Resistive magnetohydrodynamic equilibrium and stability of a rotating plasma with particle sources

Yi-Min Huang and A. B. Hassam

Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742

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Particle sources and resulting density profiles are shown to play an important role in the magnetohydrodynamic (MHD) stability of azimuthally rotating axially magnetized plasmas. In the absence of particle sources, density profiles relax under resistive diffusion to pile up to the outside of the system consistent with the outward centrifugal forces. In particular, particle sources would have to be placed appropriately to maintain desired density profiles for fusion applications of centrifugally confined systems. Tailoring of the density profiles could, however, be used to achieve control over MHD stability. Long wavelength Kelvin–Helmholtz modes as well as short wavelength interchange modes are studied in relation to profile tailoring and efficacy of velocity shear. It is concluded that judiciously placed particle sources could be used to enhance stability.

I. INTRODUCTION AND MOTIVATION

An idea recently revived and currently under investigation is to use the centrifugal force of a rotating plasma to augment magnetic confinement for thermonuclear fusion plasmas. In this scheme, a magnetic mirror type plasma is made to rotate azimuthally at supersonic speeds; thus, centrifugal forces along magnetic field lines confine the plasma to the center section. One of the central questions for the success of this scheme is the magnetohydrodynamic (MHD) stability of the rapidly rotating configuration. This is a rather complicated problem because of the various ingredients involved. It is well-known that flute interchanges could be driven unstable by the pressure gradient in the mirror field with an unfavorable curvature. The centrifugal force from the rotation is also a potential driving force for interchanges. In recent years, it was recognized that the strong velocity shear accompanying the rotation could be utilized to stabilize the interchanges. Yet the velocity shear could also drive the Kelvin–Helmholtz (KH) instability. In a previous study the above mentioned issues were addressed analytically with a simplified Dean flow model, wherein the effect of the curvature of the field lines was modeled by an outwardly pointing effective gravity acting on the pressure. A subsequent three-dimensional (3D) MHD simulation demonstrated the stability of a centrifugally confined plasma; in particular, the velocity shear was identified as the stabilizer of the interchanges.

The present study is motivated by two observations from the above mentioned 3D simulation. First, the two-dimensional (2D) steady state in that study was in fact, not steady; it was still slowly evolving on transport time scales. Since the ultimate goal of the centrifugally confined fusion plasma is to operate in a steady state, a closer examination is needed. If the axial magnetic field is straight, one can easily show that in the absence of particle sources or an auxiliary electric current drive, resistivity (which was explicitly included in Ref. 9) will eventually bring the toroidal current down to zero (this can be shown as follows: the toroidal electric field \( E_\phi = 0 \) in a steady state. If the flow is toroidal \( \mathbf{u} = u_\phi \hat{\phi} \) and the magnetic field is axial \( \mathbf{B} = B_z \hat{z} \), then the Ohm’s law (3) implies that the toroidal current \( J_\phi = 0 \); hence in a steady state the centrifugal force can only be balanced by the pressure gradient and the confinement is essentially lost, as the density has to pile up toward the outside of the system to provide the necessary pressure gradient. In a centrifugally confined plasma, however, the magnetic field is a curved mirror field. The effect of the magnetic curvature on resistive relaxation is not all obvious. In an axisymmetric rotating plasma with a curved poloidal magnetic field, one can demonstrate the existence of poloidal convection cells as follows. Suppose there is no poloidal convection, then the toroidal current \( J_\phi = 0 \) from the Ohm’s law (3); therefore, the centrifugal force is balanced by the pressure gradient alone, i.e., \( \rho \Omega^2 \hat{r} = \nabla p \). However, the curl of this equation yields \( \partial_z (\rho \Omega^2) = 0 \) which, in general, is not true when the magnetic field is curved; therefore, there is a contradiction. The poloidal convection, together with the poloidal magnetic field, drives a toroidal current via the electromotive force (e.m.f) \( \mathbf{u} \times \mathbf{B} \), where \( \mathbf{u} \) is the plasma flow and \( \mathbf{B} \) is the magnetic field. The operative question then is: Is this toroidal current sufficient to maintain a well confined plasma in the absence of particle sources? In Sec. II we address this question by studying the slowly diffusing plasma, in the spirit of the classic work of Kruskal and Kulsrud. We show that in spite of the nonvanishing toroidal e.m.f, the conclusion is essentially the same; i.e., in the absence of particle sources, the resistivity relaxes the magnetic field to nearly a potential field and the density piles up toward the outside of the system as a result. Therefore, to have a confined steady

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a) Electronic address: yopology@umd.edu
b) Electronic address: hassam@umd.edu
state one has to introduce particle sources (and sinks) into the system. The governing equations for the slowly diffusing plasma then determine the density profile for a given particle source.

The second observation from the study of Ref. 9 is that the system in that case was very stable. There was no sign of the type of wobbles observed in a previous Z pinch simulation. The superior stability should nevertheless not be regarded as a general result, and we will try to give an explanation in this paper. Based on theoretical understandings, there are at least two factors that could be relevant to the overall stability of the system: First, the stability may weakly depend on dissipation such as the viscosity and the resistivity, etc. Second, the density, the pressure, and the flow profiles, as well as the interplay between them might be important for the stability. For example, the generalized Rayleigh’s criterion depends on both the density and the flow profiles. Another example is that found by Benilov et al. and in Ref. 11 that unstable modes could be regarded as a general result, and we will try to give an explanation in this paper. Based on theoretical predictions in Ref. 8. We also discuss the in-plane instabilities, namely, hysteresis, asymmetric particle profiles, and dissipation coefficients; the results of those simulations are presented in subsequent Secs. IV and V.

Taking Eq. (3) along three independent directions \( \mathbf{B} \), \( \nabla \psi \), and \( \phi \), we have

\[
\begin{align*}
  \mathbf{B} &= \nabla \times (\nabla \psi), \\
  \nabla \psi \cdot \nabla \Phi &= |\nabla \psi|^2 \mathbf{u} \cdot \nabla \phi, \\
  \mathbf{u} \cdot \nabla \psi &= \eta r^2 \nabla \left( \frac{\nabla \psi}{r^2} \right).
\end{align*}
\]

From Eq. (7), \( \Phi \) is a flux function \( \Phi(\phi) \). If we define the angular velocity \( \Omega = u_\phi / r \), Eq. (8) gives \( \Omega = -d\Phi/d\psi \), which means each field line rotates as a rigid rotor. The physical meaning of Eq. (9) is that the toroidal electric current is driven by the e.m.f. from the poloidal flow together with the toroidal magnetic field. From Eq. (9) we can see that poloidal flow scales as \( \eta \), which we assume to be small. Hence we assume the ordering \( \mathbf{u} \ll u_\phi \), where the “\( \ll \)” denotes the poloidal component. Under this assumption, the momentum Eq. (2) is approximately

\[
-\rho r \Omega^2 \hat{b} = -\nabla p - \nabla \left( \frac{\nabla \psi}{r^2} \right) \nabla \psi.
\]

Taking Eq. (10) along two independent directions \( \hat{b} \) and \( \nabla \psi \), where \( \hat{b} \) is the unit vector along \( \mathbf{B} \), we obtain

\[
\rho r \Omega^2 \hat{b} = \hat{b} \cdot \nabla p,
\]

\[
(\rho r \Omega^2 \hat{b} - \nabla p) \cdot \nabla \psi = \nabla \cdot \left( \frac{\nabla \psi}{r^2} \right) |\nabla \psi|^2.
\]

Now we assume that the plasma is enclosed by surfaces (e.g., the walls of the chamber) on which \( \mathbf{u} = 0 \). Integrating Eq. (1) over the volume where \( \psi < c \), with any constant \( c \), we get

\[
\int_{\psi < c} S d\tau = \int_{\psi < c} \frac{\rho \mathbf{u} \cdot \nabla \psi}{|\nabla \psi|} d\sigma,
\]

where \( d\tau \) is the volume element and \( d\sigma \) the surface area element. Using (9) and (12) into (13), we get
If there is no particle source, the left-hand side (LHS) of (14) is zero. The right-hand side (RHS) then implies that the centrifugal force is essentially balanced by the pressure gradient alone, although in some average sense on each flux surface. Therefore, we conclude that even when the magnetic field is curved, particle sources are still necessary to have a resistive MHD equilibrium without density piling-up. For a solution of Eq. (10), we can use Eq. (14) to calculate the particle source needed to maintain that equilibrium. We may estimate the amount of the particle source needed to maintain a steady state from Eq. (14). If there is no rotation, Ω=0, then $S \sim \eta \beta \rho / a^2$, where $\beta=2p/B^2$ as usual, and $a$ is the radial size of the system. When the system is rotating supersonically, however, the centrifugal force term in Eq. (14) dominates the pressure gradient term. In that case, we have $S \sim (1+M^2_\text{S}a/r) \eta \beta \rho / a^2$, where $M_\text{S}$ is the sonic Mach number; i.e., the cross-field particle loss is enhanced approximately by a factor of $(1+M^2_\text{S}a/r)$ because of the centrifugal force.

If we assume an equation of state $p=\rho T$ with $T=T(\psi)$, the system can be further simplified. Under these assumptions, Eq. (11) can be simplified as

$$\frac{1}{T} = \frac{\ln(\rho)}{T} - \frac{r^2 \Omega^2}{2T} = 0.$$  \hspace{1cm} (15)

Therefore, we can express $\rho$ as

$$\rho = f(\psi) \exp(r^2 \Omega^2 / 2T).$$  \hspace{1cm} (16)

Substituting (16) into (12) and (14), after some algebra, we get

$$-T \frac{df}{d\psi} + \left( r^2 \Omega \frac{d\Omega}{d\psi} \right) \frac{df}{d\psi} + \left( 1 - \frac{r^2 \Omega^2}{2T} \right) \frac{d\psi}{d\psi} \exp \left( \frac{r^2 \Omega^2}{2T} \right) = \nabla \cdot \left( \frac{\nabla \psi}{r} \right),$$  \hspace{1cm} (17)

and

$$q(\psi) f \frac{df}{d\psi} + g(\psi) f^2 = h(\psi),$$  \hspace{1cm} (18)

where $q(\psi)$, $g(\psi)$, and $h(\psi)$ are flux functions defined on each flux surface $\psi=c$ as

$$q(c) = \int_{\phi=c}^{} T \frac{r^2 \exp(r^2 \Omega^2 / T)}{\nabla \psi} \, d\sigma,$$  \hspace{1cm} (19)

$$g(c) = \int_{\phi=c}^{} r^2 \exp(r^2 \Omega^2 / T) \left( r^2 \Omega \frac{d\Omega}{d\psi} \right) \frac{d\sigma}{\nabla \psi},$$  \hspace{1cm} (20)

and

$$h(c) = -\int_{\phi=c}^{} S \, d\sigma.$$  \hspace{1cm} (21)

The solution of Eq. (18) can be formally written as

$$f^2 = \frac{1}{k} \int \left( \frac{2khq}{a} \right) d\psi + \text{const},$$  \hspace{1cm} (22)

where $k$ is a flux function defined as

$$k(\psi) = \exp \left( \int \frac{2gq}{a} d\psi \right),$$  \hspace{1cm} (23)

and the constant of integration is determined by the total mass.

Our system of equations now consists of a Grad–Shafranov-like equation (17) and an auxiliary condition (18). The general procedure to solve this system for given $T(\psi)$, $\Omega(\psi)$, and $S$ is as follows. First we have to solve $\psi$ and $f(\psi)$ by solving (17) and (18) simultaneously. Then we can use (16) and (9) to determine $\rho$ and the poloidal flow perpendicular to the field line. Finally, to complete the pattern of convection cells, we can use (1) to determine the poloidal flow along the field line. This solution represents a slowly diffusing equilibrium of a rotating plasma with poloidal convection cells. In the low-$\beta$ (i.e., $p/B^2 \ll 1$, $\rho r^2 \Omega^2/B^2 \ll 1$) limit, we can approximate $\psi$ by the vacuum flux function provided by the external field. Therefore, we do not have to solve the coupled equations (17) and (18) simultaneously, and the solution can be largely simplified.

### III. NUMERICAL MODEL

In the previous section we set up the governing equations for slowly diffusing equilibria. Solving the equilibrium for given particle sources and external coils is in general complicated. We make no attempt to solve it in this study. Rather, we would like to address an even more important issue, viz., the stability, since such an equilibrium is physically realizable only when it is stable. In the following sections we will address this issue by numerical simulations. For the reasons mentioned in the Introduction, our simulation will be limited to a 2D Dean flow model on the $r-\phi$ plane, namely, an annular flow which is no-slip at edges, with a threaded axial magnetic field. The equilibrium is therefore one dimensional (1D), with $r$ dependence only.

Our numerical model is governed by the following set of equations:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = S,$$  \hspace{1cm} (24)

$$\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho - \frac{\nabla B^2}{2} + \mathbf{B} \cdot \nabla \mathbf{B} + \mu \nabla^2 \mathbf{u} + \mathbf{F},$$  \hspace{1cm} (25)

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E},$$  \hspace{1cm} (26)

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \nabla \times \mathbf{B}.$$  \hspace{1cm} (27)

The plasma is assumed isothermal for simplicity. A force in the azimuthal direction is applied to spin up the plasma. Assuming $\partial_z=0$, we numerically solve the system in the $r-\phi$ plane. The numerical algorithm is described in detail in Ref. 12. The simulation was done in the region $[R, R+1] \times [0, \phi_0]$, which represents a section of the full annulus. We impose periodic boundary conditions in the $\phi$ direction. The
chosen simulation box length \( L = (R + 1/2) \phi_0 \) was sufficient to study both the long wavelength KH modes and the short wavelength interchange modes. The boundaries in \( r \) are no-slip, perfectly conducting hard walls. The resolution of the simulations reported here is 100×100.

Assuming an axisymmetric particle source, the 1D equilibrium with \( u = u_r + u_\phi \hat{\phi} \) and \( B = B \hat{z} \) is determined by

\[
\frac{1}{r} \frac{d}{dr} \left( r \rho u_r \right) = S, \quad \frac{d}{dr} \left( \rho T + \frac{B^2}{2} \right) + \rho u_\phi^2 \frac{r}{2} = 0, \quad \rho u_r \frac{du_\phi}{dr} + \frac{\rho u_r u_\phi}{r} = F + \mu \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{du_\phi}{dr} \right) - \frac{u_\phi}{r^2} \right), \quad E_\phi = u_r B - \eta \frac{dB}{dr} = 0,
\]

where in Eq. (29) we neglect some small terms proportional to \( u_r \). This is justified since Eq. (31) implies \( u_r \sim \eta \) and \( \eta \) is small.

From the hard wall boundary condition, \( u_r \big|_{r=0} = 0 \). Hence, (31) implies \( d/br \big|_{r=0} = 0 \), which, together with the no-slip boundary condition \( u_\phi \big|_{r=0} = 0 \) and (29), imply \( \partial \rho \big|_{r=0} = 0 \). Except for those just mentioned, there are no further constraints on the equilibrium profile. For any given \( \rho(r) \) and \( u_\phi(r) \), the required particle source \( S \) and applied force \( F \) can be determined as follows. Equation (29) determines \( B^2 \) up to a integration constant; with the \( B \) obtained, Eq. (31) determines \( u_r \); Eq. (28) then determines the particle source \( S \). The applied force \( F \) can be determined by using Eq. (30).

\section*{IV. VELOCITY SHEAR STABILIZATION}

Velocity shear stabilization of interchanges in Dean flow was studied analytically in Ref. 8. The stability criterion, based on a conservative estimate of negligible growth, is

\[
R^2 \Omega^{1/2} > \gamma_s^2 \ln(R\mu),
\]

where

\[
\gamma_s = (-R \Omega^{1/2} \rho' / \rho)^{1/2}
\]

is the growth rate without velocity shear, \( \mu \Omega^{1/2} \) is a Reynolds number, and primes denote derivative with respect to \( r \). Since we do not include the effective gravity introduced in Ref. 8, the \( \gamma_s \) here does not depend on \( \rho' \). Our first task is to test this theory in more detail. Given the somewhat arbitrary nature of the criterion, we limit ourselves to a qualitative confirmation. Two qualitative predictions can be deduced from the criterion (32). First, for the same unstable density profile and shear flow, larger \( R \) (aspect ratio) systems would be more stable, since \( \gamma_s \sim 1/\sqrt{R} \). Second, the efficacy of the velocity shear stabilization decreases with a decrease of dissipative coefficients, somewhat counter-intuitively. We will examine these two predictions in this section.

We choose the equilibrium density profile

\[
\rho_0 = 1 + A \cos(\pi x),
\]

where \( x = r - R \). And we choose the flow profile to be the solution of

\[
F_0 + \mu \left( \frac{1}{r} \frac{d}{dr} \left( \frac{r \, du_\phi}{dr} \right) - \frac{u_\phi}{r^2} \right) = 0,
\]

i.e., the velocity profile driven by a constant applied force, subject to no-slip boundary conditions. In the limit \( R \rightarrow 1 \), the flow is approximately parabolic, \( u_\phi = (F_0/2\mu) x (1 - x) \). We choose \( A = 0.5 \), \( F_0 / \mu = 32 \) for this simulation. The temperature is set to be unity. As mentioned, \( B^2 \) is determined up to an integration constant from Eq. (29). We choose the constant such that \( B(x = 1/2) = 5 \). For the parameters we choose, the sound speed \( C_s = 1 \); the peak of \( u_\phi \) corresponds to Mach number \( M_S = u_\phi / C_s = 4 \).

\section*{A. Cases with no velocity shear}

As a benchmark, we first did some simulations with no velocity shear. Since no background flow is present, the centrifugal force from the rotation was replaced by an artificial gravity pointing outwardly. More precisely, Eq. (2) was replaced by

\[
\rho \partial_r u + \rho u \cdot \nabla u = -\nabla \rho - \frac{\nabla B^2}{2} + B \cdot \nabla B + \mu \nabla^2 u + \rho g \hat{z},
\]

where \( g = u_\phi^2 / r \), with \( u_\phi \) being the steady flow. In this way, we can use the same source term to maintain the same density profile in this artificial system.

We ran the simulation for \( R = 2, 4, 6, 8, \) and \( 10 \); and we took \( \mu = \eta = 0.002 \). As we varied the radius \( R \), we kept the box length \( L = (R + 1/2) \phi_0 = 2 \); i.e., the simulation box was approximately kept at the same size. Initially the density is higher inside. We started the simulation by adding an initial random perturbation of the order \( 10^{-3} \) to the momentum density in the \( r \) direction. For all cases, the initial perturbation grew exponentially, and the equilibrium was completely destroyed. Figure 1 shows the time sequence of the density for the case \( R = 4 \) in six gray-scale frames. The unstable mode first showed up as the characteristic “mushrooms” of interchanges; as the plasma continued to swirl around the chamber, the density was further mixed up, resulting in a slightly higher density on the outside. This behavior is typical, for all cases; only the time scales are different.

To quantify the deviation of the density profile from the equilibrium, we define the following function:

\[
W(t) = \langle [1 - \rho / \rho_0] \rangle,
\]

where the angular brackets denote an average taken over the whole volume at a given time \( t \). Figure 2 plots the time evolution of \( W(t) \) for different \( R \). The growth rate of the unstable mode can be estimated from the slope of the \( W(t) \) curve in Fig. 2, which is a semi-log plot) during linear growing regime. The growth rate scales as \( 1/\sqrt{R} \), as expected.

\section*{B. Cases with velocity shear}

We next repeated the same numerical experiment, but with the velocity shear. As will be shown, the interchange
mode was largely mitigated, though the stabilization was incomplete. A series of simulations have been done to test the theoretical prediction.

1. Dependence on aspect ratio

First we test the aspect ratio dependence of the stabilization by shear flow. The theoretical prediction is that large aspect ratio systems should be more stable, essentially due to the fact that the mode growth rate without flow shear scales as $1/\sqrt{R}$. We repeated the same simulation for $R = 2, 4, 6, 8,$ and $10$; $\eta = \mu = 0.002$; but this time with the shear flow. Figure 3 plots the time evolution of $W(t)$ for different $R$. The initial perturbation still grew, but now saturated to a much lower level compared to Fig. 2. For larger $R$ the saturation level is lower, in agreement with the theoretical prediction. Figure 4 plots the $\phi$ averaged density profile $\bar{\rho}$ (the overbar denotes averaging over $\phi$) after the saturation of the interchange instability. The top left panel is the laminar density, shown for reference. The remaining five panels correspond to increasing $R$, with the laminar density overlaid (dashed). The large flattened portion in the middle for the case $R = 2$ is evident. As $R$ becomes larger, the saturated density profile gets closer to the laminar profile. At $R = 10$, the laminar profile is almost completely retained. The localization of the disturbance near the middle is, presumably due to the fact that the flow shear is weakest there. This is consistent with previous simulations$^{11}$ and the analytic result of Ref. 13.

As a measurement of the turbulent flow, Fig. 5 plots the time evolution of the average radial kinetic energy, $\langle \rho u_r^2/2 \rangle$, where the average is taken over the volume. A noteworthy phenomenon is the “oscillatory” behavior, which could be understood as follows. In the beginning the instability led to turbulent flow and started mixing up the density. After the free energy was tapped, the density profile was flattened; the turbulence then lost its driving force and started to decay, followed by a “quiescent” period. As the system became quiescent, the particle source would try to “rebuild” the density profile, which made the system go unstable again. Since the density profile never rebuilt to the original level before the instability destroyed it again, the subsequent peaks were always lower than the first one. Figure 6 plots the time sequence of the density in six gray-scale frames, in which the above mentioned behavior is evident. It should be mentioned...
that although the interchanges do not destroy the whole density profile, the residual wobble could imply an enhanced cross-field transport in a real system.\textsuperscript{11}

2. Dependence on Reynolds number

To test the Reynolds number dependence of velocity shear stabilization, we simply re-did the same simulation with different dissipation coefficients. We fixed \( R = 4 \) throughout these simulations and varied \( \eta \) and \( \mu \). We wish to look at the effect of viscosity and resistivity separately. In one set of simulations, we kept \( \mu = 0.0005 \), and set \( \eta \) to three different values: 0.0005, 0.0005, and 0.00005. In another set of simulations, we kept \( \eta \) but changed \( \mu \). Figure 7 plots the saturated density profile (averaged along \( \phi \)) for each case after the onset of interchanges, which clearly shows that the deviation from the laminar density profile gets larger as either \( \mu \) or \( \eta \) gets smaller. Although both resistivity and viscosity seem to affect the stability in the same way, a more detailed look reveals some difference. Figure 8 depicts the time evolution of the average radial kinetic energy, for both sets of simulations. All the cases (except the one with \( \mu = 0.005 \) and \( \eta = 0.0005 \)) seem to have the same initial growth rate, yet the dissipation affects the subsequent decay of the disturbance. The plots show that the peak of the radial kinetic energy roughly scales as \( \mu \) (the bottom), and is independent of \( \eta \) (the top). We observe that although the peak of \( \left< \rho u_r^2/2 \right> \) is roughly independent of \( \eta \), smaller \( \eta \) causes a

![FIG. 4. The saturated density profile (averaged over \( \phi \)) after the onset of the interchange instability. The top left panel is the laminar density, shown for reference. The remaining five panels correspond to increasing \( R \), with the laminar density overlaid (dashed). At \( R = 2 \), the density profile is heavily flattened at the middle (where the flow shear is weakest); however, at \( R = 10 \), the laminar profile is almost completely retained. \( \eta = \mu = 0.002 \) in these simulations.](image1)

![FIG. 5. The time sequence of the density for the case \( R = 4 \). Flute interchanges occurred but were localized at the middle (\( t = 10,14 \)). The interchanges flattened the density and started to decay (\( t = 24 \)), leading to a quiescent period (\( t = 44 \)). As the particle source tried to rebuild the density profile, the system went unstable again (\( t = 60 \)).](image2)
slower decay of the disturbance. This is presumably the cause of the large deviation from the laminar density profile. Viscosity, on the other hand, affects the turbulent flow more significantly, as it directly dissipates the kinetic energy.

In summary, although the stability criterion for interchange modes is essentially local, it nevertheless provides a fairly intuitive picture of the stabilizing mechanism. Its theoretical predictions are qualitatively borne out by direct simulations.

C. The “weakest” point

We observe that the residual wobbles were localized about the radius where \(d\Omega/dr = 0\). Therefore, one is led to suggest that a system could be completely laminar provided \(d\rho_0/dr > 0\) at this “weakest” point. To test for this, we now changed the laminar density profile to

\[
\rho_0(x) = 1 - C \cos(2\pi(x - Dx(1-x))
\]

where the parameters \(C\) and \(D\) determine the amplitude and the position of the central density peak, respectively. Several different parameters have been tried. Here we only report the result of two cases: \(C=0.25, D=0.5\) and \(C=0.25, D=-0.5\). We took \(R=4, L=2,\) and \(\eta = 0.0005\). The amplitude \(C=0.25\) was chosen so that the Rayleigh’s inflection point criterion is not violated (to be discussed in more detail later). Both cases have destabilizing stratification somewhere, hence both are unstable to the “no-shear” test. However, the former has a stabilizing stratification at the weakest point, while the latter has a destabilizing one. When we ran both cases with flow shear, the former is completely stable, while unstable modes developed in the latter, resulting in a flattop of the density profile. This is clearly shown in Fig. 9, where we plot the laminar density profile of the two cases, with the saturated density profile of the unstable case overlaid. Therefore, we conclude that it is highly desirable to have a stabilizing stratification at the weakest point. By judiciously placing particle sources we presumably have some control over that.

V. KELVIN–HELMHOLTZ INSTABILITY

It is well-known that flow shear could drive the Kelvin–Helmholtz instability. For the Dean flow system, one can
prove that in the limit $R \to \infty$, the following "generalized" Rayleigh’s inflection point theorem holds, namely
\[
\frac{d}{dr} \left( \frac{\rho}{r} \frac{d}{dr} (r^2 \bar{\Omega}) \right) \neq 0
\] (39)
is a sufficient condition for ideal stability. For finite $R$ systems one cannot expect the same condition to hold. An immediate counterexample is that, the cases in Sec. IV all satisfy the Rayleigh’s criterion, yet all of them are unstable. Nonetheless one may still expect the Rayleigh’s criterion to be a somewhat useful indicator for long wavelength modes. One should also bear in mind that even in the limit $R \to \infty$, violation of Rayleigh’s criterion does not always imply instabilities.

To demonstrate the utility of Rayleigh’s criterion, we report the result of two simulations. Both have density profiles of the form of (38); one with $C=0.5$, $D=0.5$ and the other with $C=0.5$, $D=-0.5$. We took $R=4$, $L=2$, $\eta=\mu =0.002$. Both cases violate Rayleigh’s criterion; in addition, the latter has a destabilizing density stratification at the weakest point. Now we see a different kind of unstable behavior. Figure 10 shows the time sequence of the former case in six gray-scale frames. The characteristic Kelvin cat’s eye of the Kelvin–Helmholtz instability are clearly visible. Figure 11 shows the time sequence of the latter one. Since the density stratification in this case is destabilizing at the weakest point, the interchanges set in at first, followed by a Kelvin–Helmholtz type of behavior. Taking a look at the time evolution of the average radial kinetic energy (Fig. 12) reveals more about the difference between the two cases. After the initial, violent stage, both cases settled down to a somewhat quiescent new state. However, we observed that in the latter case, the average radial kinetic energy fluctuated about $10^{-5}$, due to the residual wobbles about the weakest point, in a manner described in Sec. IV. On the other hand, the former case enjoyed a long period (about 100 sound times) of a nearly stable state, with $\langle \rho u_r^2/2 \rangle < 10^{-7}$. At $t = 170$, the system started going unstable again, and $\langle \rho u_r^2/2 \rangle$ finally settled to about $10^{-4}$. To elaborate, in Fig. 13 we plot $\bar{\rho}$, $\langle (\bar{\rho}/r)(r^2 \bar{\Omega}) \rangle'$, and $\rho u_r^2/2$ at four representative times. At $t=0$, $\langle (\bar{\rho}/r)(r^2 \bar{\Omega}) \rangle'$ is greater than zero in a region around $r=R+0.9$. After the free energy was released, at $t=100$, $\langle (\bar{\rho}/r)(r^2 \bar{\Omega}) \rangle'$ is less than zero everywhere. At this stage the system is very stable, as one can see from the very small radial kinetic energy. However, as the particle source rebuilds the density profile, $\langle (\bar{\rho}/r)(r^2 \bar{\Omega}) \rangle'$ about the point $r=R+0.8$ was again approaching zero at $t=175$. It was about this time unstable modes started to grow again. Finally, after the unstable modes saturated, at $t=250$, the density profile was about “marginal” to the Rayleigh’s criterion. We also notice that at this stage the radial convection was “global” instead of being localized at some small region (which was the case for the residual wobbles of interchanges). Compared to the localized residual wobbles, this global radial flow could be more harmful to the cross-field heat transport. To assess that we have to use a nonisothermal equation of state, which is beyond the scope of the present study.

VI. OTHER ISSUES

A. Hysteresis

In all the simulations mentioned above, we first calculated the 1D equilibrium, seeding it with a 2D random noise as the starting point for the 2D stability test. We then looked for the saturated state after the onset of instabilities. This approach, rather than starting from some nonequilibrium 2D state, saves us some time since the density adjustment takes place on a resistive time scale, which is usually long. However, this does not preclude the possibility that the system could run into violent turbulence and get “clamped” somewhere during the formation. To test for this, we reran the case $R=2$, $\eta=\mu =0.002$ in 2D with the initial density and the magnetic field both flat (both were calculated from the equilibrium we expected to achieve, since our particle source conserves total mass and the perfect conducting boundary condition conserves total magnetic flux). The initial flow was set to zero, with some random noise added. If there had been any hysteresis, the final state would have been qualitatively
different from what we reported previously. This was found not to be the case. We, therefore, concluded that there was no hysteresis.

B. Asymmetric source

It is well-known that asymmetric sources drive convection cells. For a system close to marginal stability, the driven convection could be very large. Thus far our derivation and simulation assume axisymmetric particle sources. The question how an asymmetric source may affect the result is relevant, since an axisymmetric source could be more difficult and costly to realize in practice. If the particle source is fixed in the laboratory frame, one may naturally suggest that the fast rotation could smear out the asymmetric source and effectively make it “quasi”-symmetric. Furthermore, since velocity shear is effective in tearing apart the convection caused by the interchange mode, the same effect may apply to the asymmetric source driven convection. To test for this, we re-did several simulations with asymmetric source by replacing $S(r) \rightarrow S(r)(1 + 0.5 \cos(4\pi\phi/d_0))$. For those completely laminar cases, the asymmetric source caused a small fluctuation of $u_r$ along the $\phi$ direction. The fluctuation in $u_r$ was of the order of the laminar $u_r$; therefore, the asymmetry did not seem to significantly enhance the cross-field transport. The density profile remained intact as well. On the other hand, for those wobbling cases, the asymmetric source...
introduced no discernible effect. We conclude that an asymmetric source may not cause problems. If this is indeed the case, the implementation could be easier.

C. Generalized Ohm’s law and thermoelectric effect

In this study, we use the Ohm’s law in its simplest form. To better model the real system the generalized Ohm’s law should be taken into account, and the thermoelectric effect should also be considered. Including this physics will certainly change the detail of the slowly diffusing equilibrium described in Sec. II. However, we observe that none of this extra physics can provide a unidirectional toroidal current drive. Since such a toroidal current is required to balance the centrifugal force, including this physics does not seem to change the main conclusion of this paper, namely that particle sources are necessary to maintain a centrifugally confined steady state, and that particle sources may be utilized to control the density profile, therefore, affect the stability.

VII. SUMMARY AND CONCLUSION

In this paper we studied the equilibrium and the stability of a slowly diffusing rotating plasma with particle sources. We summarize our findings as follows:

1. Particle sources (and sinks) are necessary for a steady state of centrifugally confined plasmas. The density profile of the steady state depends on the placement of particle sources.

2. Such a slowly diffusing steady state is realizable only when it is stable. Our simulation shows velocity shear stabilization of interchanges, and the steady state is largely maintained. The velocity shear stabilization may not be complete, noticeably around the “weakest point” where \( \Omega' = 0 \). In those cases the interchanges flatten the density profile about the weakest point, therefore bring the system close to marginal stability. Some residual wobbles still remain; however, they are well localized and the “transport barrier” near the walls is still maintained. Furthermore, by adjusting the particle source we could have a stabilizing stratification about the weakest point, therefore, achieve a completely stable equilibrium.

3. The Kelvin–Helmholtz instability is also studied. When the generalized Rayleigh’s criterion is violated, the KH modes could occur. The KH instability again will not destroy the steady state completely; it rather brings the system close to marginal stability, with some residual convection.

We are now in a position to address the question why the system of Ref. 9 was so stable. In that case both the density and the pressure stratification were stabilizing at the weakest point (in a rotating mirror both the density and the pressure could drive interchanges; unfortunately, only the pressure profile was published in Ref. 9). It will then be interesting to see how the residual wobbles may affect the cross-field transport in a centrifugally confined plasma, if one can produce a unfavorable profile by “injudiciously” placing the source.

In conclusion, introducing a shear flow complicates the stability of the system, yet this complication also means more flexibility. In particular, the density profile plays a significant role in the stability of a rotating system, while it is completely irrelevant in a static system. Certainly a strong velocity shear is not always stabilizing. However, since the stability is profile dependent, by adjusting various profiles via various sources we may have some control over it.

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