A Torso-Moving Balance Control Strategy for a Walking Biped Robot Subject to External Continuous Forces

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Moving the torso laterally in a walking biped robot can be mechanically more torque-efficient than not moving the torso according to recent research. Motivated by this observation, a torque-efficient torso-moving balance control strategy of a walking biped robot subject to a persistent continuous external force is suggested and verified in this paper. The torso-moving balance control strategy consists of a preliminary step and two additional steps. The preliminary step (disturbance detection) is to perceive the application of an external force by a safety boundary of zero moment point, detected approximately from cheap pressure sensors. Step 1 utilizes center of gravity (COG) Jacobian, centroidal momentum matrix and linear quadratic problem calculation to shift the zero moment point to the center of the support polygon. Step 2 makes use of $H_\infty$ controllers for a more stable state shift from single support phase to double support phase. By comparing the suggested torso moving control strategy to the original control strategy that we suggested previously, a mixed balance control strategy is suggested. The strategy is verified through numerical simulation results.

Keywords: Biped robot; walking; balance control; external force; torque efficient; torso.

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1. Introduction

Presently, with the maturity of humanoid locomotion research, some questions have arisen regarding the dynamic stability of the humanoid robot when some external force is applied to the robot. The stabilizing method is important because a biped robot may be substantially damaged when it tips over being subject to an externally applied force. This method of stabilizing the robot is the so-called “push recovery” method. The push recovery method was first studied when the robot was standing still. The push recovery method has been categorized as an “ankle strategy”, a “momentum (hip) strategy” and a “step out strategy”. Ankle strategy is utilized when the external force is relatively small, and the robot can withstand this small external force with only ankle torque control. Huang suggested an ankle torque compensation technique for a small disturbance force in which zero moment point fluctuation is compensated for during walking. Hemami et al. utilized a feedback linearization technique using hip and ankle torque against the external disturbance force. The effects of discretization and quantization are studied in the simulation of a compound inverted pendulum model. The momentum (hip) strategy involves the use of upper body momentum to compensate for intermediate external force. Lee et al. suggested a momentum control strategy for the external force disturbance on nonlevel, nonstationary ground. This strategy utilized the optimization technique and a centroidal momentum matrix. Asmar et al. applied the ankle strategy and momentum strategy with virtual mode control (VMC) and a linearization technique for the external disturbance force. The step out strategy is used when the external force is so large that the ankle and momentum strategy cannot withstand the externally applied force. Pratt et al. suggested a capture point for the stepping out strategy. At present capture point is widely used in the balance control strategy for a humanoid robot. These three control strategies are summarized by Stephens.

Meanwhile, some research results involving a balance method for a walking humanoid robot subject to an external force was recently put forward. Yasin et al. suggested a step out control strategy for a walking biped robot subject to an external applied force using the capture point. Yi et al. suggested a balance strategy using the capture point and a reinforcement learning technique. Li et al. created a dynamic balance in a walking biped robot using sensor fusion, the Kalman filter and fuzzy logic to compensate for the joint angle of a biped robot walking subject to an external disturbance. Adiwahono et al. made a control method to perform a push recovery during walking through walking phase modification utilizing linear inverted pendulum mode (LIPM) and energy analysis. Also, Cho et al. developed a control method to achieve a dynamic balance of a hopping humanoid robot using a linearization method.

Also, active compliance controllers are successful in dealing with external disturbances. Hyon used an active compliance control in rough terrain adaptation. Ott et al. suggested a posture and balance controller utilizing the active compliance
control schematics. Ugurlu et al.\textsuperscript{17} made a dynamically equilibrated one-legged running experiments utilizing a position-based active compliance control.

All the above balance control strategies assume that the external force is impulsive except the active compliance controller-based control strategy and that the robot need an inertial measurement unit (IMU) to activate the strategy. The external force can be persistently continuous or constant, in a real situation and IMU is not only expensive, but also needs additional noise-suppressing signal processing. And the active compliance controller for disturbance suppression needs to calculate the amount of external forces and torques to suppress. To overcome these points, we proposed a balance control strategy for a walking biped robot subject to a constant external force in the paper\textsuperscript{18}. The method need not use IMU and can withstand an impulsive or constant force and need not calculate the amount of an external forces.

The paper\textsuperscript{18} incorporated a whole-body cooperative control to balance against an external forces. The advantage of a whole-body cooperative control is to use all of its body parts cooperatively to balance against an external forces resulting in a less concentrated actual joint torques. There are a number of whole-body cooperative biped control studies. Yang et al.\textsuperscript{19} made a biped walking pattern considering the upper-body motion which is a result of motion captures of walking of a real human. Yamaguchi et al.\textsuperscript{20} proposed a basic control method of whole body cooperative walking that uses trunk and trunk-waist motion to compensate for moments generated by the body motion. Ugurlu et al.\textsuperscript{21,22} incorporated Eulerian zero moment point resolution to make a more precise zero moment point calculation and make use of it for yaw moment compensation during walking by the waist torque control. Englsberger et al.\textsuperscript{23} made a walking control based on a capture point\textsuperscript{7} dynamics, zero moment point and center of gravity (COG). Matsubara et al.\textsuperscript{24} make use of a reinforcement learning technique to learn to acquire whole body CoM movement to achieve dynamic tasks. As a further study of the paper,\textsuperscript{18} we suggest a torque-efficient torso-moving balance control strategy of a walking biped robot subject to an continuous external force. The main advantage of this method is that by moving the torso to comply with the external force, the torque of a stance leg is diminished compared with the method suggested in the paper.\textsuperscript{18} $H_{\infty}$ controllers are utilized in this paper for a more robust phase shift from the single support phase to double support phase. A recent paper\textsuperscript{25} also suggests that a lateral spiral motion during walking makes the robot mechanically energy efficient compared with walking with no spiral motion. The methodology in the paper\textsuperscript{25} uses a lateral spiral motion to save energy while in this paper, only the torso moves laterally to save the torque subject to an external force.

The subsequent sections are as follows. The preliminary information is presented in Sec. 2. The biped model and predefined working gait are explained in Sec. 3. The proposed balance control strategy is suggested and explained in Sec. 4. The overview of the proposed torso-moving balance control strategy is explained in Sec. 5. In Sec. 6, the numerical simulation results are presented. Finally, the conclusions are presented in Sec. 7.
2. Problem Statement

In our previous study, we have remarked that a new balance control strategy is inevitable when a persistently continuous or constant force is applied to the biped robot. A simple simulation carried out based on the open dynamics engine (ODE) substantiates that the existing methods devised to endure impulsive forces may fail to deal with continuous forces.

3. Biped Model and Predefined Walking Gait

The biped robot model which is used in the simulations is 12DOF, 6DOF per each leg. The biped robot model is depicted in Fig. 1(a). The detailed mass, length and inertia data are displayed in Tables 1 and 2. The convex hull of the biped model is depicted in Fig. 1(b). The admissible zero moment point region in Fig. 1(b) will be explained later.

We assumed that an external force is applied in the torso of the biped robot. The direction of the force is arbitrary in the X and Y directions and there is no external force in the Z direction. The force in the Z direction is excluded because the tangential

Fig. 1. The biped robot model. (a) The overall biped model schematics. (b) The convex hull of the biped model.
force on the side surface of the torso is negligible compared with the normal force. We further assumed that the joint motors of the biped robot are position-controlled motors because most state-of-the-art humanoid robots are constructed using position-controlled joint motors and there is no slip and no contact with other object during the application of an external force. All these are summarized below.

(i) The direction of the force is arbitrary in $X$ and $Y$ direction and there is no force in $Z$ direction.

(ii) The joint motors are position-controlled motors.

Table 1. Mass and length data.

<table>
<thead>
<tr>
<th>Mass number</th>
<th>Mass (kgm)</th>
<th>Length number</th>
<th>Length (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>2.34</td>
<td>$L_1$</td>
<td>0.2</td>
</tr>
<tr>
<td>$M_2$</td>
<td>5.93</td>
<td>$L_2$</td>
<td>0.27</td>
</tr>
<tr>
<td>$M_3$</td>
<td>10.9</td>
<td>$L_3$</td>
<td>0.41</td>
</tr>
<tr>
<td>$M_4$</td>
<td>58.3</td>
<td>$L_4$</td>
<td>0.41</td>
</tr>
<tr>
<td>$M_5$</td>
<td>10.9</td>
<td>$L_5$</td>
<td>0.22</td>
</tr>
<tr>
<td>$M_6$</td>
<td>5.93</td>
<td>$L_6$</td>
<td>0.41</td>
</tr>
<tr>
<td>$M_7$</td>
<td>2.34</td>
<td>$L_7$</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$W_1$</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$W_2$</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Table 2. Inertial data.

<table>
<thead>
<tr>
<th>$I$ $(m^2kg)$</th>
<th>$I_x$</th>
<th>$I_y$</th>
<th>$I_z$</th>
<th>$I_{xy}$</th>
<th>$I_{xz}$</th>
<th>$I_{yz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{xx}$</td>
<td>0.07</td>
<td>0.02</td>
<td>0.06</td>
<td>5.86</td>
<td>0.06</td>
<td>0.02</td>
</tr>
<tr>
<td>$I_{yy}$</td>
<td>0.00</td>
<td>0.47</td>
<td>0.82</td>
<td>1.11</td>
<td>0.82</td>
<td>0.47</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>0.07</td>
<td>0.47</td>
<td>0.8</td>
<td>6.75</td>
<td>0.8</td>
<td>0.47</td>
</tr>
</tbody>
</table>
(iii) There is no slip and no contact with other object during the application of an external force.

The predefined walking gait that is used in this paper is the same as the gait from our previous paper\cite{27} and it is depicted in Fig. 2. Refer to Ref. 27 for details on the walking gait used in this study. The gait from the previous paper\cite{27} consists of single support phase, double support phase 1 and double support phase 2. In this paper, we use only the single support phase walking gait because we assumed that the external force was applied in the single support phase. The overall single support phase walking time is 600 ms, and the stride is 0.595 m.

4. The Proposed Torso-Moving Balance Control Strategy

The proposed strategy consists of a preliminary step (disturbance detection) and Steps 1 and 2. Disturbance detection, which is just a preliminary step to enter Steps 1 and 2 concerns the method to perceive the abnormal situation when some arbitrary
constant external force is applied to the torso of the walking biped robot. The biped robot only measures zero moment point. An inertial measurement unit (IMU) is not needed in this preliminary step. The objective of Step 1 is to move the zero moment point to the center of the support polygon. Step 1 is activated by torso-moving and force-resisting control. Step 2 is to switch from the single support phase to the double support phase for more solid stability by a robust control method. Each step is explained fully in the next subsection.

4.1. **Preliminary step (disturbance detection): Perceiving the abnormal situation**

To activate the proposed control strategy, the biped robot must recognize the abnormal situation in which some external force is applied to the biped robot. For this objective, we defined the admissible zero moment point region as depicted in Fig. 1(b). The sole size of the biped robot is $0.2 \times 0.27$ m, and the admissible zero moment point region is a $0.14 \times 0.21$ m rectangle located at the center of the convex hull. The size of this rectangle is approximately 70% of the size of the convex hull. We established the predefined walking gait such that the zero moment point moves in the admissible zero moment point region. In normal walking without an external force, the zero moment point of the biped robot remains in the admissible zero moment point region. When some persistently continuous external force is applied to the robot, the zero moment point moves out of the admissible zero moment point region. The robot ascertains that the abnormal situation has occurred when the zero moment point is located out of the admissible zero moment point region.

4.2. **Step 1: Moving the zero moment point to the center of the support polygon**

The authors have already suggested a method to move the zero moment point to the center of the support polygon utilizing the LIPM and the linear quadratic
programming technique in a previous paper.\textsuperscript{18} There are methods such as ankle strategy and hip strategy to move the zero moment point to the center of the support polygon subject to an external force. However, in Ref. 18 the authors used whole body cooperative framework which utilizes LIPM and LQ problem technique because the method using whole body can resist an external force and moves the zero moment point to the center of the support polygon more efficiently than the conventional ankle or hip method. In this paper, we combined the method\textsuperscript{18} with torso-moving control for a more torque-efficient balance control strategy.

4.2.1. \textit{Discrete linear inverted pendulum model method}

In this section, the original strategy in the previous paper\textsuperscript{18} is represented. Because the original strategy is based on utilizing DLIP model, we refer to the original strategy as DLIP model method. In the control strategy, the controller first converts the $\Delta ZMP$ to $\Delta COG$ by using Eqs. (1) and (2). This is LIPM\textsuperscript{32} expanded by a discrete finite time difference.

$$\Delta ZMP = \Delta COG - \frac{Z_c}{g} \text{COG}\delta t^2,$$

$$\Delta ZMP = ZMP_{\text{target}} - ZMP_{\text{current}}$$

$$\Delta COG = \Delta ZMP + \frac{Z_c}{g} \{2\text{COG}(t - 1) - 5\text{COG}(t - 2)$$

$$+ 4\text{COG}(t - 3) - \text{COG}(t - 4)\}.$$  

In Eq. (2), $Z_c$ is the height of COG of the biped robot, $g$ is the gravitational acceleration, COG is second time derivative of COG and $\delta t$ is the time increment (0.001 sec). In Eq. (2), COG$(t - n)$ is COG at $t - n$ time ($n$ step before time $t$). We make $\Delta ZMP$ increment 0.1 [mm]. The sampling frequency is 1 [ms] in the simulation. With this sampling frequency, if $\Delta ZMP$ is large, $\Delta COG$ will also be large which makes the COG Jacobian constraint error large. With a safety factor, we set this increment value to 0.1 mm. After calculating $\Delta COG$, we calculate $\Delta \theta_t$, the actual joint angle increments using the COG Jacobian and constrained linear quadratic programming. The bold variable means that it is a vector quantity. The COG Jacobian and constrained linear quadratic programming in robot balance were first introduced by Sugihara et al.\textsuperscript{28} The detailed equations are displayed below.

$$\Delta \text{target} \theta_t = (\theta_{\text{target}} - \theta_{t-1}),$$

$$\Delta \theta_t = (\theta_t - \theta_{t-1}).$$

Minimize

$$\frac{1}{2} (\Delta \text{target} \theta_t - \Delta \theta_t)^T W (\Delta \text{target} \theta_t - \Delta \theta_t)$$

Subject to

$$J_{\text{COG}} \Delta \theta_t = \Delta COG,$$

$$|\Delta \theta_t| < 0.01^\circ,$$
\[ J_{\text{COG}} = \begin{pmatrix} \frac{\partial x_{\text{COG}}}{\partial \theta_1} & \frac{\partial x_{\text{COG}}}{\partial \theta_2} & \cdots & \frac{\partial x_{\text{COG}}}{\partial \theta_{13}} \\ \frac{\partial y_{\text{COG}}}{\partial \theta_1} & \frac{\partial y_{\text{COG}}}{\partial \theta_2} & \cdots & \frac{\partial y_{\text{COG}}}{\partial \theta_{13}} \\ \frac{\partial z_{\text{COG}}}{\partial \theta_1} & \frac{\partial z_{\text{COG}}}{\partial \theta_2} & \cdots & \frac{\partial z_{\text{COG}}}{\partial \theta_{13}} \end{pmatrix}, \tag{6} \]

\[ W = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 500 \end{pmatrix}. \tag{7} \]

\( \theta_{\text{target}} \) in Eq. (3) is the target (reference) joint angle. The target (reference) joint angles are determined by inverse kinematics in which the robot is in double support phase with minimal angular displacement from the current posture. \((5)\) is the objective function that minimizes the distance between \( \Delta_{\text{target}} \theta_t \) and \( \Delta \theta_t \) with two constraints. The weighting matrix is shown in \((7)\). All of the diagonal elements of the weight matrix are 1 except the five elements in the lower diagonal, which are set to 500 because the left leg must track the target angle more rigorously. For the constraints, the values of the upper and the lower range of \( \Delta \theta_t \) are set to \(+0.01\) [°] and \(-0.01\) [°], because the solution must be in this range to provide a fine control of the actuating motors. The COG Jacobian is another constraint that must be satisfied. The definition of the COG Jacobian is shown in \((6)\). In short, the control logic is what determines \( \Delta ZMP \) and calculates \( \Delta \text{COG} \) and finally, \( \Delta \theta_t \) by the linear quadratic constrained optimization technique.

4.2.2. The proposed torso-moving balance control strategy

The torso-moving strategy is rotating the torso laterally to the direction complying with the external force with the hip angle movement of the stance leg. Figure 3(c) explains this situation. In Fig. 3(c), the biped robot is viewed from the front, and the biped robot rotates its torso in the direction complying with the external force by moving the roll direction hip angle of the stance leg. The aim of this torso movement is to alleviate the contact pressure in the torso applied by the external constant force. Another aim is to alleviate the torque in the ankle, knee and hip by this torso movement. Moving the torso in the direction complying with the external force moves the COG of the biped robot to the center of the convex hull and eventually lessens the actuating control torque in the ankle, knee and hip. A detailed torque comparison between implementation of the torso-moving strategy and that of the DLIP model method is explained in Sec. 6.5.

The detailed equation is described below.

\[ \Delta \theta_t = (\theta_t - \theta_{t-1}), \tag{8} \]
Minimize \( \frac{1}{2} (\Delta \theta_t)^T U (\Delta \theta_t) \)

Subject to
\[
A_G \Delta \theta_t = \Delta \mathbf{h}_G \\
|\Delta \theta_t| < 0.01^\circ
\]

\[
\Delta \mathbf{h}_G = \begin{pmatrix}
\frac{d \text{ZeroMomentPoint}}{dt} \\
\frac{d \text{ZeroMomentPoint}}{dt} \\
0 \\
\Delta \text{COG}_x \\
\Delta \text{COG}_y \\
\Delta \text{COG}_z
\end{pmatrix},
\]

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0.01 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

Fig. 3. The overall procedure of the proposed balance control strategy.
Equation (8) is the same as (4). (9) is the linear quadratic optimization problem. The objective function is solely to minimize $\Delta \theta_t$ with the weighting matrix $U$. The weighting matrix is described in (11), only the weight of the hip angle of the stance leg is 0.01, and all the others are 1 because we need hip angle movement only for the stance leg. The constraint shown in (9) utilizes the centroidal momentum matrix $A_G$. The centroidal momentum matrix $A_G$ relates the linear and angular velocities of the centroid of the robot to the angular velocity of the joints. Its original equation is represented in (12). The $h_G$ in (12) is $6 \times 1$ centroidal momentum vector and its elements are $x$, $y$ and $z$ direction angular momentum and $x$, $y$, $z$ direction linear momentum. The $A_G$ in (12) is $6 \times n$ centroidal momentum matrix which is a function of $q$, the actual joint angle displacement. The $q$ term in (12) is the angular velocities of $n$ joints. In constraints, $\Delta h_G$ is set as shown in (10) such that the linear momentum is only $\Delta COG$ and the angular momentum is set by the zero moment point velocity at the moment when the zero moment point crosses the admissible zero moment point region. The greater the magnitude of the external force, the faster the zero moment point velocity, eventually by solving (9), the torso rotates more rigorously.

The proposed control strategy is a combination of above two linear quadratic problems (DLIP model method + torso moving strategy). The overall linear quadratic problem is described below.

$$\Delta^{\text{target}} \theta_t = (\theta_{\text{target}} - \theta_{t-1}),$$

$$\Delta \theta_t = (\theta_t - \theta_{t-1}).$$

Minimize

$$\frac{1}{2} (\Delta^{\text{target}} \theta_t - \Delta \theta_t)^T W (\Delta^{\text{target}} \theta_t - \Delta \theta_t)$$

$$+ \frac{1}{2} (\Delta \theta_t)^T U (\Delta \theta_t)$$

Subject to

$$J_{COG} \Delta \theta_t = \Delta COG$$

$$A_G \Delta \theta_t = h_G$$

$$|\Delta \theta_t| < 0.01^\circ$$

The weighting matrix in Eq. (15) is $W$ and $U$. We set $W$ and $U$ as (7) and (11), respectively. As mentioned previously, setting 500 in (7) is for the left leg to track the target angle more rigorously. $W$ and $U$ are empirically tuned. In this study, $U$ is assigned 0.1.

By solving Eq. (15), the proposed torso-moving balance control strategy is accomplished. Additionally, note that (15) is also a constrained linear quadratic problem formation. It is therefore, possible to solve (15) with various solvers. We used the solver “qld” to solve (15). This solver takes some time (order of $10 \mu s$ seconds) to calculate (15) and which is considered as a real-time in 1 kHz sampling frequency.
4.3. Step 2: Transition from single support phase to double support phase with the proposed $H_\infty$ controller

After moving the zero moment point to the center of the support polygon, the controller must switch the biped robot from single support phase to double support phase to increase stability. This movement can be realized by the inverse kinematics technique. Performing inverse kinematics to the desired double support phase gait and incrementing the actual joint angle from the current posture to the desired double support phase posture switch the biped robot from single support phase to double support phase. However, the problem occurs when incrementing the motors from the current position to the target double support phase position because the external force can change and can disturb the biped robot. In our knowledge, there is hardly any closed loop control methods applicable for this transition process as like in Step 1. Our proposed strategy is to suppress this type of external disturbance using a $H_\infty$ controller. There are two typical methods to suppress the external disturbance, the sliding mode controller and $H_\infty$ controller. We choose $H_\infty$ controller because in linear system, the sliding mode controller has the restriction that the external disturbance must be in the range space of the input matrix $B$, which means that the external disturbance must have some characteristics for the sliding mode controller to take effect.

To design the $H_\infty$ controller, we reduce the 12DOF robot model into the 2DOF and 3DOF planar manipulators separately. The 2DOF manipulator represents the lateral configuration of the biped robot and the 3DOF manipulator represents the sagittal configuration of the biped robot. In 2DOF manipulator model, the actuating motor is the roll-direction ankle and hip motors of the stance leg and in 3DOF manipulator model, it is the pitch-direction ankle, knee and hip motors of the stance leg. This is from the assumption that suppressing the movement of the stance leg also suppress the movement of the swing leg. With this model, we then linearize the acquired nonlinear equations at the equilibrium posture. The equilibrium point is every posture from the current posture to the target double support phase posture. With these linearized models, we design $H_\infty$ controllers, and the controllers can suppress the external disturbance at each equilibrium posture.

4.3.1. Modeling the biped robot to planar 2DOF and 3DOF manipulators

We must first model the biped robot to planar 2DOF and 3DOF manipulators. When the biped robot is viewed in the lateral plane, the robot can be viewed as a 2DOF manipulator. The 2 moving joints of the 2DOF manipulator correspond to the roll direction motor of the ankle and the roll direction motor of the hip joint of the biped robot. When the biped robot is viewed in the sagittal plane, the robot can be viewed as a 3DOF manipulator. The three moving joints of the 3DOF manipulator also correspond to the pitch direction motor of the ankle, the pitch direction motor of the knee and the pitch direction motor of the hip joint of the biped robot. The two manipulator models each have a $H_\infty$ controller and thus can suppress sagittal and lateral disturbance. The $H_\infty$ controllers must be designed at every posture from current to the target.
Figure 4 explains this modeling. The 12DOF biped robot is now modeled to the two link planar manipulator and the three link planar manipulator. $m_1$, $m_2$ and $m_3$ are the total mass of link 1, link 2 and link 3. $I_1$, $I_2$ and $I_3$ are the mass moment of inertia at the center of mass of link 1, link 2 and link 3. $l_{c1}$, $l_1$, $l_{c2}$, $l_2$ and $l_{c3}$, $l_3$ are the lengths to the center of mass and the total length of link 1, link 2 and link 3. $\tau_1$, $\tau_2$ and $\tau_3$ constitute the actual joint torque at each joint. $\theta_1$, $\theta_2$ and $\theta_3$ are the angular displacements at each joint.

With this model, the linearized state-space equations are constructed by exact differentiation of the governing equation. The states of the linearized state-space equation of 2DOF manipulator are $\delta \theta_1$, $\delta \theta_2$, $\delta \dot{\theta}_1$, $\delta \dot{\theta}_2$ and the inputs are $\tau_1$ and $\tau_2$.

![Diagram](image)

Fig. 4. Modeling the biped robot in lateral and sagittal planes. (a) 2DOF manipulator model and (b) 3DOF manipulator model.
outputs are $\delta \theta_1$ and $\delta \theta_2$. Equation (16) is the linearized state space equation. Each variable used in Eq. (16) is represented in Appendix A.

The states of the linearized state-space equation of the 3DOF manipulator are $\delta \theta_1$, $\delta \theta_2$, $\delta \theta_3$, $\delta \theta_4$, $\delta \theta_5$. The inputs are $\tau_1$, $\tau_2$, $\tau_3$ and the outputs are $\delta \theta_1$, $\delta \theta_2$, $\delta \theta_3$. Equations (17) and (18) are the linearized state space equations. Each variable used in (17) is represented in Appendix B.

\[
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
A & C & E & G \\
B & D & F & H
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4
\end{bmatrix} +
\frac{1}{(X_1 + X_2 \cos^2 \theta_2)}
\begin{bmatrix}
d_{22} & \alpha + \beta \cos \theta_2 \\
\alpha + \beta \cos \theta_2 & \gamma + \varepsilon \cos \theta_2
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix},
\]

\[
\begin{bmatrix}
\delta y_1 \\
\delta y_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4
\end{bmatrix}.
\]

\[
x_1 = \theta_1, \ x_2 = \theta_2, \ x_3 = \dot{\theta}_1, \ x_4 = \dot{\theta}_2,
\]

\[
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4 \\
\delta x_5 \\
\delta x_6
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
A_1 & B_1 & C_1 & D_1 & E_1 & F_1 \\
A_2 & B_2 & C_2 & D_2 & E_2 & F_2 \\
A_3 & B_3 & C_3 & D_3 & E_3 & F_3
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4 \\
\delta x_5 \\
\delta x_6
\end{bmatrix} +
\frac{1}{D}
\begin{bmatrix}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix},
\]

\[
\begin{bmatrix}
\delta y_1 \\
\delta y_2 \\
\delta y_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4 \\
\delta x_5 \\
\delta x_6
\end{bmatrix}.
\]

As stated previously, we assumed that the biped robot has position-controlled motors. In the linearized equation, the inputs are torques ($\tau_1$, $\tau_2$ and $\tau_3$) and our simulator directly send torque command. If the desired position is needed, forward dynamics can be used.
4.3.2. $H_\infty$ controller design

After obtaining each linearized state space equation, we must design the $H_\infty$ controllers and control the motor at every equilibrium point with these controllers.

Figure 5 represents the generalized plant for each $H_\infty$ controller. In Fig. 5, $P$ represents the linearized model at the equilibrium point, and $K$ represents the $H_\infty$ controller. $W_m$ and $W_M$ are output weighting matrices. In this generalized plant, we design $K$ minimizing the $H_\infty$ norms in Eq. (19).

$$\begin{align*}
\| W_M \frac{P}{I - PK} \|_\infty &< \gamma, \\
\| W_m \frac{PK}{I - PK} \|_\infty &< 1.
\end{align*}$$

(19)

The inequality in Eq. (19) is for suppressing the influence of the external disturbance $w$ on output $z_1(y)$. We explain this inequality. If we set the $W_M$ equal 1, the transfer equation from $w$ to $z_1$ is as follows:

$$z_1 = y = P(u + w), \quad u = Ky,$$

$$z_1 = (I - PK)^{-1}Pw.$$  \hfill (20)

Therefore, suppressing the influence of the external disturbance $w$ on output $z_1$ means to determine $K$ satisfying the following inequality:

$$\| (I - PK)^{-1}P \|_\infty < \gamma.$$  \hfill (21)

The smaller the $\gamma$, the better the suppressing the external disturbance. However, Eq. (21) is very severe inequality which suppress the external disturbance in all frequency spectrum. Therefore, we introduce transfer function $W_M$ which is multiplied as below to suppress the external disturbance in some specific frequency spectrum.

$$\| W_M(I - PK)^{-1}P \|_\infty < \gamma.$$  \hfill (22)

The inequality (22) is the same as the inequality in Eq. (19).

![Diagram](image)

Fig. 5. Generalized plant.
The inequality in Eq. (19) is for robust stability with respect to the multiplicative uncertainty. We induce the inequality in (19).

Figure 6 represents the plant with multiplicative uncertainty \( \Delta_m \). The actual plant \( \hat{P} \) represented as follows:

\[
\hat{P} = (I + \Delta_m)P. \tag{23}
\]

With the plant with multiplicative uncertainty, the transfer function from point a to point b in Fig. 6 is as follows:

\[
T_m = (I - PK)^{-1}PK. \tag{24}
\]

As a result, Fig. 6(a) equals Fig. 6(b). From small gain theorem, if inequality (25) is satisfied, the system can be stabilized.

\[
\| \Delta_m T_m \|_{\infty} < 1. \tag{25}
\]

However, the multiplicative uncertainty \( \Delta_m \) cannot be obtained in general, \( W_m \) which satisfies the below inequality is used instead.

\[
\sigma \{ \Delta_m(j\omega) \} < \| W_m(j\omega) \| \forall \omega. \tag{26}
\]

As a result, inequality (27) which is the same as the inequality in (19) is obtained

\[
\| W_m T_m \|_{\infty} < 1. \tag{27}
\]

The weighting matrix \( W_m \) must fulfill the inequality in Eq. (28) and \( W_M \) is set to a constant value.

\[
| \Delta_m(j\omega) | \leq | W_m(j\omega) | \forall \omega, \quad \Delta_m(j\omega) = \frac{P_u(j\omega) - P(j\omega)}{P'(j\omega)}. \tag{28}
\]

\( P_u \) in (28) is the uncertain plant. \( P_u \) is calculated by perturbing the equilibrium point angle and angular velocity with approximately 1 \(^{\circ}\) and 1 \(^{\circ}/s\).

After determining all the infinity norm, we calculate \( K \) using D–K iteration procedure.\(^{31}\)
5. The Proposed Torso-Moving Balance Control Strategy Summary

For a thorough understanding of the proposed torso-moving balance control strategy, we demonstrate a situation in which the proposed strategy is applied and provide an overview of the proposed strategy.

The biped robot walks with a pre-defined walking gait. Then, a sudden external force is applied to the torso of the biped robot. The external force is constant or persistently continuous, so the zero moment point of the biped robot is moving away from the center of the sole. Without some intervention, the robot will eventually fall down. This situation is represented in Figs. 3(a) and 3(b).

The biped robot then controls its joint motors to avoid falling down through Steps 1 and 2, as in Fig. 3(c). After the three steps, the biped robot is in the double support phase, a robust stance posture against an external force, as in Fig. 3(d).

The flowcharts that summarize the proposed torso-moving balance control strategy are also presented in this section.

The overall flowchart is in Fig. 7, and the detailed flowchart of Steps 1 and 2 is in Figs. 8 and 9. In the overall flowchart, the predefined gait is continued until the zero moment point is out of the admissible zero moment point region. If the zero moment point is out of the stability region, Steps 1 and 2 are sequentially activated.

In Step 1 flowchart, the controller first determines the zero moment point. After determining the zero moment point, the controller calculates $\Delta COG$ from $\Delta ZMP$ by Eq. (2) and further calculates $h_G$ after determining the exit velocity of zero moment point across the admissible zero moment point region. With these information, the controller solves the constrained linear quadratic problem of (15). With

![Fig. 7. Overall flowchart.](image-url)
Fig. 8. Step 1: Moving the zero moment point to the center of the support polygon.

Fig. 9. Step 2: Switch the robot to double support phase state.
the solution, the joint motors are incremented. Finally, if the zero moment point is in the center of the support polygon, Step 1 ends. If not, the controller loops back to the $\Delta \text{COG}$ calculation.

In Step 2, the controller performs the inverse kinematics of the target double support phase gait state. After this process, the controller increment the actual joint angles toward the target (reference) joint angles. The increment value is approximately $1/100$ of the $\Delta$ (target angle–current angle). Afterwards, the controller checks whether it reached the target double support phase state or not. If the target double support phase state is reached, Step 2 ends. If the target double support phase state is not reached, the controller uses the linearized state space equations at the current gait and design $H_\infty$ controllers. The robot is controlled by $H_\infty$ controllers during the control time. The magnitude of the control time is 1 [ms]. After the control time, the actual joint angles are again incremented and the loop is continued.

6. Numerical Simulation Results

The numerical simulation results consist of gait characteristics, zero moment point, torque at each joint, $H_\infty$ controller performance and the results of the comparison of torque between the DLIP model method and proposed strategy.

The simulation machine consists of Inter core i6 CPU with 3.24 Gbyte RAM. The sampling rate of the simulation is 1 [ms].

The base coordinate system is the Cartesian coordinate system, which is described in Fig. 1(b).

The time derivative of actual joint angle is numerically performed by the backward finite difference formula (error: $O(h^2)$). Zero moment point is calculated by Eq. (29).

$$\text{ZeroMomentPoint} = \frac{Z_0 \times M_0}{Z_0 \cdot F_0}, \quad (29)$$

where $Z_0$ is the $z$-direction base coordinate, and $F_0$ and $M_0$ are force and moment acting on the sole of the stance leg of the biped robot in the base coordinate system. $F_0$ and $M_0$ are calculated by iterative Newton–Euler dynamics.

The simulation conditions are summarized in Table 3. There are six cases in this simulation. The $X$-direction forces are 0 [N] and $\pm 30$ [N], and the $Y$-direction forces are $+70$ [N] and $-70$ [N]. The time to apply the force is a quarter of the walking time when $+70$ [N] is applied and half of the walking time when $-70$ [N] is applied. We selected the time to apply the force in this manner because if $+70$ [N] is applied at quarter of the walking time, the zero moment point will be out of the zero moment point stability boundary at one-third of the walking time, and the zero moment point will be out of the zero moment point stability boundary as soon as $-70$ [N] is applied at half of the walking time. A force of $\pm 30$ [N] in the $X$-direction is the perturbation force to disturb the motion in the $X$-direction.
6.1. Gait characteristics

The gait characteristics of Cases 1 and 4 are presented in this subsection. The other cases show almost the same results as Cases 1 and 4 for the addition of the ± 30 [N] X-direction perturbation force which only slightly affects the gait characteristics.

The gait characteristics in Case 1 are depicted in Fig. 10. Figures 10(a) and 10(b) are viewed in the sagittal plane, and Figs. 10(c)–10(e) are viewed in the lateral plane. The coordinates of these pictures viewed in the sagittal and lateral plane are displayed in Figs. 10(a) and 10(c). The pictures are displayed in time-ascending order. The biped robot is about to begin its single support phase walking in Fig. 10(a). The left leg is the swing leg, and the right leg is the stance leg. The swing leg is stretched behind to begin its swing during single support phase walking. In Fig. 10(b), the +Y direction external force of 70 [N] is applied to the torso of the biped robot. From this time, the zero moment point gradually moves toward the boundary of the admissible zero moment point region. In Fig. 10(c), the elapsed time is 200 [ms] and the robot recognizes the abnormal situation in which the +70 [N] force has been applied. (Disturbance detection) and starts controlling by Step 1. The controller moves the whole body against the +70 [N] external force, and the torso is being controlled to incline to the +Y direction. In Fig. 10(d), the ZeroMomentPoint_y is in the center of the support polygon and ends the Step 1. The link which connects the left and right legs (L_5 in Fig. 1(a)) is almost flat, which means that the torso is inclined to the +Y direction. Step 2 ends in Fig. 10(e), and the swing left leg is on the ground. The overall elapsed time is 1300 [ms].

The gait characteristics of Case 4 are depicted in Fig. 11. In this case, as described in Table 3, the biped robot is subject to −70 [N] force applied in the Y direction. Figure 11(a) demonstrates that the biped robot is about to begin single support phase walking. In Fig. 11(b), the robot recognizes the abnormal situation in which the zero moment point is out of the stability region and thus, begins Step 1, which moves the whole body of the biped robot against the externally applied force of −70 [N] and inclines the torso of the biped robot to the −Y direction. In Fig. 11(c), the zero moment point is in the center of the support polygon. The link between left leg and right leg (L_5 in Fig. 1(a)) is in a left point down and right point up configuration, which
Fig. 10. Gait characteristics of Case 1. (a) Beginning (0 ms), (b) external force application (150 ms), (c) Step 1 begins (200 ms), (d) Step 2 begins (800 ms), and (e) Step 2 ends (1300 ms).
means that the torso is inclined in the \(-Y\) direction. Step 2 ends in Fig. 11(d) and the swing left leg is in contact with the ground. The overall time is 950 [ms].

6.2. Zero moment point results

The zero moment point results explain our proposed strategy well because the Step 1 strategy is to move the zero moment point to the center of the support polygon.

The zero moment point results from the simulation in Case 1 are described in Fig. 12. While \(\text{ZeroMomentPoint}_x\) in Step 1 phases is almost flat, \(\text{ZeroMomentPoint}_y\) from 200 to 800 [ms] decreases to zero by the zero moment point control logic. Then, there is a slightly lowering of \(\text{ZeroMomentPoint}_y\) in Step 2 for switching to the double support phase state in Step 2 moves the \(\text{ZeroMomentPoint}_y\) slightly. The characteristics of the \(Y\)-direction Zero Moment Point of Cases 2 and 3 are almost the same as the characteristics shown in Fig. 12. The \(X\)-direction zero moment point of Cases 2 and 3 is slightly higher and lower compared with Fig. 12, respectively because there are +30 [N] and −30 [N] external \(X\)-direction force disturbances.
The zero moment point results for the simulation in Case 4 are described in Fig. 13. From 300 to 455 [ms] (Step 1 phase), the zero moment point control is in progress, and the \( \text{ZeroMomentPoint}_y \) changes from approximately \(-0.05 \text{ m}\) to \(0 \text{ m}\). During the right leg contracting process from 455 to 950 [ms] (Step 2 phase), there is a slight increase in \( \text{ZeroMomentPoint}_y \). The \( \text{X-direction zero moment point} \) in Fig. 13 is almost flat. The \( \text{X-direction zero moment point} \) in Case 5 is slightly higher than that in Fig. 13. The \( \text{X-direction zero moment point} \) of Case 6 is slightly lower than that in Fig. 13 because the perturbation force of \(\pm 30 \text{ [N]}\) directly affects the \( \text{X-direction zero moment point} \) during the entire control process. The \( \text{Y-direction zero moment point} \) is almost the same in Cases 4–6.

**6.3. Torque results**

The actual joint torque is the amount of torque that is exerted in each of the motors. The motors exerting the most torque of all the 12 motors are roll direction ankle motor, pitch direction knee motor and roll direction hip motor of the stance leg (right leg). These motors are represented in Fig. 1(a) (1, 3 and 6 coordinate system motors). The torques of these motors in Cases 1 and 4 are presented in this subsection.

The right (stance) leg major joint actuating torque in Cases 1 and 4 is described in Figs. 14 and 15. Each of the three graphs in Figs. 14 and 15 represents the roll direction ankle, pitch direction knee and roll direction hip joint actuating torque of the stance leg. The ankle joint actuating torque in Fig. 14 changed abruptly from \(-50 \text{ [Nm]}\) to \(+50 \text{ [Nm]}\) at 200 [ms] because of the sudden continuous force of 70 [N].
Fig. 13. Zero moment point results (Case 4).

Fig. 14. Torque graphs of right (stance) leg (Case 1).
applied in the Y-direction. The torque decreases from $+50$ [Nm] to 0 [Nm] during Step 1 because the Y-direction zero moment point control is activated.

During Step 2, the ankle torque decreases slightly, displaying characteristics similar to those shown in ZeroMomentPoint graph. The knee joint actuating torque in Fig. 14 during Step 1 phase is almost flat. The value is approximately 50 Nm at the end of Step 1. The hip actuating torques in Fig. 14 during Step 1 increased because the torso moving control is activated. The value is approximately 110 [Nm] at the end of Step 1. The ankle actuating torque in Fig. 15 increased from 50 [Nm] to 0 [Nm] in Step 1. The knee actuating torque in Fig. 15 is almost flat, whereas in the hip case, this value decreases during the Step 1 phase, the results of the torso moving control. The knee joint actuating torque in Figs. 14 and 15 increased linearly during Step 2 because the knee of the stance leg bends to cause the robot double support phase status to endure the weight of the robot.

6.4. $H_\infty$ controllers performance results

As explained previously, two $H_\infty$ controllers are constructed at every sampling time in Step 2, and the two controllers can robustly control each motor against external disturbances. We designate the two-link model $H_\infty$ controller as $H_\infty$ controller 1 and the three-link model $H_\infty$ controller as $H_\infty$ controller 2.

The $H_\infty$ controller 1 performance results at one equilibrium point ($\theta_1 = 1.76749$ rad ($101.27^\circ$), $\theta_2 = -3.1127$ rad ($-178.34^\circ$), $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$) are presented as an example. The $H_\infty$ controller 2 performance results at one equilibrium
point \((\theta_1 = 1.2217 \text{ rad (70.0°)}, \theta_2 = 0 \text{ rad (0°)}, \theta_3 = -2.0944 \text{ rad (120.0°)}, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0 \text{ and } \dot{\theta}_3 = 0)\) are also presented as an example.

To design the controllers, \(W_m\) and \(W_M\) values must be determined. \(W_M\) is determined by the trial and error process by changing the value from 1 to 10, 100, 1000 and 5000 and finding out the adequate value. The rule of thumb is 100, and \(W_m\) is determined by plotting the multiplicative uncertainty as described in Eq. (28). The specified values are in (30) and (31). Equation (30) corresponds to the controller 1, and (31) corresponds to controller 2.

\[
W_M = \begin{bmatrix}
100 & 0 \\
0 & 100
\end{bmatrix} \tag{30}
\]

\[
W_m = \frac{1}{5} \frac{(s^2 + 30s + 900)}{(s^2 + 1.8s + 36)} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \tag{30}
\]

\[
W_M = \begin{bmatrix}
100 & 0 & 0 \\
0 & 100 & 0 \\
0 & 0 & 100
\end{bmatrix} \tag{31}
\]

\[
W_m = \frac{1}{5} \frac{(s^2 + 5.6s + 80)}{(s^2 + 4.5s + 225)} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{31}
\]

With the weighting matrix above, \(H_{\infty}\) controllers 1 and 2 are designed by the D–K iteration procedure. The designed \(H_{\infty}\) controllers 1 and 2 characteristics are displayed in Figs. 16 and 17, respectively.

K1, K2, K3 and K4 in Fig. 16 represent the \((1,1), (1,2), (2,1)\) and \((2,2)\) element of \(H_{\infty}\) controller 1 and K1, K2, K3, K4, K5, K6, K7, K8 and K9 in Fig. 17 represent the \((1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\) and \((3,3)\) elements of \(H_{\infty}\) controller 2, respectively.

The \(\theta_1\) and \(\theta_2\) values with \(H_{\infty}\) controller 1 and conventional PI controller under impulsive torque disturbance and the \(\theta_1\), \(\theta_2\) and \(\theta_3\) values with \(H_{\infty}\) controller 2 and conventional PI controller under impulsive torque disturbance are depicted in Figs. 18 and 19, respectively.

If there are no controllers, the \(\theta_1\), \(\theta_2\) and \(\theta_3\) diverges rapidly under impulsive torque disturbance. With PI controllers, the three value stabilizes with some oscillation as in Figs. 18 and 19. However, with \(H_{\infty}\) controllers, they converges more rapidly compared with PI controller under impulsive torque disturbance as depicted in Figs. 18 and 19.

6.5. Torque requirements between Step 1 phase of the DLIP model method (Ref. 18) and the proposed strategy

The advantage in Step 1 of the proposed strategy compared with the DLIP model method\(^{18}\) is torque-efficiency. The point is that less torque is needed in executing the proposed strategy than in executing the DLIP model method. In this subsection, we verify this point by comparing the torques.
Fig. 16. $H_\infty$ controller 1 characteristics.

Fig. 17. $H_\infty$ controller 2 characteristics.
The conditions of the cases in Table 4 correspond to the conditions of the cases in Table 3. There are six cases in Table 4. The ankle torque means the roll direction ankle torque, the knee torque means the pitch direction knee torque, and the hip torque means the roll direction hip torque of the stance leg (right leg). The average torque is the average of the three torques (ankle, knee and hip).

There are three strategies in Table 4. One is the DLIP model method and another is the proposed strategy and the other is the ankle strategy. The ankle strategy is just a conventional ankle strategy for comparing the results. The bold numeric value means that it is the lowest value in the three strategies Fig. 20 is the graphical representation of the average torques of the three strategies.

The average torque of the Step 1 phase of the proposed strategy decreases in Cases 4–6 compared with the average torque of the DLIP model method and increases in Cases 1 and 3. The average torque of the Step 1 phase of the two strategies is almost the same in Case 2. Actually the average torque is decreased in the case where the external continuous force is $-Y$ direction and is increased in the case in which the external force is $+Y$ direction. The main reason of the increase of the torque of $+Y$ direction case is the increase of hip torque. It increased because the COG of the torso is moving away from the center of the right hip, causing more hip torque to move the torso. In this simulation, the left leg is the swing leg, and the right leg is the stance leg.
Fig. 19. Impulsive response results of $H_\infty$ controller 2.

Fig. 20. Average torque comparison between strategies.
The ankle torque of the Step 1 phase of the proposed strategy is decreased in all cases compared with the ankle torque of the DLIP model method. Clearly, the proposed strategy decreases the ankle torque. The ankle torque decreases because the fact that by moving the torso in a manner complying with the external force, the center of gravity is getting close to the ankle of the stance leg, thus less ankle torque is needed.

The hip torque of Step 1 of the proposed strategy is increased in Cases 1–3 compared with the hip torque of the DLIP model method and is decreased in Cases 4–6. The decrease of the hip torque in Cases 4–6 occurs because the COG of the torso is getting close to the center of the right hip, requiring less hip torque to move the torso. In Cases 1–3, the proposed strategy increases the torque because the COG of the torso is moving away from the center of the right hip, causing more hip torque to move the torso.

In summary, when the left leg is the swing leg and the right leg is the stance leg in walking, the Step 1 phase of the proposed strategy is torque-efficient in the $-Y$ direction of external force application compared with the DLIP model method. In the $+Y$ direction in the external force application case, the Step 1 phase of the proposed strategy is not torque-efficient compared with the DLIP model method but only diminishes the ankle torque significantly. The mixed Step 1 strategy, in which Step 1 of the DLIP model method is executed in the $+Y$ external force application case, and Step 1 of the proposed strategy is executed in the $-Y$ external force application case, can be suggested. If only $\pm X$ direction external force application, both the DLIP model method and proposed Step 1 strategy can be executed. The disturbance detection phase is the same in the DLIP model method and proposed strategy and

Y.-J. Kim, J.-Y. Lee & J.-J. Lee

<table>
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<th>Case</th>
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<th>Ankle torque</th>
<th>Knee torque</th>
<th>Hip torque</th>
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Table 4. Torque comparison (Unit: [Nm]).
Step 2 of the proposed strategy is superior to the DLIP model method because Step 2 of the proposed strategy can suppress the external force disturbance. The overall flowchart of this mixed strategy is shown in Fig. 21.

7. Conclusion

In this paper, we proposed a torso-moving balance control strategy of a walking biped robot subject to external continuous force. First, the reason why some balance control strategy is needed in a continuous external force case is explained by applying a capture point strategy to the case of a continuous external force. The proposed strategy consists of preliminary step (disturbance detection), Steps 1 and 2, which are fully explained in Sec. 4. Step 1 makes use of COG Jacobian, centroidal momentum matrix and linear quadratic problem solving to move zero moment point to the center of the support polygon. We explained that the reason for moving torso in Step 1 is to diminish the torque on the stance leg and the contact force. Step 2 makes use of 2 and 3 link biped modeling, linearization and $H_{\infty}$ controllers to switch the biped from single support phase state to double support phase state more stably.
The proposed torso-moving strategy is verified by numerical simulations. Six individual cases are simulated. Zero moment point results show that the proposed strategy moves the out-going zero moment point to the center of the support polygon effectively and switch the robot from single support phase status to double support phase status more stably. $H_\infty$ controller performance is presented to investigate the validity of the $H_\infty$ controller design.

The primary reason of the torso-moving in proposed balance control strategy is torque-efficiency. The torque-efficiency is investigated by comparing the Step 1 phase of the proposed strategy with the Step 1 phase of the DLIP model method. The results show that the proposed strategy is torque-efficient in $-Y$ external continuous force application and not in $+Y$ external continuous force application when the right leg is the stance leg and the left leg is the swing leg. With this result, a mixed strategy is suggested where both the proposed strategy and DLIP model method are used in a torque-efficient manner.

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Appendix A. Parameters of Eq. (16)

$$\alpha = -(m_2 l_{1,2}^2 + I_2), \quad \beta = -m_2 l_1 l_{2,2}$$

$$\gamma = (I_1 + I_2) + m_1 l_{c1}^2 + m_2 (l_1^2 + l_{2,2}^2)$$

$$\varepsilon = 2m_2 l_1 l_{2,2}$$

$$d_{11} = m_1 l_{c1}^2 + m_2 (l_1^2 + l_{2,2}^2 + 2l_1 l_{2,2} \cos \theta_2) + I_1 + I_2$$

$$d_{12} = d_{21} = m_2 (l_1^2 + l_1 l_{2,2} \cos \theta_2) + I_2$$

$$d_{22} = m_2 l_{2,2}^2 + I_2$$

$$h_{11} = h_{21} = m_2 l_1 l_{2,2}, \quad h_{12} = 2m_2 l_1 l_{2,2}$$

$$\phi_{11} = (m_1 l_{c1} + m_2 l_1)g, \quad \phi_{12} = \phi_{21} = m_2 l_{2,2}g$$

$$X_1 = m_1 m_2 l_{c1}^2 l_{2,2}^2 + m_1 l_{c1}^2 I_2 + m_2 l_{2,2}^2 (l_1^2 + l_{c2}^2) + m_2 I_2 (l_1^2 + l_{c2}^2)$$

$$+ m_2 l_{2,2}^2 (I_1 + I_2) + (I_1 + I_2) I_2 - (m_2 l_{2,2}^4 + I_2^2 + 2m_2 I_2 l_{2,2}^2)$$

$$X_2 = -m_2 l_1^2 l_{2,2}^2$$
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\[ Y_1 = d_{22}h_{11} \sin \theta_2 \dot{\theta}_2^2 + d_{22}h_{12} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 - d_{22} \phi_{11} \cos \theta_1 \]
\[ - d_{22} \phi_{12} \cos(\theta_1 + \theta_2) - \alpha h_{21} \sin \theta_2 \dot{\theta}_1^2 - \alpha \phi_{21} \cos(\theta_1 + \theta_2) \]
\[ - \beta h_{21} \sin \theta_2 \cos \theta_2 \dot{\theta}_1^2 - \beta \phi_{21} \cos \theta_2 \cos(\theta_1 + \theta_2) \]
\[ Y_2 = \alpha h_{11} \sin \theta_2 \dot{\theta}_2^2 + \alpha h_{12} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 - \alpha \phi_{11} \cos \theta_1 - \alpha \phi_{12} \cos(\theta_1 + \theta_2) \]
\[ + \beta h_{11} \sin \theta_2 \cos \theta_2 \dot{\theta}_2 + \beta h_{12} \sin \theta_2 \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2 - \beta \phi_{11} \cos \theta_1 \cos \theta_2 \]
\[ - \beta \phi_{12} \cos \theta_2 \cos(\theta_1 + \theta_2) - \gamma h_{21} \sin \theta_2 \dot{\theta}_1^2 - \gamma \phi_{21} \cos(\theta_1 + \theta_2) \]
\[ - \varepsilon h_{21} \sin \theta_2 \cos \theta_2 \dot{\theta}_1^2 - \varepsilon \phi_{21} \cos \theta_2 \cos(\theta_1 + \theta_2) \]

\[ A = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)} \left( d_{22} \phi_{11} \sin \theta_1 + d_{22} \phi_{12} \sin(\theta_1 + \theta_2) \right) \]
\[ + \alpha \phi_{21} \sin(\theta_1 + \theta_2) + \beta \phi_{21} \cos \theta_2 \sin(\theta_1 + \theta_2) \]

\[ B = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)} \left( \alpha \phi_{11} \sin \theta_1 + \alpha \phi_{12} \sin(\theta_1 + \theta_2) + \beta \phi_{11} \cos \theta_2 \sin \theta_1 \right) \]
\[ + \beta \phi_{12} \cos \theta_2 \sin(\theta_1 + \theta_2) + \gamma \phi_{21} \sin(\theta_1 + \theta_2) + \varepsilon \phi_{21} \cos \theta_2 \sin(\theta_1 + \theta_2) \]

\[ C = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)^2} \left( d_{22} \dot{h}_{11} \dot{\theta}_2 \cos \theta_2 + d_{22} \dot{h}_{12} \dot{\theta}_1 \dot{\theta}_2 + \alpha \phi_{11} \sin(\theta_1 + \theta_2) \right) \]
\[ + d_{22} \phi_{12} \sin(\theta_1 + \theta_2) - \alpha h_{21} \dot{\theta}_1 \dot{\theta}_2 + \alpha \phi_{21} \sin(\theta_1 + \theta_2) \]
\[ - \beta h_{21} \dot{\theta}_1^2 (\cos^2 \theta_2 - \sin^2 \theta_1) - \beta \phi_{21} (-\sin \theta_2 \cos(\theta_1 + \theta_2)) \]
\[ - \cos \theta_2 \sin(\theta_1 + \theta_2)) (X_1 + X_2 \cos^2 \theta_2) + 2 Y_1 X_2 \sin \theta_2 \cos \theta_2 \]

\[ D = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)^2} \left( (\alpha h_{11} \dot{\theta}_2 \cos \theta_2 + \alpha h_{12} \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 + \alpha \phi_{12} \sin(\theta_1 + \theta_2) \right) \]
\[ + \beta h_{11} \dot{\theta}_2 (\cos^2 \theta_2 - \sin^2 \theta_2) + \beta h_{12} \dot{\theta}_1 \dot{\theta}_2 (\cos^2 \theta_2 - \sin^2 \theta_2) \]
\[ - \beta \phi_{11} \cos \theta_1 \sin(\theta_1 + \theta_2) + \beta \phi_{12} (\sin \theta_2 \cos(\theta_1 + \theta_2)) + \cos \theta_2 \sin(\theta_1 + \theta_2) \]
\[ - \gamma h_{21} \dot{\theta}_1^2 \cos \theta_2 + \gamma \phi_{21} \sin(\theta_1 + \theta_2) \]
\[ - \varepsilon h_{21} \dot{\theta}_1^2 (\cos^2 \theta_2 - \sin^2 \theta_2) + \varepsilon \phi_{21} (\sin \theta_2 \cos(\theta_1 + \theta_2)) + \varepsilon \phi_{21} (\cos \theta_2 \sin(\theta_1 + \theta_2)) \]
\[ \times (X_1 + X_2 \cos^2 \theta_2) + 2 Y_2 X_2 \sin \theta_2 \cos \theta_2 \]

\[ E = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)} \left( d_{22} \dot{h}_{12} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 - 2 \alpha h_{21} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 - 2 \beta h_{21} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 \right) \]

\[ F = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)} \left( \alpha h_{12} \sin \theta_2 \dot{\theta}_2 + \beta h_{12} \sin \theta_2 \cos \theta_2 \dot{\theta}_2 \right) \]
\[ - 2 \gamma h_{21} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 - 2 \varepsilon h_{21} \sin \theta_2 \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2 \]

\[ G = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)} \left( 2 d_{22} h_{11} \sin \theta_2 \dot{\theta}_2 + d_{22} \dot{h}_{12} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 \right) \]
\[ H = \frac{1}{(X_1 + X_2 \cos^2 \theta_2)} \left( 2 \alpha h_{11} \sin \theta_2 \dot{\theta}_2 + \alpha h_{12} \sin \theta_2 \dot{\theta}_1 ight) 
+ \beta h_{11} \sin \theta_2 \cos \theta_2 + \beta h_{12} \sin \theta_2 \cos \theta_2 \dot{\theta}_1 \right) \] (A.1)

**Appendix B. Parameters of Eq. (17)**

\[
n_1 = I_1 + I_2 + I_3 + m_1 l_1 + m_2 (l_1^2 + l_2^2) + m_3 (l_2^2 + l_3^2)
\]

\[
n_2 = 2(m_2 l_1 l_2 + m_3 l_1 l_2)
\]

\[
n_3 = 2m_3 l_1 l_3
\]

\[
n_4 = 2m_3 l_2 l_3
\]

\[
n_5 = I_2 + I_3 + m_2 l_2^2 + m_3 (l_2^2 + l_3^2)
\]

\[
n_6 = (m_2 l_1 l_2 + m_3 l_1 l_2)
\]

\[
n_7 = 2m_3 l_2 l_3
\]

\[
n_8 = m_3 l_1 l_3
\]

\[
n_9 = I_3 + m_3 l_3^2
\]

\[
n_{10} = m_3 l_1 l_3
\]

\[
n_{11} = m_3 l_2 l_3
\]

\[
n_{12} = I_2 + I_3 + m_2 l_2^2 + m_3 (l_2^2 + l_3^2)
\]

\[
n_{13} = 2m_3 l_2 l_3
\]

\[
n_{14} = I_3 + m_3 l_3^2
\]

\[
n_{15} = m_3 l_2 l_3
\]

\[
h_{11} = -2(m_2 l_1 l_2 + m_3 l_1 l_2)
\]

\[
h_{12} = -2m_3 l_1 l_3
\]

\[
h_{13} = -2m_3 l_2 l_3
\]

\[
h_{14} = -(m_2 l_1 l_2 + m_3 l_1 l_2)
\]

\[
h_{15} = -2m_3 l_2 l_3
\]

\[
h_{16} = -m_3 l_1 l_3
\]

\[
h_{17} = m_3 l_1 l_3
\]

\[
h_{18} = -m_2 l_2 l_3
\]

\[
h_{21} = -2m_3 l_2 l_3
\]

\[
h_{22} = -(m_2 l_1 l_2 + m_3 l_1 l_2)
\]

\[
h_{23} = -2m_3 l_2 l_3
\]

\[
h_{24} = -m_3 l_1 l_3
\]

\[
h_{25} = -m_3 l_1 l_3
\]

\[
h_{26} = -m_3 l_2 l_3
\]

\[
h_{27} = (m_2 l_1 l_2 + m_3 l_1 l_2)
\] (B.1)
\[ h_{28} = m_3 l_1 l_c \]
\[ h_{29} = (m_2 l_1 l_c + m_3 l_1 l_c) \]
\[ h_{210} = m_3 l_1 l_c \]
\[ h_{211} = m_3 l_1 l_c \]
\[ h_{31} = -m_3 l_1 l_c \]
\[ h_{32} = -m_3 l_2 l_c \]
\[ h_{33} = -m_3 l_2 l_c \]
\[ h_{34} = m_3 l_1 l_c \]
\[ h_{35} = m_3 l_2 l_c \]
\[ h_{36} = 2m_3 l_2 l_c \]
\[ h_{37} = m_3 l_1 l_c \]
\[ h_{38} = m_3 l_2 l_c \]
\[ h_{39} = m_3 l_1 l_c \]
\[ h_{310} = m_3 l_2 l_c \]
\[ h_{311} = m_3 l_2 l_c \]
\[ g_{11} = (m_1 l_1 + m_2 l_1 + m_3 l_1) g \]
\[ g_{12} = (m_2 l_1 + m_3 l_1) g \]
\[ g_{13} = m_3 l_1 g \]
\[ g_{21} = (m_2 l_2 + m_3 l_2) g \]
\[ g_{22} = m_3 l_2 g \]
\[ g_{31} = m_3 l_3 g \]
\[ M_{11} = n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3 \]
\[ M_{12} = n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3) \]
\[ M_{13} = n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3 \]
\[ M_{22} = n_{12} + n_{13} \cos \theta_3 \]
\[ M_{23} = n_{14} + n_{15} \cos \theta_3 \]
\[ M_{33} = I_3 + m_3 l_3^2 \]

\[ D = M_{11} M_{22} M_{33} + M_{12} M_{23} M_{13} + M_{12} M_{23} M_{13} - M_{22} M_{13}^2 \]
\[ - M_{11} M_{23}^2 - M_{13} M_{12}^2 \]

\[ M = \begin{bmatrix}
  M_{22} M_{33} - M_{23}^2 & -(M_{12} M_{33} - M_{13} M_{23}) & M_{12} M_{23} - M_{13} M_{22} \\
  -(M_{12} M_{33} - M_{23} M_{13}) & M_{11} M_{33} - M_{13}^2 & -(M_{11} M_{23} - M_{12} M_{13}) \\
  M_{12} M_{23} - M_{13} M_{22} & -(M_{11} M_{23} - M_{12} M_{13}) & M_{11} M_{22} - M_{12}^2
\end{bmatrix} \]
\[ A_1 = \left( (n_{14} + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3)M_{33} \right)(h_{11} \sin \theta_2 \hat{\theta}_1 \hat{\theta}_2) \\
+ h_{12} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) + h_{13} \sin \theta_3 \hat{\theta}_1 + h_{14} \cos \theta_2^2 + h_{15} \sin \theta_3 \hat{\theta}_2 \\
+ h_{16} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) - h_{17} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_2 + h_{18} \sin \theta_3 \hat{\theta}_3 \]

\[ A_2 = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))M_{33} \\
- (n_0 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \right)(h_{21} \sin \theta_3 \hat{\theta}_2 \\
+ h_{22} \sin \theta_2 \hat{\theta}_2 \hat{\theta}_1 + h_{23} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_1 + h_{24} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_1 \\
+ h_{25} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) + h_{26} \sin \theta_3 \hat{\theta}_3^2 + h_{27} \sin \theta_2 \hat{\theta}_1^2 + h_{28} \sin(\theta_2 + \theta_3) \hat{\theta}_1^2 \\
+ h_{29} \sin \theta_2 \hat{\theta}_1 \hat{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 + h_{211} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_3 \]

\[ A_3 = \left( (n_0 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
+ n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3) \right)(h_{31} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_1 \\
+ h_{32} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_1 + h_{33} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_2 + h_{34} \sin(\theta_2 + \theta_3) \hat{\theta}_1^2 + h_{35} \sin \theta_3 \hat{\theta}_1^2 \\
+ h_{36} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_1 + h_{37} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_1 + h_{38} \sin \theta_3 \hat{\theta}_3^2 + h_{39} \sin(\theta_2 + \theta_3) \hat{\theta}_3 \hat{\theta}_1 \\
+ h_{310} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_3 + h_{311} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_3 \]

\[ A_4 = \left( (n_{14} + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3)M_{33} \right)(g_{11} \cos \theta_1 + g_{12} \cos(\theta_1 + \theta_2) \\
+ g_{13} \cos(\theta_1 + \theta_2 + \theta_3)) \]

\[ A_5 = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) \right)M_{33} - (n_0 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \right)(g_{21} \cos \theta_1 + g_{22} \cos(\theta_1 + \theta_2 + \theta_3)) \]

\[ A_6 = \left( (n_0 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
+ n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3) \right)(g_{31} \cos \theta_1 + g_{32} \cos(\theta_1 + \theta_2 + \theta_3)) \] (B.2)

\[ B_1 = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3) \right)M_{33} \\
- (n_{14} + n_{15} \cos \theta_3)(n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) \right)(h_{11} \sin \theta_2 \hat{\theta}_1 \hat{\theta}_2 \\
+ h_{12} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_1 + h_{13} \sin \theta_3 \hat{\theta}_1 \hat{\theta}_2 + h_{14} \sin \theta_2 \hat{\theta}_2 \hat{\theta}_2 \\
+ h_{15} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_2 + h_{16} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_2 + h_{17} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_2 \\
+ h_{18} \sin \theta_3 \hat{\theta}_3 \]

\[ B_2 = \left( (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)^2 - (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) \\
+ n_4 \cos \theta_3 M_{33} \right)(h_{21} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_2 + h_{22} \sin \theta_2 \hat{\theta}_2 \hat{\theta}_1 + h_{23} \sin \theta_3 \hat{\theta}_2 \hat{\theta}_1 \\
+ h_{24} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_1 + h_{25} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_1 + h_{26} \sin \theta_3 \hat{\theta}_3 \]

\[ \quad + h_{27} \sin \theta_2 \hat{\theta}_1^2 + h_{28} \sin(\theta_2 + \theta_3) \hat{\theta}_1^2 + h_{29} \sin \theta_2 \hat{\theta}_1 \hat{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 \\
+ h_{211} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_3 \]
\[ B_3 = ((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \\
- (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{10} + n_{11} \cos(\theta_2 + \theta_3)) \\
+ n_{11} \cos \theta_3)) (h_{31} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{32} \sin \theta_3 \dot{\theta}_3 \dot{\theta}_1 + h_{33} \sin \theta_3\ddot{\theta}_2 \\
+ h_{34} \sin(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{35} \sin(\theta_3\dot{\theta}_1^2 + h_{36} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_2 + h_{37} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_2 \\
+ h_{38} \sin \theta_3 \dot{\theta}_2^2 + h_{39} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_3 + h_{310} \sin \theta_3 \dot{\theta}_3 \dot{\theta}_3 + h_{311} \sin \theta_3\ddot{\theta}_2 \dot{\theta}_3) \\
B_4 = ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))M_{33} \\
- (n_{14} + n_{15} \cos \theta_3)(n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)) \\
\times (g_{11} \cos \theta_1 + g_{12} \cos(\theta_1 + \theta_2) + g_{13} \cos(\theta_1 + \theta_2 + \theta_3)) \\
B_5 = ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)^2 - (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) \\
+ n_4 \cos \theta_3)M_{33}) (g_{21} \cos(\theta_1 + \theta_2) + g_{22} \cos(\theta_1 + \theta_2 + \theta_3)) \\
B_6 = ((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \\
- (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3)) (g_{31} \cos(\theta_1 + \theta_2 + \theta_3)) \\
C_1 = ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) \\
- (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3)) \\
\times (h_{11} \sin(\theta_2\dot{\theta}_1 \dot{\theta}_2 + h_{12} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \theta_3)\dot{\theta}_1 + h_{13} \sin \theta_3\dot{\theta}_3 \dot{\theta}_1 \\
+ h_{14} \sin \theta_2 \dot{\theta}_2^2 + h_{15} \sin \theta_3 \dot{\theta}_3 \dot{\theta}_2 + h_{16} \sin(\theta_2 + \theta_3)(\ddot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_2 \\
- h_{17} \sin(\theta_2 + \theta_3)(\ddot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_2 + h_{18} \sin \theta_3\dot{\theta}_3) \\
C_2 = ((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \\
- (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3))(h_{21} \sin \theta_2 \dot{\theta}_3 \dot{\theta}_2 + h_{22} \sin \theta_3 \dot{\theta}_1 \dot{\theta}_2 + h_{23} \sin \theta_3\dot{\theta}_3 \dot{\theta}_1 \\
+ h_{24} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{25} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{26} \sin \theta_3\dot{\theta}_3^2 \\
+ h_{27} \sin \theta_2 \dot{\theta}_1^2 + h_{28} \sin(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{29} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_2 \\
+ h_{211} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_3) \\
C_3 = ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))^2 - (n_1 + n_2 \cos \theta_2 \\
+ n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{12} + n_{13} \cos \theta_3))(h_{31} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1 \\
+ h_{32} \sin \theta_3\dot{\theta}_3 \dot{\theta}_1 + h_{33} \sin \theta_3\dot{\theta}_3 \dot{\theta}_2 + h_{34} \sin(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{35} \sin \theta_3\dot{\theta}_3^2 \\
+ h_{36} \sin \theta_3\dot{\theta}_1 \dot{\theta}_2 + h_{37} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_2 + h_{38} \sin \theta_3\dot{\theta}_3^2 \\
+ h_{39} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_3 + h_{310} \sin \theta_3\dot{\theta}_1 \dot{\theta}_3 + h_{311} \sin \theta_3\dot{\theta}_2 \dot{\theta}_3) \\
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\[ C_4 = \left( (n_0 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) (n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
+ n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3))(g_{11} \cos(\theta_1 + \theta_2) + g_{12} \cos(\theta_1 + \theta_2) \\
+ g_{13} \cos(\theta_1 + \theta_2 + \theta_3)) \right) \]

\[ C_5 = \left( (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \\
- (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3))(g_{21} \cos(\theta_1 + \theta_2) + g_{22} \cos(\theta_1 + \theta_2 + \theta_3)) \right) \]

\[ C_6 = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))^2 - (n_1 + n_2 \cos \theta_2 \\
+ n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{12} + n_{13} \cos \theta_3))(g_{31} \cos(\theta_1 + \theta_2 + \theta_3)) \right) \]

\[ \bar{\theta}_1 A = \left( (n_{14} + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3)M_{33} \right) \left( -g_{11} \sin \theta_1 - g_{12} \sin(\theta_1 + \theta_2) \\
- g_{13} \sin(\theta_1 + \theta_2 + \theta_3) + \left((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))M_{33} \right) \\
- \left((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{14} + n_{15} \cos \theta_3)\right) \left(-g_{21} \sin(\theta_1 + \theta_2) \right) \\
- g_{22} \sin(\theta_1 + \theta_2 + \theta_3) + \left((n_9 + n_{10} \cos(\theta_2 + \theta_3) - (n_5 + n_{10} \cos(\theta_2 + \theta_3) \right) \\
+ n_{11} \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \left(h_{21} \cos \theta_2 B_1 + h_{22} \cos \theta_3 B_1 + h_{23} \cos \theta_3 B_3 \right) \right) \\
+ (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) \sin \theta_3) \\
+ g_{31} \sin(\theta_1 + \theta_2 + \theta_3) \right) \]

\[ \bar{\theta}_1 B = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))M_{33} - (n_{14} + n_{15} \cos \theta_3) \right) \]

\[ \times \left((n_5 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) \left(-g_{11} \sin \theta_1 - g_{12} \sin(\theta_1 + \theta_2) \right) \\
- g_{13} \sin(\theta_1 + \theta_2 + \theta_3) + \left((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)^2 \right) \right) \\
- \left((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)M_{33} \right) \left(-g_{21} \sin(\theta_1 + \theta_2) \right) \\
- g_{22} \sin(\theta_1 + \theta_2 + \theta_3) + \left((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3) \right) \right) \\
\times \left((n_{14} + n_{15} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) \right) \\
+ g_{31} \sin(\theta_1 + \theta_2 + \theta_3) \right) \]

\[ \bar{\theta}_1 C = \left( (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) \right) \\
- \left((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3) \right) \\
\times \left( -g_{11} \sin(\theta_1 + \theta_2 + \theta_3) - g_{12} \sin(\theta_1 + \theta_2) - g_{13} \sin(\theta_1 + \theta_2 + \theta_3) \right) \\
+ \left((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \right) \]
\[
\theta_1 D = 0
\]

\[
\theta_2 A_1 = \left( (n_{14} + n_{15} \cos \theta_3) - (n_{12} + n_{13} \cos \theta_3) M_{33} \right) (h_{11} \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{12} \cos(\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{14} \cos \theta_2 \dot{\theta}_2^2 + h_{16} \cos(\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_2 \\
- h_{17} \cos(\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 \dot{\theta}_3
\]

\[
\theta_2 A_2 = \left( (-n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3)) M_{33} - (-n_{10} \sin(\theta_2 + \theta_3)) (n_{14} + n_{15} \cos \theta_3) \right) \\
\times (h_{21} \sin \theta_3 \dot{\theta}_1 \dot{\theta}_2 + h_{22} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 + h_{23} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_1 \\
+ h_{24} \sin(\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{25} \sin(\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{26} \sin \theta_2 \dot{\theta}_2^2 \\
+ h_{27} \sin \theta_2 \dot{\theta}_2^2 + h_{28} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 + h_{29} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{211} \cos(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_3 + ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 \\
+ n_8 \cos(\theta_2 + \theta_3)) M_{33} + h_{215} \cos(\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{27} \cos \theta_2 \dot{\theta}_1^2 \\
+ h_{28} \cos(\theta_2 + \theta_3) \dot{\theta}_1^2 + h_{29} \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2 + h_{210} \cos(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{211} \cos(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_3) \right)
\]

\[
\theta_2 A_3 = \left( (-n_{10} \sin(\theta_2 + \theta_3)) (n_{12} + n_{13} \cos \theta_3) - (-n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3)) \right) \\
\times (n_{14} + n_{15} \cos \theta_3) (h_{31} \sin \theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{32} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_3 \\
+ h_{33} \sin \theta_2 \dot{\theta}_2 \dot{\theta}_3 + h_{34} \sin(\theta_2 + \theta_3) \dot{\theta}_1^2 + h_{35} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{37} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 + h_{38} \sin \theta_2 \dot{\theta}_2^2 + h_{39} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_3 + h_{310} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_3 \\
+ h_{311} \sin \theta_2 \dot{\theta}_3 \dot{\theta}_2 \dot{\theta}_3 + ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) (n_{12} + n_{13} \cos \theta_3) \\
- (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) (n_{14} + n_{15} \cos \theta_3) \right) \\
\times (h_{31} \cos(\theta_2 + \theta_3) \dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{34} \cos(\theta_2 + \theta_3) \dot{\theta}_1^2 + h_{37} \cos(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{39} \cos(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_3)
\]

\[
\theta_2 A_4 = \left( (n_{14} + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3) M_{33} \right) (-g_{12} \sin(\theta_1 + \theta_2) \\
- g_{13} \sin(\theta_1 + \theta_2 + \theta_3))
\]

\[
\theta_2 A_5 = \left( (-n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3)) M_{33} - (-n_{10} \sin(\theta_2 + \theta_3)) (n_{14} + n_{15} \cos \theta_3) \right) \\
\times (g_{21} \cos(\theta_1 + \theta_2) + g_{22} \cos(\theta_1 + \theta_2 + \theta_3)) + ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) M_{33} - (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) \\
\times (n_{14} + n_{15} \cos \theta_3) (-g_{21} \sin(\theta_1 + \theta_2) - g_{22} \sin(\theta_1 + \theta_2 + \theta_3))
\]
\[ \theta_2A_6 = (-n_{10}\sin(\theta_2 + \theta_3)(n_{12} + n_{13}\cos\theta_3) - (-n_6\sin\theta_2 - n_8\sin(\theta_2 + \theta_3)) \\
\times (n_{14} + n_{15}\cos\theta_3))(g_{31}\cos(\theta_1 + \theta_2 + \theta_3)) + ((n_9 + n_9\cos(\theta_2 + \theta_3) \\
+ n_{11}\cos\theta_3)(n_{12} + n_{13}\cos\theta_3) - (n_5 + n_6\cos\theta_2 + n_7\cos\theta_3 \\
+ n_8\cos(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3))(g_{31}\sin(\theta_1 + \theta_2 + \theta_3)) \]

\[ \theta_2B_1 = (((-n_6\sin\theta_2 - n_8\sin(\theta_2 + \theta_3))M_{33} - (n_{14} + n_{15}\cos\theta_3)(-n_{10}\sin(\theta_2 + \theta_3))) \\
\times (h_{11}\sin\theta_2\dot{\theta}_1\dot{\theta}_2 + h_{12}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{13}\sin\theta_3\dot{\theta}_1\dot{\theta}_3 \\
+ h_{14}\sin\theta_2\dot{\theta}_2^2 + h_{15}\sin\theta_3\dot{\theta}_2\dot{\theta}_3 + h_{16}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_2 \\
- h_{17}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_2 + h_{18}\sin\theta_3\dot{\theta}_3 + ((n_5 + n_6\cos\theta_2 + n_7\cos\theta_3 \\
+ n_8\cos(\theta_2 + \theta_3))M_{33} - (n_{14} + n_{15}\cos\theta_3)(n_9 + n_{10}\cos(\theta_2 + \theta_3) \\
+ n_{11}\cos\theta_3))(h_{11}\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + h_{12}\cos(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{14}\cos\theta_2\dot{\theta}_2^2 \\
+ h_{16}\cos(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_2 - h_{17}\cos\theta_3\dot{\theta}_3)(\theta_2 + \theta_3)\dot{\theta}_3) \]

\[ \theta_2B_2 = (2(n_0 + n_{10}\cos(\theta_2 + \theta_3) + n_{11}\cos\theta_3) - n_{10}\sin(\theta_2 + \theta_3) \\
- (-n_2\sin\theta_2 - n_3\sin(\theta_2 + \theta_3))M_{33})(h_{21}\sin\theta_2\dot{\theta}_2\dot{\theta}_2 + h_{22}\sin\theta_2\dot{\theta}_2\dot{\theta}_1 \\
+ h_{23}\sin\theta_2\dot{\theta}_3\dot{\theta}_1 + h_{24}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{25}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_1 \\
+ h_{26}\sin\theta_3\dot{\theta}_3^2 + h_{27}\sin\theta_2\dot{\theta}_2^2 + h_{28}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_2 \\
+ h_{29}\sin\theta_2\dot{\theta}_2\dot{\theta}_1 + h_{21}\sin(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_3 + ((n_9 + n_{10}\cos(\theta_2 + \theta_3) \\
+ n_{11}\cos\theta_3)^2 - (n_1 + n_2\cos\theta_2 + n_3\cos(\theta_2 + \theta_3) + n_4\cos\theta_3)M_{33} \\
\times (h_{22}\cos\theta_2\dot{\theta}_3\dot{\theta}_1 + h_{24}\cos(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{25}\cos(\theta_2 + \theta_3)(\theta_2 + \dot{\theta}_3)\dot{\theta}_1 \\
+ h_{26}\cos\theta_3\dot{\theta}_3^2 + h_{28}\cos(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{29}\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + h_{210}\cos(\theta_2 + \theta_3)\dot{\theta}_1\dot{\theta}_2 \\
+ h_{211}\cos(\theta_2 + \theta_3)\dot{\theta}_1\dot{\theta}_3) \]

\[ \theta_2B_3 = (((-n_2\sin\theta_2 - n_3\sin(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3) - ((-n_7\sin\theta_2 \\
- n_6\sin(\theta_2 + \theta_3))(n_9 + n_{10}\cos(\theta_2 + \theta_3) + n_{11}\cos\theta_3) + (n_5 + n_6\cos\theta_2 \\
+ n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3)) \\
\times (h_{31}\sin(\theta_2 + \theta_3)\dot{\theta}_2\dot{\theta}_3\dot{\theta}_1 + h_{32}\sin(\theta_3\dot{\theta}_1\dot{\theta}_2 + h_{33}\sin\theta_3\dot{\theta}_2\dot{\theta}_3 \\
+ h_{34}\sin(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{35}\sin\theta_3\dot{\theta}_2^2 + h_{36}\sin\theta_3\dot{\theta}_2\dot{\theta}_3 + h_{37}\sin(\theta_2 + \theta_3)\dot{\theta}_1\dot{\theta}_2 \\
+ h_{38}\sin\theta_3\dot{\theta}_2^2 + h_{39}\sin(\theta_2 + \theta_3)\dot{\theta}_1\dot{\theta}_3 + h_{310}\sin\theta_2\dot{\theta}_1\dot{\theta}_3 + h_{311}\sin\theta_3\dot{\theta}_2\dot{\theta}_3 \\
+ (n_1 + n_2\cos\theta_2 + n_3\cos(\theta_2 + \theta_3) + n_4\cos\theta_3)(n_{14} + n_{15}\cos\theta_3) \\
- (n_5 + n_6\cos\theta_2 + n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))(n_9 + n_{10}\cos(\theta_2 + \theta_3) \\
+ n_{11}\cos\theta_3))(h_{31}\cos(\theta_2 + \theta_3)\dot{\theta}_2\dot{\theta}_3\dot{\theta}_1 + h_{34}\cos(\theta_2 + \theta_3)\dot{\theta}_1^2 \\
+ h_{37}\cos(\theta_2 + \theta_3)\dot{\theta}_1\dot{\theta}_2 + h_{39}\cos(\theta_2 + \theta_3)\dot{\theta}_1\dot{\theta}_3) \]

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\[ \theta_2 B_4 = \left( \left( -n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3) \right) M_{33} - (n_{14} + n_{15} \cos \theta_3)(-n_{10} \sin(\theta_2 + \theta_3)) \right) \]
\[ \times \left( g_{11} \cos \theta_1 + g_{12} \cos(\theta_1 + \theta_2) + g_{13} \cos(\theta_1 + \theta_2 + \theta_3) \right) + \left( (n_5 + n_6 \cos \theta_2 + n_r \cos \theta_3 + n_s \cos(\theta_2 + \theta_3)) M_{33} - (n_{14} + n_{15} \cos \theta_3) \right) \]
\[ \times \left( n_9 + n_{10} * \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3 \right) (-g_{12} \sin(\theta_1 + \theta_2)) \]
\[ - \frac{g_{13}}{0} \sin(\theta_1 + \theta_2 + \theta_3) \right) \right) \] (B.5)

\[ \theta_2 B_5 = (2(n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) - n_{10} \sin(\theta_2 + \theta_3)) \]
\[ - (\frac{n_9}{2} \sin \theta_2 - n_5 \sin(\theta_2 + \theta_3)) M_{33} \left( g_{21} \cos \theta_1 + \theta_2 \right) \]
\[ + g_{22} \cos(\theta_1 + \theta_2 + \theta_3) + (n_9 + n_{10} \cos(\theta_1 + \theta_2) + n_{11} \cos \theta_3) \]
\[ - (n_{14} + n_{15} \cos \theta_3) \left( n_9 + n_{10} \cos(\theta_2 + \theta_3) \right) \]
\[ - g_{22} \sin(\theta_1 + \theta_2 + \theta_3) \]

\[ \theta_2 B_6 = (\frac{-n_9}{2} \sin \theta_2 - n_3 \sin(\theta_2 + \theta_3)) (n_{14} + n_{15} \cos \theta_3) - (n_{10} \sin(\theta_2 + \theta_3) \]
\[ - n_8 \sin(\theta_2 + \theta_3) \right) M_{33} \left( g_{31} \cos(\theta_1 + \theta_2 + \theta_3) \right) \]
\[ + (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) \left( n_9 + n_{10} \cos(\theta_2 + \theta_3) \right) \]
\[ - (n_9 + n_{10} \cos \theta_2 + \theta_3) \left( n_9 + n_{10} \cos(\theta_2 + \theta_3) \right) \]
\[ + n_{11} \cos \theta_3) \left( n_9 + n_{10} \cos(\theta_2 + \theta_3) \right) \]

\[ \theta_2 C_1 = ((-n_{10} \sin(\theta_2 + \theta_3)) (n_{12} + n_{13} \cos \theta_3) - (n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3)) \]
\[ \times (n_{14} + n_{15} \cos \theta_3) \right) (h_{11} \sin \theta_2 \hat{\theta}_2 + h_{12} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{13} \sin \theta_3 \hat{\theta}_1 + h_{14} \sin \theta_2 \hat{\theta}_2 + h_{15} \sin \theta_3 \hat{\theta}_2 + h_{16} \sin(\theta_2 + \theta_3) \right) \]
\[ - h_{17} \sin(\theta_2 + \theta_3) \right) (\hat{\theta}_2 + \hat{\theta}_3) \right) \]
\[ + n_{11} \cos \theta_3) \left( n_{12} + n_{13} \cos \theta_3 \right) \]
\[ + n_8 \cos(\theta_2 + \theta_3) \left( n_{14} + n_{15} \cos \theta_3 \right) \]
\[ + n_{11} \cos \theta_3) \left( n_{12} + n_{13} \cos \theta_3 \right) \]
\[ + h_{12} \cos(\theta_2 + \theta_3) \right) (\hat{\theta}_2 + \hat{\theta}_3) \right) \]
\[ - h_{17} \cos(\theta_2 + \theta_3) \right) (\hat{\theta}_2 + \hat{\theta}_3) \right) \]

\[ \theta_2 C_2 = (\frac{-n_9}{2} \sin \theta_2 - n_3 \sin(\theta_2 + \theta_3)) (n_{14} + n_{15} \cos \theta_3) - (n_{10} \sin \theta_2 \]
\[ - n_8 \sin(\theta_2 + \theta_3) \right) (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3) \]
\[ + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3) \right) (n_{14} + n_{15} \cos \theta_3) \right) \]
\[ + h_{21} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{22} \sin \theta_2 \hat{\theta}_2 + h_{23} \sin \theta_2 \hat{\theta}_2 + h_{24} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{25} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{26} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{27} \sin(\theta_2 + \theta_3) \]
\[ + h_{28} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{29} \sin \theta_2 \hat{\theta}_2 + h_{30} \sin(\theta_2 + \theta_3) \right) \]
\[ + h_{211} \sin(\theta_2 + \theta_3) \right) \]
\[ + (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3) \]

\[ A Torso-Moving Balance Control Strategy for a Walking Biped Robot \]
\[
\begin{align*}
\theta_2 C_3 &= (2(n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(-n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3)) \\
&- (-n_2 \sin \theta_2 - n_3 \sin(\theta_2 + \theta_3))(n_{12} + n_{13} \cos \theta_3))(h_{31} \sin(\theta_2 + \theta_3) \\
&\times (n_{14} + n_{15} \cos \theta_3))(g_{11} \cos \theta_1 + g_{12} \cos(\theta_1 + \theta_2) + g_{13} \cos(\theta_1 + \theta_2 + \theta_3)) \\
&+ (n_5 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(-n_{10} \sin(\theta_2 + \theta_3)))(g_{21} \cos(\theta_1 + \theta_2) \\
&\times (n_{14} + n_{15} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
&- n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3))(n_{12} + n_{13} \cos \theta_3))(g_{31} \cos(\theta_1 + \theta_2 + \theta_3)) \\
&- g_{22} \cos(\theta_1 + \theta_2 + \theta_3) + (n_5 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3)(-g_{21} \sin(\theta_1 + \theta_2) \\
&- g_{22} \sin(\theta_1 + \theta_2 + \theta_3)) \\
\theta_2 C_6 &= (2(n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(-n_6 \sin \theta_2 - n_8 \sin(\theta_2 + \theta_3)) \\
&- (-n_2 \sin \theta_2 - n_3 \sin(\theta_2 + \theta_3))(n_{12} + n_{13} \cos \theta_3))(g_{31} \cos(\theta_1 + \theta_2 + \theta_3)) \\
&+ (n_5 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))2 - (n_5 + n_7 \cos \theta_2 \\
&+ n_3 \cos \theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{12} + n_{13} \cos \theta_3)(-g_{31} \sin(\theta_1 + \theta_2 + \theta_3)) \\
\theta_2 D_1 &= (-2(m_2 l_1 l_2 + m_3 l_1 l_2) \sin \theta_2 - 2m_3 l_1 l_3 \sin(\theta_2 + \theta_3))(I_2 + I_3 + m_2 l_2^2 \\
&+ m_3(l_2^2 + l_3^2) + 2m_3 k_2 l_3 \cos \theta_3)(I_3 + m_3 l_3^2)
\end{align*}
\]
\[ \dot{\theta}_2(t_2) = (\dot{I}_3 + m_2 l_2^2 + m_3 l_3^2 \cos \theta_3) \left( - \left( m_2 l_2 l_3 + m_3 l_3^2 \right) \sin \theta_2 + \left( m_3 l_3^2 \sin \theta_2 + \theta_3 \right) (\dot{I}_3 + m_2 l_2^2 + m_3 l_3^2 \cos \theta_2 + \theta_3) + m_2 l_3 \right) \cos \theta_2 + 2 m_3 l_3 \cos \theta_3 + m_3 l_3^2 \cos \theta_2 (\dot{I}_3 + \theta_3) \right) (\dot{I}_3 + m_2 l_2^2 + m_3 l_3^2 \cos \theta_2 + \theta_3) + m_3 l_3 \sin \theta_2 (\dot{I}_3 + m_2 l_2^2 + m_3 l_3^2 \cos \theta_2 + \theta_3) \right) \]
\[
\theta_3A_3 = \left( (-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin(\theta_3))(n_{12} + n_{13} \cos(\theta_3)) + (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_1 \cos(\theta_3) - (n_9 - n_{11} \sin(\theta_2 + \theta_3))(n_{14} + n_{15} \cos(\theta_3)) \\
+ (n_9 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3))(-n_{15} \sin(\theta_3)) \right) \times (h_{31} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{32} \sin(\theta_2 \dot{\theta}_2 \dot{\theta}_1 + h_{33} \sin(\theta_3 \dot{\theta}_2 \dot{\theta}_2) \right. \\
+ h_{34} \sin(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{35} \sin(\theta_3 \dot{\theta}_2 \dot{\theta}_1 + h_{36} \sin(\theta_2 + \theta_3)\dot{\theta}_2 + h_{37} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \dot{\theta}_2) \\
+ h_{38} \sin(\theta_3 \dot{\theta}_2 \dot{\theta}_1 + h_{39} \sin(\theta_2 + \theta_3)\dot{\theta}_2 \dot{\theta}_1 + h_{310} \sin(\theta_2 \dot{\theta}_2 \dot{\theta}_1 + h_{311} \sin(\theta_3 \dot{\theta}_2 \dot{\theta}_2) \right. \\
+ (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3))(n_{12} + n_{13} \cos(\theta_3)) - (n_9 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos(\theta_3))(h_{31} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1) \\
+ h_{32} \cos(\theta_3 \dot{\theta}_2 \dot{\theta}_1 + h_{33} \cos(\theta_3 \dot{\theta}_2 \dot{\theta}_2 + h_{34} \cos(\theta_2 + \theta_3)\dot{\theta}_1^2 + h_{35} \cos(\theta_3 \dot{\theta}_2 \dot{\theta}_1) \\
+ h_{36} \cos(\theta_3 \dot{\theta}_2 \dot{\theta}_1 + h_{37} \cos(\theta_2 + \theta_3)\dot{\theta}_2 \dot{\theta}_1 + h_{38} \cos(\theta_2 + \theta_3)\dot{\theta}_2 \dot{\theta}_1 + h_{39} \cos(\theta_2 + \theta_3)\dot{\theta}_3 \dot{\theta}_1) \\
+ h_{310} \cos(\theta_3 \dot{\theta}_2 \dot{\theta}_1 + h_{311} \cos(\theta_3 \dot{\theta}_2 \dot{\theta}_3) \\
\theta_3A_4 = (-2(n_{14} + n_{15} \cos(\theta_3))n_{15} \sin(\theta_3) + M_{33}n_{13} \sin(\theta_3))(g_{11} \cos(\theta_1 + g_{12} \cos(\theta_1 + \theta_2) \\
+ g_{13} \cos(\theta_1 + \theta_2))(\theta_3) + (n_{14} + n_{15} \cos(\theta_3))^2 - (n_{12} + n_{13} \cos(\theta_3)M_{33}) \\
\times (-g_{13} \sin(\theta_1 + \theta_2 + \theta_3)) \\
\theta_3A_5 = ((-n_7 \sin(\theta_3) - n_8 \sin(\theta_2 + \theta_3))M_{33} - (-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin(\theta_3) \\
\times (n_{14} + n_{15} \cos(\theta_3) + (n_9 + n_{10} \cos(\theta_1 + \theta_2) + n_{11} \cos(\theta_3)(-n_{15} \sin(\theta_3)))) \\
\times (g_{21} \cos(\theta_1 + \theta_2) + g_{22} \cos(\theta_1 + \theta_2) + \theta_3) + (n_9 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) \\
+ n_8 \cos(\theta_2 + \theta_3))M_{33} - (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3) \\
\times (n_{14} + n_{15} \cos(\theta_3))(g_{22} \sin(\theta_1 + \theta_2 + \theta_3)) \\
\theta_3A_6 = (((-n_{10} \sin(\theta_3 + \theta_3) - n_{11} \sin(\theta_3))(n_{12} + n_{13} \cos(\theta_3)) + (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3) \\
+ n_{14} \cos(\theta_3)(-n_{13} \sin(\theta_3)) - ((-n_7 \sin(\theta_3 - n_8 \sin(\theta_2 + \theta_3))(n_{14} + n_{15} \cos(\theta_3) \\
+ (n_9 + n_6 \cos(\theta_2 + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3))(-n_{15} \sin(\theta_3)) \\
\times (g_{31} \cos(\theta_1 + \theta_2 + \theta_3)) + (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3) \\
\times (n_{12} + n_{13} \cos(\theta_3) - (n_9 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3)) \\
\times (n_{14} + n_{15} \cos(\theta_3))(g_{31} \sin(\theta_1 + \theta_2 + \theta_3)) \\
\theta_3B_1 = (((-n_7 \sin(\theta_3 - n_8 \sin(\theta_2 + \theta_3))M_{33} - ((-n_{15} \sin(\theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos(\theta_3) + (n_{14} + n_{15} \cos(\theta_3)(-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin(\theta_3))) \\
\times (h_{11} \sin(\theta_2 \dot{\theta}_2 \dot{\theta}_1 + h_{12} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_1 + h_{13} \sin(\theta_3 \dot{\theta}_2 \dot{\theta}_1 + h_{14} \sin(\theta_2 \dot{\theta}_2 \dot{\theta}_2) \\
+ h_{15} \sin(\theta_3 \dot{\theta}_2 \dot{\theta}_2 + h_{16} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_2 - h_{17} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_2 \\
+ h_{13} \cos(\theta_2 \dot{\theta}_2 \dot{\theta}_1 + h_{15} \cos(\theta_2 \dot{\theta}_2 \dot{\theta}_2 + h_{16} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_2 \\
- h_{17} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_3 + h_{18} \cos(\theta_2 \dot{\theta}_2 \dot{\theta}_2) \\
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\[
\theta_3 B_2 = (2(n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin \theta_3) \\
- (-n_3 \sin(\theta_2 + \theta_3) - n_4 \sin \theta_3)M_{33})(h_{21} \sin \theta_2 \dot{\theta}_2 + h_{22} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{23} \sin \theta_2 \dot{\theta}_1 + h_{24} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + h_{25} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 \\
+ h_{26} \sin \theta_2 \dot{\theta}_3 + h_{27} \sin \theta_2 \dot{\theta}_1 + h_{28} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + h_{39} \sin \theta_2 \ddot{\theta}_1 \dot{\theta}_2 \\
+ h_{210} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + h_{220} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 \\
+ h_{220} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{260} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{280} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \\
+ h_{210} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{220} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{260} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{280} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \\
+ h_{210} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{220} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{260} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{280} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \\
+ h_{210} \cos(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \dot{\theta}_1 + h_{220} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{260} \cos(\theta_2 + \theta_3) \dot{\theta}_1 + h_{280} \sin(\theta_2 + \theta_3) \dot{\theta}_1)
\]

\[
\theta_3 B_3 = (((-n_3 \sin(\theta_2 + \theta_3) - n_4 \sin \theta_3)(\dot{n}_{14} + n_{15} \cos \theta_3) + (n_1 + n_2 \cos \theta_2 \\
+ n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3(-n_{15} \cos \theta_3)) - ((-n_7 \sin \theta_3 - n_8 \sin \theta_2 + \theta_3)) \\
\times ((n_6 + n_{10} \cos(\theta_2 + \theta_3) + n_11 \cos \theta_3) + (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 \\
+ n_8 \cos(\theta_2 + \theta_3)(-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin \theta_3)) \\
\times (h_{31} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + h_{32} \sin \theta_2 \dot{\theta}_1 + h_{33} \sin \theta_2 \theta_3 \\
+ h_{34} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 + h_{35} \sin \theta_2 \dot{\theta}_2 + h_{36} \sin(\theta_2 + \theta_3) \dot{\theta}_2 \dot{\theta}_3 \\
+ h_{37} \sin(\theta_2 + \theta_3) \dot{\theta}_2 \dot{\theta}_3 + h_{38} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_2 \\
+ h_{39} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \dot{\theta}_3 + h_{310} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_3 + h_{311} \sin \theta_2 \dot{\theta}_2 \dot{\theta}_3 \\
+ h_{312} \sin \theta_2 \dot{\theta}_2 \dot{\theta}_3 + h_{313} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_3 + h_{314} \sin \theta_2 \dot{\theta}_1 \dot{\theta}_3 + h_{315} \sin \theta_2 \dot{\theta}_2 \dot{\theta}_3)
\]

\[
\theta_3 B_4 = (((-n_7 \sin \theta_3 - n_8 \sin(\theta_2 + \theta_3))M_{33} - ((-n_{15} \sin \theta_3)(\dot{n}_9 + n_{10} \cos \theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3) + (n_1 + n_2 \cos \theta_2 + \theta_3) (-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin \theta_3)) \\
\times (g_{11} \cos \theta_1 + g_{12} \cos(\theta_1 + \theta_2) + g_{13} \cos \theta_1 + \theta_2 + \theta_3)) + ((n_5 + n_6 \cos \theta_2 \\
+ n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))M_{33} - (n_4 + n_{15} \cos \theta_3)(\dot{n}_9 + n_{10} \cos \theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3)(-n_{13} \sin \theta_1 + \theta_2 + \theta_3))
\]

\[
\theta_3 B_5 = (2(n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin \theta_3) \\
- (n_3 \sin(\theta_2 + \theta_3) - n_4 \sin \theta_3)M_{33})(g_{21} \cos \theta_1 + \theta_2) \\
+ g_{22} \cos \theta_1 + \theta_2 + \theta_3) + (n_9 + n_{10} \cos \theta_2 + \theta_3) + n_{11} \cos \theta_3)^2 \\
- (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)M_{33})(-g_{22} \sin \theta_1 + \theta_2 + \theta_3))
\]
\[ \theta_3 B_6 = \left( (-n_3 \sin(\theta_2 + \theta_3) - n_4 \sin(\theta_3))(n_{14} + n_{15} \cos(\theta_3)) + (n_1 + n_2 \cos(\theta_2) \\
+ n_3 \cos(\theta_2 + \theta_3) + n_4 \cos(\theta_3)(-n_{15} \cos(\theta_3))) - ((-n_7 \sin(\theta_2 + \theta_3) - n_8 \sin(\theta_2 + \theta_3)) \\
\times (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3)) + (n_5 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) \\
+ n_8 \cos(\theta_2 + \theta_3))(-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin(\theta_3))))(g_{31} \cos(\theta_1 + \theta_2 + \theta_3)) \\
+ ((n_1 + n_2 \cos(\theta_2 + \theta_3) + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos(\theta_3))(n_{14} + n_{15} \cos(\theta_3)) \\
- (n_5 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos(\theta_3))(-g_{31} \sin(\theta_1 + \theta_2 + \theta_3))) \right) \]

\[ \theta_3 C_1 = \left( (-n_10 \sin(\theta_2 + \theta_3) - n_{11} \sin(\theta_3))(n_{12} + n_{13} \cos(\theta_3)) + (n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos(\theta_3))(-n_3 \sin(\theta_3)) - ((-n_7 \sin(\theta_2 + \theta_3)) \sin(\theta_2 + \theta_3))(-n_{14} + n_{15} \cos(\theta_3)) \\
+ (n_5 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3))(-n_{15} \sin(\theta_3)))(h_{11} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 \\
+ h_{12} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_3 \hat{\theta}_4 + h_{13} \sin(\theta_3) \hat{\theta}_1 \hat{\theta}_3 + h_{14} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_2 \\
+ h_{16} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_2 - h_{17} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_2 + h_{18} \sin(\theta_3) \hat{\theta}_3 \\
+ ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3))(n_{12} + n_{13} \cos(\theta_3)) - (n_5 + n_6 \cos(\theta_2 + \theta_3) \\
+ n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos(\theta_3)))h_{12} \cos(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_1 \\
+ h_{13} \cos(\theta_2 \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_1 + h_{15} \cos(\theta_2 \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_2 + h_{16} \cos(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) \hat{\theta}_2 \\
- h_{17} \cos(\theta_2 + \theta_3)(\hat{\theta}_2 + \theta_3) \hat{\theta}_3 + h_{18} \cos(\theta_3) \hat{\theta}_3)) \right) \]

\[ \theta_3 C_2 = \left( (-n_3 \sin(\theta_2 + \theta_3) - n_4 \sin(\theta_3))(n_{14} + n_{15} \cos(\theta_3)) + (n_1 + n_2 \cos(\theta_2) \\
+ n_3 \cos(\theta_2 + \theta_3) + n_4 \cos(\theta_3)(-n_{15} \sin(\theta_3)) - ((-n_7 \sin(\theta_2 + \theta_3) - n_8 \sin(\theta_2 + \theta_3)) \\
\times (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3)) + (n_5 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) \\
+ n_8 \cos(\theta_2 + \theta_3))(-n_{10} \sin(\theta_2 + \theta_3) - n_{11} \sin(\theta_3))))(h_{21} \sin(\theta_3) \hat{\theta}_3 \hat{\theta}_2 \\
+ h_{22} \sin(\theta_2 \hat{\theta}_1 \hat{\theta}_2 + h_{23} \sin(\theta_3) \hat{\theta}_1 \hat{\theta}_3 + h_{24} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_1 \\
+ h_{25} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_1 + h_{26} \sin(\theta_3) \hat{\theta}_3 \hat{\theta}_2 + h_{27} \sin(\theta_2) \hat{\theta}_1 \hat{\theta}_2 \\
+ h_{28} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 + h_{29} \sin(\theta_2 \hat{\theta}_1 \hat{\theta}_2 + h_{30} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \theta_3) \hat{\theta}_1 \\
+ h_{211} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_3 + ((n_1 + n_2 \cos(\theta_2 + \theta_3) + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos(\theta_3) \\
\times (n_{14} + n_{15} \cos(\theta_3)) - (n_5 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos(\theta_3) + n_8 \cos(\theta_2 + \theta_3)) \\
\times (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos(\theta_3))))(h_{21} \cos(\theta_3) \hat{\theta}_2 \hat{\theta}_3 + h_{23} \cos(\theta_2 \hat{\theta}_3 \hat{\theta}_2 \hat{\theta}_3 \\
+ h_{24} \cos(\theta_2 + \theta_3) \hat{\theta}_2 \hat{\theta}_3 \hat{\theta}_1 + h_{25} \cos(\theta_2 + \theta_3)(\hat{\theta}_2 + \theta_3) \hat{\theta}_1 + h_{26} \cos(\theta_3) \hat{\theta}_3 \hat{\theta}_2 \\
+ h_{28} \cos(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 + h_{210} \cos(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 + h_{211} \cos(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_3) \right) \]
\[ \theta_3 C_3 = (2(n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_5 \sin(\theta_2 + \theta_3) - n_8 \sin(\theta_2 + \theta_3)) - \left((n_3 \sin(\theta_2 + \theta_3) - n_4 \sin \theta_3)(n_{12} + n_{13} \cos \theta_3) + (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3))(n_3 + n_4 \cos(\theta_2 + \theta_3))\right)\) \\
+ h_{32} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_1 + h_{33} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_2 + h_{34} \sin(\theta_2 + \theta_3) \gamma_1^2 + h_{35} \sin \theta_3 \gamma_1^2 \\
+ h_{36} \sin \theta_3 \hat{\theta}_2 + h_{37} \sin(\theta_2 + \theta_3) \hat{\theta}_1 \hat{\theta}_2 + h_{38} \sin \theta_2 \gamma_2^2 + h_{39} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \\
+ h_{310} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_1 + h_{311} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_2 + \left((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))\right) \gamma_1^2 \\
\times (n_2 + n_3 \cos(\theta_2 + \theta_3)) + h_{312} \cos(\theta_1 + \theta_2) + h_{313} \cos(\theta_1 \theta_2 + \theta_3) + \left((n_9 + n_{10} \cos(\theta_2 + \theta_3))\right) \gamma_2^2 \\
+ h_{314} \cos(\theta_2 + \theta_3) + h_{315} \cos(\theta_2 \theta_3 + n_8 \cos(\theta_2 + \theta_3)) \right) \gamma_3^2 \\
+ h_{316} \cos(\theta_2 + \theta_3) \gamma_4^2 = 2(n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(-n_7 \sin(\theta_2 + \theta_3) - n_8 \sin(\theta_2 + \theta_3)) \\
- \left((n_3 \sin(\theta_2 + \theta_3) - n_4 \sin \theta_3)(n_{12} + n_{13} \cos \theta_3) + (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3))\right) \gamma_1^2 \\
+ h_{310} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_1 + h_{311} \sin \theta_3 \hat{\theta}_3 \hat{\theta}_2 + \left((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))\right) \gamma_1^2 \\
\times (n_2 + n_3 \cos(\theta_2 + \theta_3)) + h_{312} \cos(\theta_1 + \theta_2) + h_{313} \cos(\theta_1 \theta_2 + \theta_3) + \left((n_9 + n_{10} \cos(\theta_2 + \theta_3))\right) \gamma_2^2 \\
+ h_{314} \cos(\theta_2 + \theta_3) + h_{315} \cos(\theta_2 \theta_3 + n_8 \cos(\theta_2 + \theta_3)) \right) \gamma_3^2 \\
+ h_{316} \cos(\theta_2 + \theta_3) \gamma_4^2 \]
\[ \theta_3 D_1 = (-2m_3 l_1 l_3 \sin(\theta_2 + \theta_3) - 2m_3 l_2 l_3 \sin \theta_3) M_{22} M_{33} \\
+ M_{11}(-2m_3 l_1 l_3 \sin \theta_3) M_{33} \]
\[ \theta_3 D_2 = (-2m_3 l_2 l_3 \sin \theta_3 - m_3 l_1 l_3 \sin(\theta_2 + \theta_3)) M_{23} M_{13} + M_{12}(-m_3 l_2 l_3 \sin \theta_3) M_{13} \\
+ M_{12} M_{23}(-m_3 l_1 l_3 \sin(\theta_2 + \theta_3) - m_3 l_2 l_3 \sin \theta_3) \]
\[ \theta_3 D_3 = 2M_{13}(-m_3 l_1 l_3 \sin(\theta_2 + \theta_3) - m_3 l_2 l_3 \sin \theta_3) M_{22} + M_{13}^2(-2m_3 l_2 l_3 \sin \theta_3) \]
\[ \theta_3 D_4 = (-2m_3 l_1 l_3 \sin(\theta_2 + \theta_3) - 2m_3 l_2 l_3 \sin \theta_3) M_{22}^2 + 2M_{11} M_{23}(-m_3 l_2 l_3 \sin \theta_3) \]
\[ \theta_3 D_5 = 2M_{33} M_{13}(-2m_3 l_2 l_3 \sin \theta_3 - m_3 l_1 l_3 \sin(\theta_2 + \theta_3)) \]
\[ \overline{\theta}_3 A_1 = (n_4 + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3) M_{33}(h_{11} \sin \theta_2 \hat{\theta}_2 \\
+ h_{12} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) + h_{13} \sin \theta_3 \hat{\theta}_3) \]
\[ \overline{\theta}_3 A_2 = ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) M_{33} - (n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3)(n_4 + n_{15} \cos \theta_3)) (h_{22} \sin \theta_2 \hat{\theta}_2 + h_{23} \sin \theta_3 \hat{\theta}_3) \\
+ h_{24} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) + h_{25} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) + 2h_{27} \sin \theta_2 \hat{\theta}_1 \\
+ 2h_{28} \sin(\theta_2 + \theta_3)(\hat{\theta}_1 + h_{29} \sin \theta_2 \hat{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3) \hat{\theta}_2 \\
+ h_{211} \sin(\theta_2 + \theta_3) \hat{\theta}_3) \]
\[ \overline{\theta}_3 A_3 = ((n_5 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
+ n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3)) (h_{31} \sin \theta_2 \hat{\theta}_2 + h_{33}^2 \sin \theta_2 \hat{\theta}_3) \\
+ h_{32} \sin \theta_3 \hat{\theta}_3 + 2h_{34} \sin(\theta_2 + \theta_3) \hat{\theta}_1 + 2h_{35} \sin \theta_3 \hat{\theta}_1 + h_{36} \sin \theta_3 \hat{\theta}_2 \\
+ h_{37} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + h_{38} \sin(\theta_2 + \theta_3) \hat{\theta}_3 + h_{310} \sin \theta_3 \hat{\theta}_3) \]
\[ \hat{\theta}_1 A_4 = 0.0 \]
\[ \hat{\theta}_1 A_5 = 0.0 \]
\[ \hat{\theta}_1 A_6 = 0.0 \]
\[ \hat{\theta}_1 B_1 = ((n_1 + n_{10} \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) M_{33} - (n_{14} + n_{15} \cos \theta_3) \\
\times (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(h_{11} \sin \theta_2 \hat{\theta}_2 \\
+ h_{12} \sin(\theta_2 + \theta_3)(\hat{\theta}_2 + \hat{\theta}_3) + h_{13} \sin \theta_3 \hat{\theta}_3) \]
$$\dot{B}_2 = ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)^2 - (n_4 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3)
 + n_4 \cos \theta_3)M_{23})h_{22} \sin \theta_2 \dot{\theta}_2 + h_{23} \sin \theta_3 \dot{\theta}_3 + h_{24} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)
 + h_{25} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \theta_3) + 2h_{27} \sin \theta_2 \dot{\theta}_1 + 2h_{28} \sin(\theta_2 + \theta_3)\dot{\theta}_1
 + h_{29} \sin \theta_2 \dot{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3)\dot{\theta}_2 + h_{211} \sin(\theta_2 + \theta_3)\dot{\theta}_3) \quad (B.11)$$

$$\dot{B}_3 = ((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3)
 - (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3)
 + n_{11} \cos \theta_3))h_{31} \sin \theta_2 \dot{\theta}_2 + h_{32} \sin \theta_3 \dot{\theta}_3 + 2h_{34} \sin(\theta_2 + \theta_3)\dot{\theta}_1
 + 2h_{35} \sin \theta_3 \dot{\theta}_1 + h_{36} \sin \theta_3 \dot{\theta}_2 + h_{37} \sin(\theta_2 + \theta_3)\dot{\theta}_2
 + h_{39} \sin(\theta_2 + \theta_3)\dot{\theta}_3 + h_{310} \sin \theta_2 \dot{\theta}_3)
$$

$$\dot{B}_4 = 0.0$$
$$\dot{B}_5 = 0.0$$
$$\dot{B}_6 = 0.0$$

$$\dot{C}_1 = ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2
 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3))h_{11} \sin \theta_2 \dot{\theta}_2
 + h_{12} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + h_{13} \sin \theta_3 \dot{\theta}_3)$$

$$\dot{C}_2 = ((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3)
 - (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3)
 + n_{11} \cos \theta_3))h_{22} \sin \theta_2 \dot{\theta}_2 + h_{23} \sin \theta_3 \dot{\theta}_3 + h_{24} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3)
 + h_{25} \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + 2h_{27} \sin \theta_2 \dot{\theta}_1 + 2h_{28} \sin(\theta_2 + \theta_3)\dot{\theta}_1
 + h_{29} \sin \theta_2 \dot{\theta}_2 + h_{210} \sin(\theta_2 + \theta_3)\dot{\theta}_2 + h_{211} \sin(\theta_2 + \theta_3)\dot{\theta}_3)$$

$$\dot{C}_3 = ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))^2 - (n_4 + n_2 \cos \theta_2
 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{12} + n_{13} \cos \theta_3))h_{31} \sin \theta_2 \dot{\theta}_2 + h_{32} \sin \theta_3 \dot{\theta}_3 + 2h_{34} \sin(\theta_2 + \theta_3)\dot{\theta}_1 + 2h_{35} \sin \theta_3 \dot{\theta}_1 + h_{36} \sin \theta_3 \dot{\theta}_2
 + h_{37} \sin(\theta_2 + \theta_3)\dot{\theta}_2 + h_{39} \sin(\theta_2 + \theta_3)\dot{\theta}_3 + h_{310} \sin \theta_3 \dot{\theta}_3)$$

$$\dot{C}_4 = 0.0$$
$$\dot{C}_5 = 0.0$$
$$\dot{C}_6 = 0.0$$
$$\dot{D} = 0.0$$
\[
\dot{\theta}_1 A = \dot{\theta}_1 A_1 + \dot{\theta}_1 A_2 + \dot{\theta}_1 A_3 + \dot{\theta}_1 A_4 + \dot{\theta}_1 A_5 + \dot{\theta}_1 A_6 \\
\dot{\theta}_1 B = \dot{\theta}_1 B_1 + \dot{\theta}_1 B_2 + \dot{\theta}_1 B_3 + \dot{\theta}_1 B_4 + \dot{\theta}_1 B_5 + \dot{\theta}_1 B_6 \\
\dot{\theta}_1 C = \dot{\theta}_1 C_1 + \dot{\theta}_1 C_2 + \dot{\theta}_1 C_3 + \dot{\theta}_1 C_4 + \dot{\theta}_1 C_5 + \dot{\theta}_1 C_6 \\
\dot{\theta}_2 A_1 = \left( (n_{14} + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3) M_{33} \right) (h_{11} \sin \theta_2 \dot{\theta}_1 \\
+ h_{12} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + 2 h_{14} \sin \theta_2 \dot{\theta}_2 + h_{15} \sin \theta_3 \dot{\theta}_3 \\
+ h_{16} \sin(\theta_2 + \theta_3)(2 \dot{\theta}_2 + \dot{\theta}_3) - h_{17} \sin(\theta_2 + \theta_3) \dot{\theta}_3 \right) \\
\dot{\theta}_2 A_2 = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) M_{33} - (n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3)(n_{14} + n_{15} \cos \theta_3))(h_{21} \sin \theta_3 \dot{\theta}_3 + h_{22} \sin \theta_2 \dot{\theta}_1 \\
+ h_{24} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + h_{25} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + h_{29} \sin \theta_2 \dot{\theta}_1 + h_{210} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \right) \\
\dot{\theta}_2 A_3 = \left( (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
+ n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3))(h_{31} \sin \theta_2 \dot{\theta}_3 + \theta_3 \dot{\theta}_1 + h_{37} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + 2 h_{38} \sin \theta_3 \dot{\theta}_2 \\
+ h_{311} \sin \theta_3 \dot{\theta}_3 \right) \\
\dot{\theta}_2 A_4 = 0.0 \\
\dot{\theta}_2 A_5 = 0.0 \\
\dot{\theta}_2 A_6 = 0.0 \\
\dot{\theta}_2 B_1 = \left( (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3)) M_{33} - (n_{14} + n_{15} \cos \theta_3) \\
\times (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3))(h_{11} \sin \theta_2 \dot{\theta}_1 + h_{12} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \\
+ 2 h_{14} \sin(\theta_2 \dot{\theta}_2 + h_{15} \sin \theta_2 \dot{\theta}_3 + h_{16} \sin(\theta_2 + \theta_3)(2 \dot{\theta}_2 + \dot{\theta}_3) \\
- h_{17} \sin(\theta_2 + \theta_3) \dot{\theta}_3 \right) \\
\dot{\theta}_2 B_2 = \left( (n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)^2 - (n_1 + n_2 \cos \theta_2 + n_4 \cos(\theta_2 + \theta_3) \\
+ n_4 \cos \theta_3)M_{33})(h_{21} \sin \theta_3 \dot{\theta}_3 + h_{22} \sin \theta_2 \dot{\theta}_1 + h_{24} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \\
+ h_{25} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + h_{29} \sin \theta_2 \dot{\theta}_1 + h_{210} \sin(\theta_2 + \theta_3) \dot{\theta}_1 \right) \\
\dot{\theta}_2 B_3 = \left( (n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \\
- (n_5 + n_6 \cos(\theta_2 + \theta_3) + n_7 \cos \theta_3 + n_8 \cos\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
+ n_{11} \cos \theta_3))(h_{31} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + h_{33} \sin \theta_3 \dot{\theta}_3 + h_{36} \sin \theta_3 \dot{\theta}_1 \\
+ h_{37} \sin(\theta_2 + \theta_3) \dot{\theta}_1 + 2 h_{38} \sin \theta_3 \dot{\theta}_2 + h_{311} \sin \theta_3 \dot{\theta}_3 \right) \\
\]
A Torso-Moving Balance Control Strategy for a Walking Biped Robot

\[
\begin{align*}
\dot{\theta}_2 B_4 &= 0.0 \\
\dot{\theta}_2 B_5 &= 0.0 \\
\dot{\theta}_2 B_6 &= 0.0 \\
\dot{\theta}_2 C_1 &= ((n_9 + n_{10} \cos(\theta_2 + \theta_3) + n_{11} \cos \theta_3)(n_{12} + n_{13} \cos \theta_3) - (n_5 + n_6 \cos \theta_2 \\
&\quad + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_{14} + n_{15} \cos \theta_3))(h_{11} \sin \theta_2 \dot{\theta}_1 \\
&\quad + h_{12} \sin(\theta_2 + \theta_3)\dot{\theta}_1 + 2h_{14} \sin \theta_2 \dot{\theta}_2 + h_{15} \sin \theta_3 \dot{\theta}_3 + h_{16} \sin(\theta_2 + \theta_3)(2\dot{\theta}_2 + \dot{\theta}_3) \\
&\quad - h_{17} \sin(\theta_2 + \theta_3)\dot{\theta}_3) \\
\dot{\theta}_2 C_2 &= ((n_1 + n_2 \cos \theta_2 + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{14} + n_{15} \cos \theta_3) \\
&\quad - (n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))(n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
&\quad + n_{11} \cos \theta_3))(h_{21} \sin \theta_3 \dot{\theta}_3 + h_{22} \sin(\theta_2 \dot{\theta}_2 + h_{24} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \\
&\quad + h_{25} \sin(\theta_2 + \theta_3)\dot{\theta}_1 + h_{26} \sin \theta_2 \dot{\theta}_1 + h_{210} \sin(\theta_2 + \theta_3)\dot{\theta}_1) \\
\dot{\theta}_2 C_3 &= ((n_5 + n_6 \cos \theta_3 + n_7 \cos \theta_2 + n_8 \cos(\theta_2 + \theta_3))^2 - (n_1 + n_2 \cos \theta_2 \\
&\quad + n_3 \cos(\theta_2 + \theta_3) + n_4 \cos \theta_3)(n_{12} + n_{13} \cos \theta_3))(h_{31} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \\
&\quad + h_{33} \sin \theta_3 \dot{\theta}_3 + h_{34} \sin \theta_3 \dot{\theta}_1 + h_{37} \sin(\theta_2 + \theta_3)\dot{\theta}_1 + 2h_{38} \sin(\theta_2 \dot{\theta}_2 \\
&\quad + h_{311} \sin \theta_2 \dot{\theta}_3) \\
\dot{\theta}_2 C_4 &= 0.0 \\
\dot{\theta}_2 C_5 &= 0.0 \\
\dot{\theta}_2 C_6 &= 0.0 \\
\overline{\dot{\theta}_2 D} &= 0.0 \\
\overline{\dot{\theta}_2 A} &= \dot{\theta}_2 A_1 + \dot{\theta}_2 A_2 + \dot{\theta}_2 A_3 + \dot{\theta}_2 A_4 + \dot{\theta}_2 A_5 + \dot{\theta}_2 A_6 \\
\overline{\dot{\theta}_2 B} &= \dot{\theta}_2 B_1 + \dot{\theta}_2 B_2 + \dot{\theta}_2 B_3 + \dot{\theta}_2 B_4 + \dot{\theta}_2 B_5 + \dot{\theta}_2 B_6 \\
\overline{\dot{\theta}_2 C} &= \dot{\theta}_2 C_1 + \dot{\theta}_2 C_2 + \dot{\theta}_2 C_3 + \dot{\theta}_2 C_4 + \dot{\theta}_2 C_5 + \dot{\theta}_2 C_6 \\
\dot{\theta}_3 A_1 &= ((n_{14} + n_{15} \cos \theta_3)^2 - (n_{12} + n_{13} \cos \theta_3)M_{33})(h_{12} \sin(\theta_2 + \theta_3)\dot{\theta}_1 \\
&\quad + h_{13} \sin \theta_3 \dot{\theta}_1 + h_{15} \sin \theta_3 \dot{\theta}_2 + h_{16} \sin(\theta_2 + \theta_3)\dot{\theta}_2 \\
&\quad - h_{17} \sin(\theta_2 + \theta_3)(2\dot{\theta}_2 + \dot{\theta}_3) + h_{18} \sin \theta_3) \\
\dot{\theta}_3 A_2 &= ((n_5 + n_6 \cos \theta_2 + n_7 \cos \theta_3 + n_8 \cos(\theta_2 + \theta_3))M_{33} - (n_9 + n_{10} \cos(\theta_2 + \theta_3) \\
&\quad + n_{11} \cos \theta_3)(n_{14} + n_{15} \cos \theta_3))(h_{21} \sin \theta_3 \dot{\theta}_2 + h_{23} \sin \theta_3 \dot{\theta}_1 \\
&\quad + h_{24} \sin(\theta_2 + \theta_3)\dot{\theta}_1 + h_{25} \sin(\theta_2 + \theta_3)\dot{\theta}_1 + 2h_{26} \sin \theta_3 \dot{\theta}_3 + h_{11} \sin(\theta_2 + \theta_3)\dot{\theta}_4)
\end{align*}
\]
\[
\dot{\theta}_3A_3 = ((n_9 + n_{10}\cos(\theta_2 + \theta_3) + n_{11}\cos\theta_3)(n_{12} + n_{13}\cos\theta_3) - (n_5 + n_6\cos\theta_2
+ n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3))(h_{31}\sin(\theta_2 + \theta_3)\dot{\theta}_1
+ h_{32}\sin\theta_3\dot{\theta}_1 + h_{33}\sin\theta_2\dot{\theta}_2 + h_{39}\sin(\theta_2 + \theta_3)\dot{\theta}_1 + h_{310}\sin\theta_3\dot{\theta}_1
+ h_{311}\sin\theta_3\dot{\theta}_2)
\]
\[
\dot{\theta}_3A_4 = 0.0
\]
\[
\dot{\theta}_3A_5 = 0.0
\]
\[
\dot{\theta}_3A_6 = 0.0
\]
\[
\dot{\theta}_3B_1 = ((n_9 + n_{10}\cos\theta_2 + n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))M_{33} - (n_{14} + n_{15}\cos\theta_3)
\times (n_9 + n_{10}\cos(\theta_2 + \theta_3) + n_{11}\cos\theta_3))(h_{12}\sin(\theta_2 + \theta_3)\dot{\theta}_1 + h_{13}\sin\theta_3\dot{\theta}_1
+ h_{15}\sin\theta_3\dot{\theta}_2 + h_{16}\sin(\theta_2 + \theta_3)\dot{\theta}_2 - h_{17}\sin(\theta_2 + \theta_3)(2\dot{\theta}_2 + \dot{\theta}_3) + h_{18}\sin\theta_3)
\]
\[
\dot{\theta}_3B_2 = ((n_9 + n_{10}\cos(\theta_2 + \theta_3) + n_{11}\cos\theta_3)^2 - (n_1 + n_2\cos\theta_2 + n_3\cos(\theta_2 + \theta_3)
+ n_4\cos\theta_3)M_{33})(h_{21}\sin\theta_3\dot{\theta}_2 + h_{22}\sin\theta_3\dot{\theta}_1 + h_{24}\sin(\theta_2 + \theta_3)\dot{\theta}_1
+ h_{25}\sin(\theta_2 + \theta_3)\dot{\theta}_1 + 2h_{26}\sin\theta_3\dot{\theta}_3 + h_{311}\sin(\theta_2 + \theta_3)\dot{\theta}_1)
\]
\[
\dot{\theta}_3B_3 = ((n_1 + n_2\cos\theta_2 + n_3\cos(\theta_2 + \theta_3) + n_4\cos\theta_3)(n_{14} + n_{15}\cos\theta_3)
- (n_5 + n_6\cos\theta_2 + n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3))(n_{11}\cos\theta_3)(h_{31}\sin(\theta_2 + \theta_3)\dot{\theta}_1 + h_{32}\sin\theta_3\dot{\theta}_1 + h_{33}\sin\theta_2\dot{\theta}_2
+ h_{39}\sin(\theta_2 + \theta_3)\dot{\theta}_1 + h_{310}\sin\theta_3\dot{\theta}_1 + h_{311}\sin\theta_3\dot{\theta}_2)
\]
\[
\dot{\theta}_3B_4 = 0.0
\]
\[
\dot{\theta}_3B_5 = 0.0
\]
\[
\dot{\theta}_3B_6 = 0.0
\]
\[
\dot{\theta}_3C_1 = ((n_9 + n_{10}\cos(\theta_2 + \theta_3) + n_{11}\cos\theta_3)(n_{12} + n_{13}\cos\theta_3) - (n_5 + n_6\cos\theta_2
+ n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3))(h_{12}\sin(\theta_2 + \theta_3)\dot{\theta}_1
+ h_{13}\sin\theta_3\dot{\theta}_1 + h_{15}\sin\theta_3\dot{\theta}_2 + h_{16}\sin(\theta_2 + \theta_3)\dot{\theta}_2 - h_{17}\sin(\theta_2 + \theta_3)(2\dot{\theta}_2 + \dot{\theta}_3)
+ h_{18}\sin\theta_3)
\]
\[
\dot{\theta}_3C_2 = ((n_1 + n_2\cos\theta_2 + n_3\cos(\theta_2 + \theta_3) + n_4\cos\theta_3)(n_{14} + n_{15}\cos\theta_3)
- (n_5 + n_6\cos\theta_2 + n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))(n_{14} + n_{15}\cos\theta_3)))(n_{11}\cos\theta_3)(h_{21}\sin\theta_3\dot{\theta}_2 + h_{23}\sin\theta_3\dot{\theta}_1 + h_{24}\sin(\theta_2 + \theta_3)\dot{\theta}_1
+ h_{25}\sin(\theta_2 + \theta_3)\dot{\theta}_1 + 2h_{26}\sin\theta_3\dot{\theta}_3 + h_{11}\sin(\theta_2 + \theta_3)\dot{\theta}_1)
\]
\[
\dot{\theta}_3C_3 = ((n_5 + n_6\cos\theta_2 + n_7\cos\theta_3 + n_8\cos(\theta_2 + \theta_3))^2 - (n_1 + n_2\cos\theta_2
+ n_3\cos(\theta_2 + \theta_3) + n_4\cos\theta_3)(n_{12} + n_{13}\cos\theta_3))(h_{31}\sin(\theta_2 + \theta_3)\dot{\theta}_1
\]
\[
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\]
+ h_{32} \sin \theta_3 \dot{\theta}_1 + h_{33} \sin \theta_3 \dot{\theta}_2 + h_{39} \sin (\theta_2 + \theta_3) \dot{\theta}_1 + h_{310} \sin \theta_3 \dot{\theta}_1 \\
+ h_{311} \sin \theta_3 \dot{\theta}_2) \quad \text{(B.14)}
\begin{align*}
\dot{\theta}_3 C_4 &= 0.0 \\
\dot{\theta}_3 C_5 &= 0.0 \\
\dot{\theta}_3 C_6 &= 0.0 \\
\dot{\theta}_3 D &= 0.0 \\
\dot{\theta}_3 A &= \dot{\theta}_3 A_1 + \dot{\theta}_3 A_2 + \dot{\theta}_3 A_3 + \dot{\theta}_3 A_4 + \dot{\theta}_3 A_5 + \dot{\theta}_3 A_6 \\
\dot{\theta}_3 B &= \dot{\theta}_3 B_1 + \dot{\theta}_3 B_2 + \dot{\theta}_3 B_3 + \dot{\theta}_3 B_4 + \dot{\theta}_3 B_5 + \dot{\theta}_3 B_6 \\
\dot{\theta}_3 C &= \dot{\theta}_3 C_1 + \dot{\theta}_3 C_2 + \dot{\theta}_3 C_3 + \dot{\theta}_3 C_4 + \dot{\theta}_3 C_5 + \dot{\theta}_3 C_6
\end{align*}
\begin{align*}
A_1 &= \frac{1}{D^2} \dot{\theta}_1 A D - A \ddot{\theta}_1 D \\
A_2 &= \frac{1}{D^2} \dot{\theta}_1 B D - B \ddot{\theta}_1 D \\
A_3 &= \frac{1}{D^2} \dot{\theta}_1 C D - C \ddot{\theta}_1 D \\
B_1 &= \frac{1}{D^2} \dot{\theta}_2 A D - A \ddot{\theta}_2 D \\
B_2 &= \frac{1}{D^2} \dot{\theta}_2 B D - B \ddot{\theta}_2 D \\
B_3 &= \frac{1}{D^2} \dot{\theta}_2 C D - C \ddot{\theta}_2 D \\
C_1 &= \frac{1}{D^2} \dot{\theta}_3 A D - A \ddot{\theta}_3 D \\
C_2 &= \frac{1}{D^2} \dot{\theta}_3 B D - B \ddot{\theta}_3 D \\
C_3 &= \frac{1}{D^2} \dot{\theta}_3 C D - C \ddot{\theta}_3 D \\
D_1 &= \frac{1}{D^2} \dot{\theta}_1 A D - A \ddot{\theta}_1 D \\
D_2 &= \frac{1}{D^2} \dot{\theta}_1 B D - B \ddot{\theta}_1 D \\
D_3 &= \frac{1}{D^2} \dot{\theta}_1 C D - C \ddot{\theta}_1 D
\[ E_1 = \frac{1}{D^2} \theta_2 A D - A \theta_2 D \]
\[ E_2 = \frac{1}{D^2} \theta_2 B D - B \theta_2 D \]
\[ E_3 = \frac{1}{D^2} \theta_2 C D - C \theta_2 D \]
\[ F_1 = \frac{1}{D^2} \theta_3 A D - A \theta_3 D \]
\[ F_2 = \frac{1}{D^2} \theta_3 B D - B \theta_3 D \]
\[ F_3 = \frac{1}{D^2} \theta_3 C D - C \theta_3 D \]
\[ G_{11} = G_{12} = G_{13} = G_{21} = G_{22} = G_{23} = 0 \]
\[ G_{31} = \frac{1}{D} M_{11} \]
\[ G_{32} = \frac{1}{D} M_{12} \]
\[ G_{33} = \frac{1}{D} M_{13} \]

(B.15)

References


