-1) + a^m(m-1) + b(m)^2(2m-4), c(m)) \)) then shows that any such \( c_m \) must satisfy \( c_m = O(2^m(c_2 + A) + \sum_{j=1}^{m} 2^m jB(j)) \). So, knowing that \( \zeta(2) = O((\log(D) \log \log \frac{R}{\varepsilon}) [\text{Ye94, p. 279}] \), we must then have that

\[
\zeta(m) = O \left( 2^m \left\{ \log(D) \log \left( \frac{R}{\varepsilon} \right) + m^2 k(D, m) \right\} \right).
\]

**Proof of Theorem 3.** The complexity bound follows immediately from the upper bound on \( K \), assuming algorithm \( \text{FASTERCOUNT} \) is correct. The bound on \( K \) and the correctness of algorithm \( \text{FASTERCOUNT} \) are proved below.

At this point, we need only show that the subroutines \( \text{HYBRID} \) and \( \text{FASTERCOUNT} \) are correct, and that \( \text{FASTERCOUNT} \) satisfies a sufficiently good complexity bound.

**Proof of Correctness and Complexity Analysis of \( \text{HYBRID} \).** First note that by assumption, there is a unique \( j \in \{0, \ldots, 2^{K-1}\} \) such that \( (\beta^j \varepsilon, \beta^{j+1} \varepsilon] \) contains the sole root of \( \phi \) in \((0, R)\). It then becomes clear that Steps 0–2 are merely an implementation of bisection that finds this \( j \): In particular, Steps 0–2 essentially find the binary expansion of \( j \) (from most significant bit to least significant bit), and \( \hat{x} = \beta^j \) once Step 3 is reached. By Theorem 4, \( \hat{x} \) is then an approximate root of \( \phi \), and Step 3 indeed finds an \( \varepsilon \)-approximation of our root.

As for the complexity of our algorithm, it is clear that \( M = \lceil \log_2(\log_2 \frac{R}{\varepsilon}) - \log_2 \log_2 \beta \rceil \) and we thus need only \( O(\log(\log \frac{R}{\varepsilon}) - \log \log \beta) \) evaluations of \( \phi \) (and inequality checks) until Step 3. Step 3 then clearly takes just \( O(\log \log \frac{R}{\varepsilon}) \) arithmetic operations and evaluations of \( \phi \) and \( \phi' \).

To conclude, recall the elementary inequalities \( \frac{1}{\log(1 + x)} \ll x \), valid for all \( x \geq 0 \). Observe then that \( -\log \log \beta = \log \frac{1}{\log \beta} \ll \log x \). Since \( \phi \) an \( m \)-nomial of degree \( D \) implies that \( \phi \) and \( \phi' \) can be evaluated within \( O(m \log D) \) arithmetic operations (cf. Remark 7), we are done.
Proof of Correctness and Complexity Analysis of FASTERCOUNT. The correctness of FASTERCOUNT when $m > 3$ is clear from any of the standard references [Roy96,LM01] so there is nothing to prove.

For $m = 3$, correctness follows easily from Lemma 1 upon noting a simple fact: The remainder of $\frac{x_a^1 v_1 x_b^1 - v_0 x_b^0}{u_1 x_a^1 - u_0 x_b^0}$ is nothing more than the reduction of $v_1 x_b^1 - v_0 x_b^0$ modulo $u_1 x_a^1 - u_0 x_b^0$. In particular, Step 2 of FASTERCOUNT is just a compact representation of a sufficient number of applications of the identity $x^{a_1} = C_0 x^{e_0}$ to reduce $-(v_1 x_b^1 - v_0 x_b^0)$ modulo $u_1 x^{a_1} - u_0 x^{e_0}$ to a polynomial of degree $< a_1$. So correctness follows immediately.

As for the complexity bound, we need only observe that (by recursive squaring) every execution of Steps 2 and 3 takes only $O(\log D)$ arithmetic operations, and that $K = O(\log D)$ at the termination of the algorithm, i.e., there are only $O(\log D)$ remainders in our Sturm sequence. The first assertion is clear, so to prove the latter assertion we will need to prove a technical bound on the exponents which occur in our remainder sequence.

In particular, let $\ell_i$ be the absolute value of the difference exponents of $p_i$ (we set $\ell_i := 0$ if $p_i$ is monomial). Note that $p_i$ a monomial $\Rightarrow p_{i+1}$ is a monomial, and thus $p_{i+2}$ is constant. Note also that, via the definition of long division, we have $\ell_{i+2} \leq |\ell_{i+1} - \ell_i|$, with equality occurring iff $p_{i+2}$ is a binomial. So if we can show

$$\min\{\ell_{i+2}, \ell_{i+3}\} \leq \ell_i/2$$

for all $i$, then we will easily obtain our bound on $K$ (since $\ell_0, \ell_1 \leq D$ and all the $\ell_i$ are integers).

To prove (\star) observe that $\ell_{i+1} \leq \frac{1}{2} \ell_i \Rightarrow \ell_{i+2} \geq \frac{1}{2} \ell_{i+1}$ (if $\ell_{i+2} \neq 0$) and thus $\ell_{i+3} \leq \frac{1}{2} \ell_i$. Similarly, $\ell_{i+1} \geq \frac{1}{2} \ell_i \Rightarrow \ell_{i+2} \leq \frac{1}{2} \ell_{i+1}$. So $K \leq 3\lceil \log_2 D \rceil + 2 = O(\log D)$ and we are done. \qed

5. The proof of Theorem 2

Dividing by a suitable monomial term, we can clearly assume without loss of generality that $f$ has a constant term. Note then that $C$ is diffeomorphic to a line iff $f$ is not the square of a binomial, via [LRW03, Proposition 2 and Lemma 1]. So there are actually no isolated singularities for trinomial curves.

It is then clear that $x$ is a vertical tangent of $C \Rightarrow x$ is a root of $F_x := (f, \frac{\partial f}{\partial x_2})$. The roots of $F_x$ with $x$ or $y$ coordinate equal to 0 or $R$ can be found easily by suitably fixing one of the variables and applying Theorem 1, so let us concentrate on the roots of $F_x$ in $(0,R)^2$.

By Rojas [Roj00a, Lemma 4.2] and van der Kallen [vdK00] we can then find (for any constant $\delta > 0$) a monomial change of variables, using just $O(\log^{2+\delta} D)$ bit operations, such that the resulting sparse polynomial system $G := (g_1, g_2)$ has $g_1 \in \mathbb{R}[x_1, x_2]$ a trinomial, $g_2 \in \mathbb{R}[x_1]$ a binomial, and all underlying exponents have
bit-length $O(\log^2 D)$. We can then solve $g_2$ to accuracy $\min\{\varepsilon^{O(D^2)}, 1\}$ via Theorem 1, back-substitute the roots into $g_1$, solve the univariate specialized $g_1$ to accuracy $\min\{\varepsilon^{O(D^2)}, 1\}$, then invert our monomial map via [Roj00a, Main Theorem 1.3] and [vdK00] to recover the roots of $F_v$, using just $O(\log(D) \log(D \log R))$ additional arithmetic operations and $O(\log^{2+\delta} D)$ additional bit operations. So we can indeed find isolated vertical tangents within the stated complexity bound.

To conclude, note that [LRW03, Lemma 9] tells us that if $x$ is an isolated inflection point of $C$ then $[\partial^2 f \cdot (\partial f)^2 - 2\partial_1 \partial_2 f \cdot \partial f \cdot \partial_2 f + \partial^2_2 f \cdot (\partial_1 f)^2] = 0$, where $\partial_i := \frac{\partial}{\partial x_i}$. In the special case of a trinomial, say $f(x_1, x_2) := 1 + Ax_1^2x_2^3 + Bx_1^4x_2^4$, the preceding polynomial in derivatives is exactly:

$$-ab(a + b)R^3 + (b^2c^2 + a^2d^2 - b^2c - 2abcd - a^2d - 2abc - 2abcd)RS^2 + (b^2c^2 - 2abcd - ad^2 - 2acd - 2bcd - bc^2 + a^2d^2)S^3,$$

where $R := Ax_1^2x_2^3$ and $S := Bx_1^4x_2^4$, thanks to a Maple calculation. (Note that we must also have $1 + R + S = 0$ since we’re assuming $f = 0$.) So we can efficiently solve for $x$ as follows: (1) solve the above cubic for $R$ up to accuracy $\min\{\varepsilon^{O(D)}, 1\}$ via the main algorithm of [NR96] or [BP94], (2) solve for $(R, S)$ via the additional relation $1 + R + S = 0$, (3) solve the resulting binomial systems in $(R, S, x_1, x_2)$ for $x$ via [Roj00a, Main Theorem 1.3] and [vdK00]. So we can $\varepsilon$-approximate the inflection points of $C$ within the stated asymptotic complexity bound as well, and we are done. \(\square\)

6. The proof of Theorem 5

**Definition 8.** For any $a \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, let $(a)_k := a(a - 1) \cdots (a - k + 1)$. (So $(a)_0 = 1$, $(a)_1 = a$, $a \in \mathbb{N} \Rightarrow (a)_a = a!$, and $a, k \in \mathbb{N}$ with $k > a \Rightarrow (a)_k = 0$.)

**Proof of Theorem 5.** The first bound is immediate since $\mathcal{F}(D, m) \subseteq \mathcal{F}(D, m + 1)$.

The formula for $\tilde{\omega}(D, 2)$ follows easily since $m = 2$ and $f(0) \neq 0$ implies that $f'$ has no positive roots. This in turn implies that $N(f') = 1$, so the only interval we may need to majorize over is $(u_0(f), u_1(f)) = (0, \infty)$. In particular, the quantity we majorize is just $\gamma(f, x)$, and the formula for $\tilde{\omega}(D, 2)$ then follows immediately from Proposition 4. So we can assume $m = 3$. Since $\gamma(f, x) = \gamma(cf, x)$ for any nonzero constant $c$, we can also clearly assume that $f(x) = x^{a_3} - Ax^{a_2} + B$ where $A$ and $B$ are nonzero real constants and $a_3 = D > a_2 \geq 1$. Since we assume $f'(0) = 0$ in the definition of $\tilde{\omega}(D, m)$, we must also have $D \geq 3$ and $a_2 > 1$.

Next, note that $A < 0 \Rightarrow f'$ has no positive roots. So, similar to the $m = 2$ case, one either majorizes over the empty set or $(0, \infty)$. In the former case there is no contribution to $\tilde{\omega}(D, m)$, while in the second case, we see that we are again majorizing $x\gamma(f, x)$ over $(0, \infty)$. Since the latter supremum is no greater than $\frac{D - 1}{2}$ by Propositions 3 and 4, we can now assume additionally that $A > 0$. 

Now note that
\[ f'(x) = a_3x^{a_3-1} - a_2Ax^{a_2-1} \]
\[ = a_3x^{a_3-1}\left(x^{a_3-a_2} - \frac{a_2A}{a_3}\right) \]
and
\[ f''(x) = a_3(a_3-1)x^{a_3-2} - a_2A(a_2-1)x^{a_2-2} \]
\[ = a_3(a_3-1)x^{a_3-2}\left(x^{a_3-a_2} - \frac{a_2A(a_2-1)}{a_3(a_3-1)}\right). \]

Clearly then, \( x_1 := \left(\frac{a_2A}{a_3}\right)^{1/(a_3-a_2)} \) is the unique positive root of \( f' \) and \( x_2 := \left(\frac{a_2(a_2-1)A}{a_3(a_3-1)}\right)^{1/(a_3-a_2)} \) is the unique positive root of \( f'' \). Furthermore, since \( 1 < a_2 < a_3 \), it is clear that \( 0 < x_2 < x_1 \). In particular, we see that we are left with a collection of 3 intervals to consider in the definition of \( \tilde{a}(D,m) \):
\[ \{(0,x_2), (x_2,x_1), (x_1, \infty)\} \]
and the quantities we majorize on these intervals are, respectively:
\[ \{x\gamma(f,x), (x_1-x)\gamma(f,x), (x-x_1)\gamma(f,x)\}. \]

The intervals \( (x_2,x_1) \) and \( (x_1, \infty) \) can be handled via a unified calculation, so let us complete our analysis by examining the intervals \( (x_2, \infty) \) and \( (0,x_2) \).

The Interval \( (x_2, \infty) \). Note that \( f(y+x_1) = (y+x_1)^{a_3} - A(y+x_1)^{a_2} + B \) and thus \( f^{(k)}(y+x_1) = (a_3)_k(y+x_1)^{a_3-k} - (a_2)_kA(y+x_1)^{a_2-k} \).

So, letting \( \phi(y) := f(y+x_1) \), we have for all \( k \geq 2 \):
\[ \left|\frac{\phi^{(k)}(y)}{k!\phi'(y)}\right|^{1/(k-1)} = \left|\frac{(a_3)_k(y+x_1)^{a_3-k} - (a_2)_kA(y+x_1)^{a_2-k}}{k!(y+x_1)^{a_3-1} - a_2A(y+x_1)^{a_2-1}}\right|^{1/(k-1)} \]
\[ = \left|\frac{(a_3)_k(y+x_1)^{a_3-a_2} - (a_2)_kA}{k!a_3(y+x_1)^{a_3-1} - (a_2)_kA(y+x_1)^{a_2-1}}\right|^{1/(k-1)} \]
\[ = \frac{1}{|y|} \left|\frac{(a_3)_k}{k!a_3}\right|^{1/(k-1)} \left|\frac{y^{a_3-1}(y+x_1)^{a_3-a_2} - (a_2)_kA}{(y+x_1)^{a_3-1} - (a_2)_kA(y+x_1)^{a_2-1}}\right|^{1/(k-1)}, \]

which in turn is bounded above by
\[ \frac{1}{|y|} \frac{a_3 - 1}{2} \left|\frac{y^{a_3-1}(y+x_1)^{a_3-a_2} - (a_2)_kA}{(y+x_1)^{a_3-1} - (a_2)_kA(y+x_1)^{a_2-1}}\right|^{1/(k-1)} \]
since \( \frac{(a_3)_k}{k!a_3} \leq \frac{a_3-1}{2} \).
Now let $z := y/x_1$ and let $\rho$ be $(a_2 - 1)/(a_3 - 1)$ or 0 according as $k \leq a_2$ or not. Then for $y \neq -x_1$ (or $z \neq -1$, equivalently) we have:

$$
\left| \frac{y^{k-1}}{(y + x_1)^{k-1}} \cdot \frac{(y + x_1)^{a_3 - a_2} - (a_2)_k}{(y + x_1)^{a_1 - a_2} - x_1^{a_1 - a_2}} \right|^{1/(k-1)} = \left| \frac{z^{k-1}}{(z + 1)^{k-1}} \cdot \frac{(z + 1)^{a_3 - a_2} - \rho^{a_3 - a_2}}{(z + 1)^{a_1 - a_2} - 1} \right|^{1/(k-1)}
$$

(5)

$$
= \left| \frac{z^{k-2}}{(z + 1)^{k-1}} \cdot \frac{(z + 1)^{a_3 - a_2} - \rho^{a_3 - a_2}}{1 + (z + 1) + \cdots + (z + 1)^{a_3 - a_2 - 1}} \right|^{1/(k-1)}
$$

(6)

$$
= \left| \frac{z}{z + 1} \right|^{(k-2)/(k-1)} \left| \frac{(z + 1)^{a_3 - a_2} - \rho^{a_3 - a_2}}{(z + 1) + (z + 1)^2 + \cdots + (z + 1)^{a_3 - a_2}} \right|^{1/(k-1)},
$$

(7)

where the last equality follows from the identity $\frac{z}{(z+1)^{k-1}} = \frac{z+1-1}{(z+1)^{k-1}} = \frac{z+1-1}{(z+1)^{k-1}} = 1/(z+1)^{k-1}$. Since $\frac{z}{z+1} = 1 - \frac{1}{z+1}$ is clearly an increasing function of $z$ for all $z > -1$ (or $y > -x_1$, equivalently), and since $0 > -\frac{x_2}{x_1} > -1$, we must have for all $y > x_2 - x_1$ (or $z > \frac{x_2}{x_1} - 1$, equivalently) that

$$
\left| \frac{z}{z + 1} \right|^{(k-2)/(k-1)} \leq \max\left\{ \left( \frac{x_1}{x_2} - 1 \right)^{(k-1)/(k-2)}, 1 \right\}
$$

$$
= \max\left\{ \left( \frac{a_3 - 1}{a_2 - 1} \right)^{1/(a_3 - a_2)} - 1 \right\}^{(k-1)/(k-2)}, 1 \right\}
$$

(★)

$$
\leq \max\left\{ \frac{a_3 - 1}{a_2 - 1} - 1, 1 \right\}.
$$

Furthermore, since $(a_2 - 1)/(a_3 - 2) \leq 1$ for $2 \leq k \leq a_2 - 2$, we must have $\frac{a_2}{x_1} \geq \rho$ and thus the last factor of Equality (7) is bounded above by 1. So by Equalities (1)–(7) and (★) we obtain that

$$
|y| \left| \frac{\phi(k)(y)}{k!\phi'(y)} \right|^{1/(k-1)} \leq \frac{1}{2} \max\{(a_3 - 1)(a_3 - 2), a_3 - 1\} \leq (D - 1)(D - 2)/2,
$$

since $D \geq 3$. Thus, the contribution of the interval $(x_2, \infty)$ to $\tilde{g}(D, m)$ is no more than $(D - 1)(D - 2)/2$. 
The Interval \((0, x_2)\). Similar to the last case, we have for all \(x \in (0, x_2)\) that

\[
\frac{|f^{(k)}(x)|}{k!f'(x)} \leq \frac{1}{y} \frac{(a_3 - 1)(a_2 - 1)}{2},
\]

and the quantity we need to majorize on this interval is no more than \((D - 1)\frac{(D - 2)}{2}\).

Putting together the bounds we have found over our two preceding intervals, we see that our upper bound for \(\bar{z}(D, 3)\) holds and we are done.
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References


