Incomplete interval fuzzy preference relations and their applications

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A B S T R A C T
This paper investigates incomplete interval fuzzy preference relations. A characterization, which is proposed by Herrera-Viedma et al. (2004), of the additive consistency property of the fuzzy preference relations is extended to a more general case. This property is further generalized to interval fuzzy preference relations (IFPRs) based on additive transitivity. Subsequently, we examine how to characterize IFPRs using these new characterizations. We propose a method to construct an additive consistent IFPR from a set of preference data and an estimation algorithm for acceptable incomplete IFPRs with more known elements. Numerical examples are provided to illustrate the effectiveness and practicality of the solution process.

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1. Introduction

Fuzzy preference relations are one of the most common preference relations for expressing a decision maker’s (DM’s) preference over alternatives. In a decision making process, the DM generally needs to compare a set of decision alternatives \( n \) (i.e., \( n = 1, 2, \ldots, n \)), thereby constructing a fuzzy preference relation (Herrera-Viedma, Herrera, Chiclana, & Luque, 2004; Kacprzyk, 1986; Orlovsky, 1978; Tanino, 1984; Wang & Fan, 2007; Xu, Da, & Liu, 2009; Xu, Patnayakuni, & Wang, 2005). However, the DM may have vague knowledge about the preference degrees of one alternative over another and cannot estimate his/her preference with an exact numerical value, but with an interval number. In this case, the DM constructs an interval preference relation.

Saaty and Vargas (1987) first presented interval judgments as a way to model subjective uncertainty. Afterwards, some methods are proposed to generate weights from interval comparison matrices, such as linear programming (LP) (Arbel, 1989; Kress, 1991), lexicographic Goal Programming (LGP) (Islam, Biswal, & Alam, 1997; Wang, 2006), fuzzy preference programming (FPP) (Mikhailov, 2002; Mikhailov, 2004), two-stage logarithmic goal programming (TLGP) (Wang, Yang, & Xu, 2005), eigenvector method (EM) (Wang & Chin, 2006), Lambda-Max method (Csutora & Buckley, 2001), goal programming method (GPM) (Wang & Elhag, 2007), etc.

For IFPRs, Xu (2004b) defined the concept of compatibility degree between two IFPRs, and showed the compatibility relationship between individual IFPRs and collective IFPR. Herrera, Martínez, and Sánchez (2005) developed an aggregation process for combining IFPRs with other types of information such as numerical preference relation and linguistic preference relation. Jiang (2007) gave an index to measure the similarity degree between two IFPRs, and employed an error-propagation principle to determine a priority vector for the aggregated IFPRs. Recently, Xu and Chen (2008b) established some linear programming models for deriving priority weights from various IFPRs. Wang and Li (2012) developed goal-programming-based models for deriving interval weights from IFPRs for both individual and group decision-making situations.

The aforesaid research focused on preference relations with complete information. A complete preference relation of order \( n \) necessitates the completion of all \( n(n-1)/2 \) judgments in its entire top triangular portion. Sometimes, however, a DM may develop a preference relation with incomplete information due to a variety of reasons such as time pressure, lack of knowledge, and the DM’s limited expertise related with the problem domain (Chiclana, Herrera-Viedma, Alonso, & Herrera, 2008; Lee, Chou, Fang, Tseng, & Yeh, 2007; Xu & Da, 2008; Xu & Da, 2009; Xu, Da, & Wang, 2010; Xu, Gupta, & Wang, 2013a; Xu, Patnayakuni, & Wang, 2013b; Xu, 2004a; Xu, 2005; Xu & Chen, 2008a); In addition, when the number of the alternatives, \( n \), is large, it may be impractically to require the DM to perform all the \( n(n-1)/2 \) required comparisons for a complete the pairwise comparison matrix (Fedrizzi & Silvio, 2007); More over, it is sometimes convenient or necessary to skip some direct comparison between alternatives even if the total number of alternatives is small (Fedrizzi & Silvio, 2007); In some other...
cases, a DM is unable to express any kind of preference between two or more options, which may be due to an expert not possessing a precise or sufficient level of knowledge of part of the problem, or because that expert is unable to discriminate the degree to which one option is preferred to another (Alonso, Chiclana, Herrera, & Herrera-Viedma, 2004; Alonso et al., 2008; Herrera-Viedma, Alonso, Chiclana, & Herrera, 2007a; Herrera-Viedma, Chiclana, Herrera, & Alonso, 2007b; Xu, 2012). A critical concern for the incomplete fuzzy preference relations is to estimate the missing values. Herrera-Viedma et al. (2007a) proposed an iterative procedure to estimate the missing information in an expert’s incomplete fuzzy preference relation. The procedure is guided by the additive consistency property and only uses the preference values provided by the expert. Fedrizzi and Silvio (2007) put forward a new method for calculating missing elements in an incomplete fuzzy preference relation by maximizing global consistency. Later, Chiclana, Herrera-Viedma, and Alonso (2009) pointed out that the two methods are very similar in calculating missing values. Chiclana et al. (2008) presented a new estimation method based on the U-consistency criterion for incomplete fuzzy preference relations. Alonso et al. (2008) presented a procedure to estimate missing preference values for incomplete fuzzy, multiplicative, interval-valued, and linguistic preference relations. Liu, Pan, Xu, and Yu (2012b) developed a method to calculate missing values by minimizing the squared error of an incomplete fuzzy preference relation and its priority weight vector. Xu (2012) devised an approach to extending each incomplete multiplicative preference relation to a complete one by exploiting the multiplicative transitivity properties. Xia, Xu, and Wang (2014) furnished an algorithm to estimate missing values for an incomplete linguistic preference relation based on multiplicative consistency. Recently, research has been extended to IFPRs. For instance, Alonso et al. (2008) put forward a procedure to estimate missing information for the incomplete IFPR. Genç, Boran, Akay, and Xu (2010) examined consistency, missing value (s) and derivation of priority vectors from IFPRs based on multiplicative transitivity. Liu, Zhang, and Wang (2012a) proposed a new method to obtain priority weights from incomplete interval multiplicative preference relations. However, limited research has been devoted to incomplete IFPRs. As such, it is necessary to pay attention to this issue.

Another important issue is the consistency of the judgment provided by experts (Chiclana, Herrera, & Herrera-Viedma, 2002; Herrera-Viedma et al., 2004). It is obvious that consistent information is more relevant or important than the information containing contradictions. Consistency is associated with certain transitivity properties. Different properties have been suggested to model transitivity of fuzzy preference relations. One of these properties is the “additive transitivity”, which, as shown in (Herrera-Viedma et al., 2004), can be seen as a parallel concept of Saaty’s consistency property for multiplicative reciprocal preference relations.

The aim of this paper is to propose some methods for constructing additive consistent IFPRs based on acceptable incomplete IFPRs. We first extend an additive consistency property proposed by Herrera-Viedma et al. (2004) for the fuzzy preference relations to a general case. Then, this property is extended to IFPRs based on the additive transitivity. After further characterizing additive consistent IFPRs, we develop two algorithms for estimating missing elements from acceptable incomplete IFPRs. A procedure is then laid out for handling GDM problems with acceptable incomplete IFPRs.

The rest of this paper is organized as follows. Section 2 reviews some properties of fuzzy preference relations. Section 3 first introduces the concepts of interval multiplicative reciprocal preference relations and IFPRs as well as their transformation function. The property of additive consistent fuzzy preference relations in Section 2 is then extended to IFPRs, followed by further additive consistent IFPRs. In Section 4, we propose two approaches to construct additive consistent IFPRs based on acceptable incomplete IFPRs. A case study is furnished in Section 5 to illustrate how to apply our algorithms. We conclude the paper in Section 6.

2. Additive consistent fuzzy preference relations

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set of alternatives, where \( x_i \) denotes the \( i \)th alternative. In multiple attribute decision making problems, a DM needs to rank alternatives \( x_1, x_2, \ldots, x_n \) from the best to the worst according to preference information. A brief description of multiplicative and fuzzy preference relations is given below.

2.1. Multiplicative preference relations

A multiplicative preference relation is a positive preference relation \( A \subseteq X \times X, A = (a_{ij})_{n \times n} \), where \( a_{ij} \) denotes the relative preference of alternative \( x_i \) over \( x_j \). The measurement of \( a_{ij} \) described by a ratio scale and in particular, as shown by Saaty (1980), \( a_{ij} \in [1/9, 9, \ldots, 1, 2, \ldots, 9] \). \( a_{ij} = 1 \) denotes the DM’s indifference between \( x_i \) and \( x_j \), \( a_{ij} = 9 \) (or \( a_{ji} = 1/9 \)) denotes that \( x_i \) is absolutely preferred to \( x_j \), and \( a_{ij} \in [2, 3, \ldots, 8] \) denotes intermediate preference evaluations. This relation is multiplicative reciprocal, i.e., \( a_{ij}a_{ji} = 1 \), for all \( i,j \in \{1, 2, \ldots, n\} \) and in particular, \( a_{ii} = 1 \), \( \forall i \in \{1, 2, \ldots, n\} \). Its consistency is defined by Saaty (1980) as follows.

**Definition 1.** Let \( A = (a_{ij})_{n \times n} \) be a multiplicative preference relation, then \( A \) is called consistent (Saaty, 1980), if \( a_{ij} = a_{ik}a_{kj} \) for all \( i, j, k \).

2.2. Fuzzy preference relations

A fuzzy preference relation \( R \subseteq X \times X \), \( R = (R_{ij})_{n \times n} \) with membership function \( \mu_{ij}: X \times X \to [0, 1] \), where \( \mu_{ij}(x_i, x_j) = r_{ij} \) denotes the preference degree of alternative \( x_i \) over \( x_j \) (Kacprzyk, 1986; Tanino, 1984). \( r_{ij} = 0.5 \) denotes indifference between \( x_i \) and \( x_j \), \( r_{ij} = 1 \), denotes that \( x_i \) is definitely preferred to \( x_j \), and \( 0.5 < r_{ij} < 1 \) (or \( 0 < r_{ji} < 0.5 \)) denotes that \( x_i \) is preferred to \( x_j \) to a varying degree.

**Definition 2.** Let \( R = (R_{ij})_{n \times n} \) be a preference relation, then \( R \) is called a fuzzy preference relation, if
\[
r_{ij} \in [0, 1], \quad r_{ij} + r_{ji} = 1, \quad r_{ii} = 0.5, \quad \text{for all } i, j \in N
\]

**Definition 3.** Let \( R = (R_{ij})_{n \times n} \) be a fuzzy preference relation, then \( R \) is called an additive transitive fuzzy preference relation if the following additive transitivity (Tanino, 1984) is satisfied:
\[
r_{ij} = r_{ik} - r_{jk} + 0.5, \quad \text{for all } i, j, k \in \{1, 2, \ldots, n\}
\]

2.3. Characterizing additive consistency of fuzzy preference relations

Herrera-Viedma et al. (2004) studied the transformation between multiplicative preference relations with values in the interval scale \([1/9, 9]\) (Alonso et al., 2004) and fuzzy preference relations with values in \([0, 1]\) and furnished the following propositions.

**Proposition 1** (Herrera-Viedma et al., 2004). Consider a set of alternatives \( X = \{x_1, x_2, \ldots, x_n\} \), associated with a multiplicative reciprocal preference relation \( A = (a_{ij})_{n \times n} \), with \( a_{ij} \in [1/9, 9] \). Then, a corresponding fuzzy preference relation, \( R = (R_{ij})_{n \times n} \), with \( R_{ij} \in [0, 1] \), associated with \( A \) is given as follows:
\[
r_{ij} = g(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij})
\]
Proposition 2 (Herrera-Viedma et al., 2004). For a fuzzy preference relation \( R = (r_{ij})_{n \times n} \), the following statements are equivalent:

(a) \( r_y + r_a + r_k = \frac{3}{2} \), \( i, j, k \);
(b) \( r_y + r_a + r_k = \frac{3}{2} \), \( i < j < k \).

Proposition 3 (Herrera-Viedma et al., 2004). For a fuzzy preference relation \( R = (r_{ij})_{n \times n} \), the following statements are equivalent:

(a) \( r_y + r_a + r_k = \frac{3}{2} \), \( i < j < k \);
(b) \( r_{(y+i)} + r_{(y+j)-k} + \cdots + r_{(y-1)-j} + \cdots + r_{(y-1)-k} = \frac{i-1}{2} \), \( i < j \).

2.4. A new characterization of additive consistency

Herrera-Viedma et al. (2004) showed that Proposition 3 can be used to construct an additive consistent fuzzy preference relation from a set of \( n-1 \) values \((r_{12}, r_{23}, \ldots, r_{n-1})\). The aforesaid propositions were also used by Wang and Chen (2007, 2008). In the following, a more general result is provided.

Proposition 4. For a fuzzy preference relation \( R = (r_{ij})_{n \times n} \), the following statements are equivalent:

(a) \( r_y + r_a + r_k = \frac{3}{2} \), \( i, j, k \in N \);
(b) \( r_{j_1} + \cdots + r_{j_{i+j}} + r_{k_1} = \frac{i-1}{2} \), \( i, j \in N \), \( i = 1, 2, \ldots, t \).

Proof. (a) \( \Rightarrow \) (b) Mathematical induction is employed to prove this part of the proposition. It is obviously true for \( t = 1 \), as it is reduced to the additive reciprocity property in Definition 2. Next, if the hypothesis is true for \( t = n \)

\[ r_{j_1} + r_{j_2} + \cdots + r_{j_{n-2+1}} + r_{j_n} + r_{k_n} = \frac{n + 1}{2} \]

then it is true for \( t = n + 1 \)

\[ r_{j_1} + r_{j_2} + \cdots + r_{j_{n+1-2+1}} + r_{j_{n+1}} + r_{k_{n+1}} = \frac{n+2}{2} + \frac{n+1}{2} = \frac{n+2}{2} \]

So the result is established.

(b) \( \Rightarrow \) (a).

\[ r_y + r_a + r_k = 1 - r_y - 1 - r_k + r_k = \frac{2}{2} + \frac{r_y + r_a + r_k - r_y - r_a + r_y - r_a + r_k - r_y - r_k}{2} + r_y = 1 \]

This completes the proof. \( \square \)

Furthermore, in the proof process, it is obvious that the differences of \( j_2 - j_1, j_3 - j_2, \ldots, j_t - j_{t-1} \) are not necessarily equal to 1. As a matter of fact, the differences do not have to be identical.

Proposition 4 differs from Proposition 3 in that any the sequential values of \( r_y, r_{j_1}, r_{j_2}, r_{j_3}, \ldots, r_{j_{t-1}}, r_{j_t} \) (for example \( r_{31}, r_{15}, r_{27}, r_{73} \)) will work for Proposition 4. But Proposition 3 requires preference values to follow a consecutive order such as \( r_{12}, r_{23}, \ldots, r_{n-1n}, r_{nn} \). Therefore, Proposition 3 is a special case of Proposition 4.

3. Interval fuzzy preference relations and their characterizations

In the following we shall first introduce some operational laws of interval numbers (Hayes, 2003; Moore, 1966). We then present a relationship between an interval multiplicative preference relation and an IFPR. The characterizations of additive consistent fuzzy preference relations in Section 2 are subsequently extended to IFPRs followed by some other useful results.

Let \( a_1 = [a_1', a_1] \), \( a_2 = [a_2', a_2] \), \( a = [a', a'] \) be three positive interval numbers, then

(a) \( a_1 \odot a_2 = [a_1', a_1] \odot [a_2', a_2] = [a_1 + a_2', a_1 + a_2] \)
(b) \( a_1 \odot a_2 = [a_1', a_1] \odot [a_2', a_2] = [a_1 - a_2', a_1 - a_2] \)
(c) \( a_1 \odot a_2 = [a_1', a_1] \odot [a_2', a_2] = [a_1 \times a_2', a_1 \times a_2] \)
(d) \( \log_a(a) = \frac{\log a'}{\log a} \)
(e) \( a^{-1} = [a', a] \)

Definition 4. An interval preference relation \( \overline{A} = (a_{ij}) \) is multiplicative if and only if \( a_{ij} = a_{ij}^{-1} \).

Definition 5 (Xu, 2010). If a positive interval multiplicative reciprocal preference relation \( \overline{A} = (a_{ij}) \) satisfies,

\[ a_{ij} = a_{ji} \odot a_{ij}, \quad \text{for all } i, k, j = 1, 2, \ldots, n, \quad i < k \leq j \]

then \( \overline{A} = (a_{ij}) \) is multiplicative consistent.

Note that if an interval multiplicative reciprocal preference relation is multiplicative consistent, for all \( i, k, j \in \{1, 2, \ldots, n\} \), as pointed out in Xu (2010), it is necessary to require \( i < k \leq j \); otherwise, \( \overline{A} = (a_{ij}) \) would be reduced to a crisp number judgment matrix (Saaty’s multiplicative reciprocal preference relation). For more detail, readers are referred to Xu, 2010. That is Eq. (9) holds only for the upper (or lower) triangular of the preference relation.

Definition 6 (Xu, 2004). Let \( R = (r_{ij}) \) be a preference relation, where

\[ r_y = [r_y', r_y], \quad r_y = [r_y', r_y'], \quad r_y + r_y = r_y + r_y = 1, \quad r_y \geq r_y \geq 0 \]

then \( R \) is called an interval fuzzy preference relation (IFPR).

Definition 7. Let \( R = (r_{ij}) \) be an IFPR, if

\[ r_y \odot r_y = r_y \odot [0.5, 0.5], \quad \forall i < j < k \]

then \( R \) is called an additive consistent IFPR.

Proposition 5. For a set of alternatives \( X = \{x_1, x_2, \ldots, x_n\} \), and its associated interval multiplicative reciprocal preference relation \( \overline{A} = (a_{ij}) \) with \( a_{ij} = a_{ij}^{-1} \), a corresponding IFPR, \( R = (r_{ij}) \), associated with \( \overline{A} \) is given as

\[ r_y = g(a_{ij}) = \frac{1}{2} \left( [1.1] \odot \log a_{ij} \right) \]

such that

(a) \( r_y + r_y = 1, \quad \forall i, j \in N \);
(b) \( r_y + r_y = 1, \quad \forall i, j \in N \).

Proof. As \( \overline{A} = (a_{ij}) \) is an interval multiplicative reciprocal preference relation, by Definition 4, \( a_{ij} = a_{ij}^{-1} \), that is

\[ a_{ij} = \frac{1}{a_{ij}'} \]

Thus

\[ a_{ij} a_{ij} = 1, \quad a_{ij} a_{ij} = 1 \]

(12)

(13)

(14)
By Proposition 1 and Eq. (11) and the operational law (Eq. (7)) of interval numbers, we have
\[ r_{ij} = \frac{1}{2} \left( 1 + \log a_{ij} \right) = \frac{1}{2} \left( [1, 1] \oplus \log \left( a_j, a_i \right) \right) \]
Thus
\[ r_{ij} = \frac{1}{2} \left( 1 + \log a_{ij} \right) \]
\[ r_{ij} = \frac{1}{2} \left( 1 + \log a_{ij}^* \right) \]
Similarly,
\[ r_{ji} = \frac{1}{2} \left( 1 + \log a_{ji} \right) \]
\[ r_{ji} = \frac{1}{2} \left( 1 + \log a_{ji}^* \right) \]
Therefore,
\[ r_{ij} + r_{ji} = \frac{1}{2} \left( 1 + \log a_{ij} \right) + \frac{1}{2} \left( 1 + \log a_{ji} \right) = 1 + \log a_{ij}^* a_{ji}^* = 1 \]
\[ r_{ij} + r_{ji} = \frac{1}{2} \left( 1 + \log a_{ij} \right) + \frac{1}{2} \left( 1 + \log a_{ji} \right) = 1 + \log a_{ij}^* a_{ji}^* = 1 \]
\[ \forall i, j \in \{1, 2, \ldots, n\} \]

The proof is thus completed. \( \square \)

Next, we examine the relationship between the multiplicative consistency of an interval multiplicative reciprocal preference relation and the additive consistency of its converted IFPR as per Eq. (11).

**Proposition 6.** If an interval multiplicative reciprocal preference relation \( \mathcal{A} = (a_{ij}) \) is multiplicative consistent, then its corresponding IFPR \( \mathcal{R} = (r_{ij}) \) is additive consistent, and

(a) \( r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}, \forall i < j < k; \)
(b) \( r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}, \forall i < j < k; \)
(c) \( r_{ij} + r_{j(1\cdots i)} + \cdots + r_{ji} = \frac{n+1}{2}, \forall i < j; \)
(d) \( r_{ij} + r_{j(1\cdots i)} + \cdots + r_{ji} = \frac{n+1}{2}, \forall i < j; \)
(e) \( r_{ij} + r_{i(1\cdots j)} + \cdots + r_{ji} = \frac{n+1}{2}, \forall i < j < j; \)
(f) \( r_{ij} + r_{i(1\cdots j)} + \cdots + r_{ji} = \frac{n+1}{2}, \forall i < j < j; \)

**Proof.** Since \( \mathcal{A} = (a_{ij}) \) is multiplicative consistent by Definition 5, then \( a_{ij} \otimes a_{kj} = a_{ik}, \) for \( \forall i < j < k. \) Taking a logarithm operation (Eq. (7)) on both sides yields
\[ \log a_{ij} \oplus \log a_{kj} = \log a_{ik}, \forall i < j < k \]
Thus
\[ \frac{1}{2} \left( [1, 1] \oplus \log a_{ij} \right) \frac{1}{2} \left( [1, 1] \oplus \log a_{kj} \right) = \frac{1}{2} \left( [1, 1] \oplus \log a_{ik} \right) \oplus [0.5, 0.5] \]
By Eq. (11), we have
\[ r_{ij} \oplus r_{kj} = r_{ik} \oplus [0.5, 0.5] \]
\[ \left[ r_{ij}, r_{jk} \right] \left[ r_{jk}, r_{ki} \right] = \left[ r_{ik}, r_{jk} \right] \oplus [0.5, 0.5] \]
\[ r_{ij} + r_{jk} = r_{ik} + 0.5, \quad r_{ij} + r_{jk} = r_{ik} + 0.5 \]
By Proposition 5, we have
\[ r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}, \quad r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}, \quad \forall i < j < k \]
Thus the expressions (a) and (b) are established.

Let \( i < j, \) and \( k = j - i. \) The expression (c) can be rewritten as follows:
\[ r_{(i+1)(i+2)} + \cdots + r_{j-1} + r_{ji} = k + 1 \quad \forall i < j \]
Mathematical induction is used to prove this part. It is clearly true for \( k = 1. \) Next if the hypothesis is true for \( k = n \)
\[ r_{(i+1)(i+2)} + \cdots + r_{(j-1)(j-1)} + r_{ji} = \frac{n+1}{2} \]
then for \( k = n + 1: \)
\[ r_{(i+1)(i+2)} + \cdots + r_{(j-1)(j-1)} + r_{ji} = \frac{n+1}{2} + r_{(i+1)(i+1)} + r_{ji} \]
\[ = \frac{n+1}{2} + r_{(i+1)(i+1)} + r_{ji} \]
\[ = \frac{n+1}{2} \quad \forall i < j \]
thus the expression (c) is confirmed and (d) can be analogously asserted. Similar to Proposition 4, (e) and (f) can be verified. \( \square \)

**Proposition 7.** An IFPR \( \mathcal{R} = (r_{ij}) \) is additive consistent if and only if

(1) \( r_{ij} - r_{ik} = r_{ij} - r_{ik}, \quad \forall i < j < k, \quad l = 1, 2, \ldots, n. \)

**Proof.** If an IFPR \( \mathcal{R} = (r_{ij}) \) is additive consistent by Definition 7, then
\[ r_{ij} \oplus r_{jk} = r_{ik} \oplus [0.5, 0.5], \quad \forall i < j < k \]
\[ r_{ij} + r_{jk} = r_{ik} + \frac{3}{2}, \quad r_{ij} + r_{jk} = r_{ik} + \frac{3}{2}, \quad \forall i < j < k \]
\[ r_{ij} - r_{ik} = \frac{3}{2}, \quad r_{ij} - r_{ik} = \frac{3}{2}, \quad \forall i < j < k \]
Similarly,
\[ r_{ij} \oplus r_{jk} = r_{ik} \oplus [0.5, 0.5], \quad \forall i < j < k \]
\[ r_{ij} + r_{jk} = r_{ik} + \frac{3}{2}, \quad r_{ij} + r_{jk} = r_{ik} + \frac{3}{2}, \quad \forall i < j < k \]
\[ r_{ij} - r_{ik} = \frac{3}{2}, \quad r_{ij} - r_{ik} = \frac{3}{2}, \quad \forall i < j < k \]
thus
\[ r_{ij} - r_{ik} = r_{ij} - r_{ik}, \quad r_{ij} - r_{ik} = r_{ij} - r_{ik}, \quad \forall i < j < k \]
On the contrary,
If \( r_{ij} - r_{ik} = r_{ij} - r_{ik}, \quad r_{ij} - r_{ik} = r_{ij} - r_{ik}, \quad \forall i < j < k \)
Let \( l = j, \) since \( r_{ij} = r_{ij} = 0.5, \) then
\[ r_{ij} - r_{ik} = 0.5 - r_{ij}, \quad r_{ij} - r_{ik} = 0.5 - r_{ik}, \quad \forall i < j < k \]
That is
\[ r_{ij} + r_{jk} = 0.5 + r_{ik}, \quad r_{ij} + r_{jk} = 0.5 + r_{ik}, \quad \forall i < j < k \]
This completes the proof. \( \square \)

Proposition 7 reveals an important property of an additive consistent IFPR. For upper (or lower) triangular interval values, the difference of the lower bounds between any two columns should be a constant for all rows. The same is true for the upper bounds of the interval preference values. Propositions 6 and 7 will play an important role in devising our algorithms to construct complete IFPRs based on an incomplete relation.
Note that, if the primary values are different then we may have obtained a matrix $R$ with entries not in the interval $[0, 1]$, but in an interval $[-c, 1 + c]$, where $c > 0$. $-c$ indicates the minimum value of matrix $R$, $1 + c$ gives the maximum value of matrix $R$. In this case, the obtained values have to be converted using a transformation function that preserves reciprocity and additive consistency, i.e., $f$: $[-c, 1 + c] \rightarrow [0, 1]$, verifying

(a) $f(-c) = 0$
(b) $f(1 + c) = 1$
(c) $f(x') + f(x'^*) = 1$, $\forall x \in [-c, 1 + c]$
(d) If $x' + y' + z'^* = \frac{3}{2}$, $f(x') + f(y') + f(z'^*) = \frac{3}{2}$, $\forall x', y', z'^* \in [-c, 1 + c]$
(e) If $x' + y' + z' = \frac{3}{2}$, $f(x') + f(y') + f(z') = \frac{3}{2}$, $\forall x', y', z' \in [-c, 1 + c]$

A linear function satisfying (a) and (b) takes the form

\[
\begin{align*}
  f(x') &= \varphi x' + \beta, \quad \varphi, \beta \in R \\
  f(x'^*) &= \varphi x'^* + \beta, \quad \varphi, \beta \in R
\end{align*}
\]

These functions are

\[
\begin{align*}
  f(x') &= \frac{1}{1 + 2c} x' + \frac{c}{1 + 2c} x'^* + \frac{c}{1 + 2c} \\
  f(x'^*) &= \frac{1}{1 + 2c} x'^* + \frac{c}{1 + 2c} x'^* + \frac{c}{1 + 2c}
\end{align*}
\]

(c) can be easily verified as

\[
\begin{align*}
  f(x'^*) &= x'^* + c, \quad \frac{x'^* + c}{1 + 2c} + \frac{c}{1 + 2c} = 1 \\
  f(x') &= \frac{x'^* + c}{1 + 2c} + \frac{x'^* + c}{1 + 2c} + \frac{x'^* + c}{1 + 2c}
\end{align*}
\]

(d) and (e) are also confirmed.

**Proposition 8.** Let $R^{(1)}, R^{(2)}, \ldots, R^{(m)}$ be m IFPRs, then their weighted average

\[
\mathbf{R} = \lambda_1 R^{(1)} + \lambda_2 R^{(2)} + \cdots + \lambda_m R^{(m)}, \quad \lambda_i \in [0, 1], \quad \sum_{i=1}^{m} \lambda_i = 1
\]

is also an IFPR.

**Proof.** Since $R^{(1)}, R^{(2)}, \ldots, R^{(m)}$ are IFPRs, it follows that

\[
\begin{align*}
  r_{ij}^{(l)} + r_{ij}^{(l)^*} &= 1, \quad r_{ij}^{(l)} + r_{ij}^{(l)^*} = 1, \quad \forall i, j \in \{1, 2, \ldots, n\}, \\
  l &= 1, 2, \ldots, m
\end{align*}
\]

Then by Eq. (20), we have

\[
\begin{align*}
  r_{ij}^c &= \lambda_1 r_{ij}^{(1)} + \lambda_2 r_{ij}^{(2)} + \cdots + \lambda_m r_{ij}^{(m)} \\
  r_{ij}^c &= \lambda_1 r_{ij}^{(1)} + \lambda_2 r_{ij}^{(2)} + \cdots + \lambda_m r_{ij}^{(m)} \\
  r_{ij}^c &= \lambda_1 r_{ij}^{(1)} + \lambda_2 r_{ij}^{(2)} + \cdots + \lambda_m r_{ij}^{(m)} \quad \forall i, j \in \{1, 2, \ldots, n\}, \quad \forall l \in \{1, 2, \ldots, m\}
\end{align*}
\]
R is called an incomplete interval IFPR, where $\Omega$ is the set of all the known elements in $R$.

**Definition 9.** The elements $\bar{r}_{ij}, \underline{r}_{ij}$ of $R$ are called adjacent, if $\{i, j\} \cap \{k, l\} = \emptyset$. For a missing element $\underline{r}_{ij}$, it can be determined indirectly if there exist a series of known elements $\bar{r}_{il}, \underline{r}_{il}, \ldots, \underline{r}_{ij}$.

**Definition 10.** Let $R = (\underline{r}_{ij})$ be an incomplete IFPR. If all missing elements of $R$ can be obtained by the known elements, then $R$ is called an acceptable incomplete IFPR. Otherwise, $R$ is an unacceptable incomplete IFPR.

Next, we extend the necessary condition (Herrera-Viedma et al., 2007b; Xu & Da, 2008) of acceptable incomplete fuzzy preference relations to the case of incomplete IFPRs.

**Proposition 10.** Let $R = (\underline{r}_{ij})$ be an incomplete IFPR. If $R$ is an acceptable incomplete IFPR, then there exists at least one known non-diagonal element in each line or each column of $R$, i.e. there exist at least $(n-1)$ judgments provided by the DM.

**Definition 11.** Let $\bar{R} = (\bar{r}_{ij})$ be an incomplete IFPR, if the known elements satisfy

\begin{alignat}{2}
\bar{r}_{i1} + \bar{r}_{i2} + \cdots + \bar{r}_{in} &= t + \frac{1}{2}, & \quad \forall i < j < \bar{j} \\
\bar{r}_{j1} + \bar{r}_{j2} + \cdots + \bar{r}_{jn} &= \bar{t} + \frac{1}{2}, & \quad \forall i < j < \bar{j}
\end{alignat}

(23)

then $\bar{R}$ is called an additive consistent incomplete IFPR.

### 4.2. An estimation procedure for acceptable incomplete IFPRs with fewest number of judgments

Next, by exploiting Propositions 5 and 6 or Proposition 7, a simple and practical method is developed for constructing a complete additive consistent IFPR based on an acceptable incomplete IFPR with fewest number of judgment data (i.e., $n-1$ preference values):

**Algorithm 1**

**Step 1.** For a decision problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of alternatives. The DM conducts pairwise comparisons among the alternatives and furnished his/her assessment as an acceptable incomplete IFPR $R = (\underline{r}_{ij})$ (if the DM provides his/her evaluation as an acceptable incomplete interval multiplicative reciprocal preference relation $\bar{R} = (\bar{r}_{ij})$, then $\bar{R} = (\bar{r}_{ij})_{\bar{R}}$ can be converted to a corresponding incomplete IFPR $R = (\bar{r}_{ij})_{\bar{R}}$ by Proposition 5), with only $n - 1$ judgments.

**Step 2.** Utilizing Proposition 6 or Proposition 7 to determine all unknown elements in $R$, and yield an interval additive consistent IFPR $\bar{R} = (\bar{r}_{ij})_{\bar{R}}$. If this preference relation contains any values falling outside the unit interval $[0, 1]$, but within the interval $[-1, 1 + \epsilon]$, then a transformation function $f(x) = \frac{x + 1}{2}$ can be applied to preserve the reciprocity and additive transitivity, resulting in an additive consistent IFPR.

**Step 3. End.**

**Example 1.** Assume that a decision problem involves evaluating seven faculties $x_i$ ($i = 1, 2, \ldots, 7$) at a university. The DM assesses these seven faculties (alternatives) by pairwise comparison and provides his/her judgment as follows:

$R_{31} = [0.4, 0.7]$, $R_{32} = [0.1, 0.3]$, $R_{34} = [0.3, 0.4]$, $R_{35} = [0.4, 0.7]$, $R_{36} = [0.3, 0.8]$, $R_{37} = [0.4, 0.9]$

**Step 1.** By Definition 8 and the aforesaid information provided by the DM, one obtains the following acceptable incomplete IFPR, where “*” denotes the unknown judgment.

$R = \begin{bmatrix}
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
\end{bmatrix}$

**Step 2.** Utilize Propositions 6 and 7 to determine all missing elements in $R$ as follows:

$R_{11} = 1 - r_{11} - 0.3$, $R_{12} = 1 - r_{12} - 0.6$, $R_{13} = 1 - r_{13} - 0.7$, $R_{14} = 1 - r_{14} - 0.9$

$R_{21} = 1 - r_{21} - 0.3$, $R_{22} = 1 - r_{22} - 0.6$, $R_{23} = 1 - r_{23} - 0.7$, $R_{24} = 1 - r_{24} - 0.9$

Thus,

$R = \begin{bmatrix}
0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 \\
\end{bmatrix}$

As the preference relation contains values falling outside the interval $[0, 1]$ and $c = 0.3$, a transformation function $f(x) = \frac{x + 1}{2}$ is applied to the lower and upper bounds of each interval value in $R$, yielding

$R = \begin{bmatrix}
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
0.25 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\
\end{bmatrix}$

Alonso et al. (2008) proposed a different procedure to estimate missing in an incomplete IFPR (see Appendix). For a comparison with our approach, their procedure is employed to determine the missing judgments in this example.
Step 1. In RL and RR, “x” denotes an unknown value, and by Eq. (35), EV of the known values are determined as:

$$\text{EV} = \{(3, 1), (3, 2), (3, 4), (3, 5), (3, 6), (3, 7)\}$$

Step 2. By applying Eqs. (36)–(46), one has:

$$H^{(1)}_{i, j} = \phi, \quad \Rightarrow c_{i, j}^{(1)} = 0, \quad c_{i, j}^{(1)} = 0$$

by the DM and estimated values during earlier iterations. In contrast, our method can estimate missing values in one step and all the missing values are estimated based on the preference values furnished by the DM. Thirdly, in the final results, we get $$rr_{22} = 1$$ and $$rr_{22} = 0$$ as this is based on Eqs. (47) and (48). This constraint makes the final estimated matrix RR do not have the additive consistent property. Finally, the derived R by Alonso et al. (2008)’s method is no longer an IFPR, because it does not satisfy the condition (a), (b), of Proposition 5, or Proposition 7, while our estimated matrix R is an additive consistent IFPR.

4.3. An estimation procedure for acceptable incomplete IFPR with more known judgments

Next, we consider an acceptable incomplete IFRP $\tilde{R} = (r_{ij})$ with more known elements (on top of the minimum n − 1 values). In this case, by Proposition 6, each missing element $r_{i,j-h}$ in $R = (r_{ij})$ can be estimated. First, find a sequence of values $r_{i,j}, r_{j,j+1}, \ldots, r_{i,j}$ that include one and only one unknown element $r_{i,j-h}$. If $j_{h-1} < j$, $r_{i,j-h}$ is located in the middle of the sequence, and this missing element can be estimated as $r_{i,j-h} = \frac{1}{2} \left( r_{i,j} + r_{j,j+1} + \cdots + r_{i,j-h+1} + r_{i,j-h} \right)$ by Eqs. (23) and (24), where

$$r_{i,j-h} = \frac{1}{\#h \leq j_{h-1} < j} \left( \frac{t+1}{2} - (r_{i,j} + r_{j,j+1} + \cdots + r_{i,j-h+1} + r_{i,j-h}) \right)$$

and $\Omega$ is the set of all the known elements in $R$. $\#H$ is the number of eligible sequence $r_{i,j}, r_{j,j+1}, \ldots, r_{i,j}$. $r_{i,j-h}$ can be automatically obtained by Eq. (22).

$$\tilde{r}_{i,j-h} \in [1, 1] \cap \tilde{r}_{i,j-h}, \quad \tilde{r}_{i,j} = [0.5, 0.5]$$

Example 2. Consider an acceptable incomplete IFPR.

$$R = \begin{bmatrix}
0.5\,0.03 & 0.3\,0.04 & 0.3\,0.04 & 0.03 & 0.3\,0.07 \\
0.5\,0.07 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
\end{bmatrix}$$

In order to estimate the missing value $r_{24}$, the only sequence that contains one and only one unknown element $r_{24}$ is $\{r_{12}, r_{24}, r_{41}\}$ by Eq. (25), we have

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.3 - 1 = 0.2$$

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.5 - 0.7 = 0.3$$

For $r_{24}$, there exist two eligible sequences $\{r_{12}, r_{23}, r_{34}, r_{41}\}, \{r_{13}, r_{24}, r_{41}\}$ that include $r_{24},$ by Eq. (25), we have

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.3 - 1 = 0.2$$

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.5 - 0.7 = 0.3$$

Example 2. Consider an acceptable incomplete IFPR.

$$R = \begin{bmatrix}
0.5\,0.05 & 0.3\,0.05 & 0.3\,0.04 & 0.03 & 0.3\,0.07 \\
0.5\,0.07 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 & 0.5\,0.05 \\
\end{bmatrix}$$

In order to estimate the missing value $r_{24}$, the only sequence that contains one and only one unknown element $r_{24}$ is $\{r_{12}, r_{24}, r_{41}\}$ by Eq. (25), we have

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.3 - 1 = 0.2$$

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.5 - 0.7 = 0.3$$

For $r_{24}$, there exist two eligible sequences $\{r_{12}, r_{23}, r_{34}, r_{41}\}, \{r_{13}, r_{24}, r_{41}\}$ that include $r_{24},$ by Eq. (25), we have

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.3 - 1 = 0.2$$

$$\tilde{r}_{24} = 1.5 - r_{12} - r_{41} = 1.5 - 0.5 - 0.7 = 0.3$$

Example 2. Consider an acceptable incomplete IFPR.
\[ L_{34} = \frac{1}{2}((2 - r_{12} - r_{23} - r_{41}) + (1.5 - r_{13} - r_{41})) = \frac{1}{2}((2 - 0.3 - 0.5 - 1) + (1.5 - 0.3 - 1)) = 0.2 \]
\[ L_{34} = \frac{1}{2}((2 - r_{12} - r_{23} - r_{41}) + (1.5 - r_{13} - r_{41})) = \frac{1}{2}((2 - 0.5 - 0.6 - 0.7) + (1.5 - 0.4 - 0.7)) = 0.3 \]

For \( L_{34} \), there exists a unique eligible sequence \( \{r_{14}, r_{45}, r_{51}\} \) containing \( r_{45} \), then we have
\[ r_{45} = 1.5 - r_{14} - r_{51} = 1.5 - 0.7 = 0.8 \]
\[ r_{51} = 1.5 - r_{14} - r_{51} = 1.5 - 0.3 = 0.9 \]

Thus, we have
\[ r_{24} = 0.2, 0.3, \quad r_{34} = 0.2, 0.3, \quad r_{45} = 0.8, 0.9 \]
\[ r_{42} = 0.7, 0.8, \quad r_{43} = 0.7, 0.8, \quad r_{54} = 0.1, 0.2 \]

Based on the aforesaid calculations, a complete additive consistent IFPR is constructed as follows:
\[
\hat{R} = \begin{bmatrix}
0.5, 0.5 & 0.3, 0.5 & 0.3, 0.4 & 0.3 & 0.3, 0.7 \\
0.5, 0.7 & 0.5, 0.5 & 0.5, 0.6 & 0.2, 0.3 & 0.3, 0.7 \\
0.4, 0.7 & 0.4, 0.5 & 0.5, 0.5 & 0.2, 0.3 & 0.3, 0.7 \\
0.7, 1 & 0.7, 0.8 & 0.7, 0.8 & 0.5, 0.5 & 0.8, 0.9 \\
0.3, 0.7 & 0.3, 0.6 & 0.4, 0.6 & 0.1, 0.2 & 0.5, 0.5
\end{bmatrix}
\]

In real world, however, many decision making processes take place in multi-person settings because the increasing complexity and uncertainty of the socio-economic environment makes it less and less possible for single decision maker to consider all relevant aspects of a decision making problem. Next, Algorithm 2 is applied to a group decision setting with incomplete IFPR.

**Algorithm 2.** For a group decision making problem, let \( X = \{x_1, x_2, \ldots, x_n\} \) be a discrete set of alternatives, \( E = \{e_1, e_2, \ldots, e_m\} \) be a group of experts, \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) be the weight vector for the DMs, where \( \lambda_i \geq 0, i = 1, 2, \ldots, m \), \( \sum_{i=1}^{m} \lambda_i = 1 \). The decision procedure with acceptable incomplete IFPR is described as follows:

**Step 1.** Each expert \( e_i (i = 1, 2, \ldots, m) \) compares each pair of alternatives and furnishes his/her assessments as an acceptable incomplete IFPR \( R^{(i)} = (\tilde{r}^{(i)}_{ij})_{n \times n} \).

**Step 2.** Estimate missing elements in \( R^{(i)} = (\tilde{r}^{(i)}_{ij})_{n \times n} \) using the known elements through Eq. (25), and obtain complete IFPRs \( \tilde{R}^{(i)} = (\tilde{r}^{(i)}_{ij})_{n \times n} \). If any resulting preference relation contains values falling outside the interval \([0,1]\), and within the interval \([-c, c+1]\), a transformation function \( f(x) = \frac{x+c}{2} \) is applied to the upper and lower bounds for the interval values to convert the preference relate into an IFPR.

**Step 3.** Use the interval fuzzy weighted averaging operator Eq. (21) to aggregate all individual complete IFPR \( \tilde{R}^{(i)} = (\tilde{r}^{(i)}_{ij})_{n \times n} \) (\( i = 1, 2, \ldots, m \)) into a collective IFPR \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} \).

**Step 4.** Utilize the interval normalizing rank aggregation method
\[
w_i = \frac{\sum_{j=1}^{n} g_{ij}}{\sum_{i=1}^{n} g_{ij}}, \quad i = 1, 2, \ldots, n
\]

to derive average degree \( w_i \) of the \( i \)th alternative over all the other alternatives.

**Step 5.** Utilize the following formula (Facchinetti, Ricci, & Muzzioli, 1998; Wang et al., 2005; Xu & Da, 2002)
\[
p(\tilde{w}_i \geq \tilde{w}_j) = \min \left\{ \max \left( \frac{w^+_i - w^-_j}{w^+_j + w^-_i - w^-_j}, 1 \right) \right\}
\]
to obtain the possibility degree \( p_i = p(\tilde{w}_i \geq \tilde{w}_j) \), and construct a complementary matrix \( P = (p_{ij})_{n \times n} \) where
\[
p_{ij} > 0, \quad p_{ij} + p_{ji} = 1, \quad p_{ii} = \frac{1}{2}, \quad i, j = 1, 2, \ldots, n
\]

**Step 6.** Utilize the normalizing rank aggregation method (Xu et al., 2009)
\[
\sigma_i = \frac{\sum_{j=1}^{n} p_{ij}}{n^2/2}, \quad i = 1, 2, \ldots, n
\]
to derive a priority vector \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^T \) based on the complementary matrix \( P \). The alternatives \( x_i (i = 1, 2, \ldots, n) \) are ranked in a descending order as per the values of \( \sigma_i (i = 1, 2, \ldots, n) \).

**Step 7.** End.

**5. Case study**

This section presents a group decision making problem that is concerned with evaluating and selecting potential suppliers for the Pars Solar Company (adapted from (Hadivi-Marghab & Mirjafari, 2011)). The firm produces solar boiler and solar water refiners. In its production process, the company needs to purchase solar panels in different sizes and voltages from different suppliers. Currently, Pars Solar Company has five potential suppliers from five different countries, namely, the U.S., Germany, China, Turkey, and Iran, denoted as \( x_i (i = 1, 2, \ldots, 5) \), respectively.

A committee consisting of three managers \( e_i (i = 1, 2, 3) \) (whose weighting vector is \( \lambda = (1/3, 1/3, 1/3)^T \)) from different departments have been set up to provide their assessment on the five suppliers \( x_i (i = 1, 2, \ldots, 5) \). Assume that the managers \( e_i (i = 1, 2, 3) \) give their assessments as the following acceptable incomplete IFPRs
\[
R^{(i)} = (\tilde{r}^{(i)}_{ij})_{n \times n} (i = 1, 2, 3):
\]

**Step 1.** Utilizing the known elements and Eqs. (23) and (24), we have:
\[
it_{ij} = \frac{1}{2}(2.5 - r_{ij}^2 - r_{ij}^2 - r_{ij}^2) + \frac{1}{2}(2 - r_{ij}^2 - r_{ij}^2 - r_{ij}^2) + \frac{1}{2}(1.5 - r_{ij}^2 - r_{ij}^2) = 0.1
\]

**Step 2.** To illustrate the solution process of Algorithm 2, the missing elements are first estimated for \( R^{(i)} = (\tilde{r}^{(i)}_{ij})_{n \times n} (i = 1, 2, 3) \).
Step 2. Utilize the interval additive weighted averaging operator Eq. (21) to fuse the constructed complete IFPRs $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \mathcal{R}^{(3)}$ into a collective complete IFPR $\mathcal{R} = \left(\hat{r}_{ij}\right)_{n \times n}$ as

$$\mathcal{R} = \begin{bmatrix}
0.5 & 0.217 & 0.389 & 0.284 & 0.311 \\
0.783 & 0.5 & 0.685 & 0.572 & 0.581 \\
0.611 & 0.315 & 0.5 & 0.387 & 0.409 \\
0.716 & 0.428 & 0.613 & 0.5 & 0.514 \\
0.689 & 0.419 & 0.591 & 0.486 & 0.5
\end{bmatrix}$$

Step 3. By employing the interval normalizing rank aggregation method Eq. (31), one can derive the average degree $w_i$ for each alternative as

$$w_1 = [0.1009, 0.2350], \quad w_2 = [0.1769, 0.3105],$$

$$w_3 = [0.1153, 0.2576], \quad w_4 = [0.1604, 0.2893],$$

$$w_5 = [0.1475, 0.2946]$$

Step 4. By using Eq. (29) to compare each $w_i$ with each $w_j (j = 1, 2, \ldots, n)$, we develop a complementary matrix $P = (p_{ij})_{n \times n}$ as

$$P = \begin{bmatrix}
0.5 & 0.217 & 0.389 & 0.284 & 0.311 \\
0.783 & 0.5 & 0.685 & 0.572 & 0.581 \\
0.611 & 0.315 & 0.5 & 0.387 & 0.409 \\
0.716 & 0.428 & 0.613 & 0.5 & 0.514 \\
0.689 & 0.419 & 0.591 & 0.486 & 0.5
\end{bmatrix}$$

Step 5. By applying the normalizing rank aggregation method (Xu et al., 2009), one has

$$\sigma = (0.136, 0.250, 0.178, 0.222, 0.215)^T$$

A final ranking of the alternatives can be thus derived as follows:

$$x_2 > x_3 > x_4 > x_3 > x_1$$

Therefore, the best alternative is $x_2$.

We can also compare the final result with what is implied in the initial incomplete input. From $\mathcal{R}^{(1)}$, it can be determined that the partial ranking order by the first manager $e_1$ is $x_2 > x_3 > x_2 > x_4$, the partial ranking order by the third manager $e_3$ is $x_2 > x_3$, $x_4 > x_5$. Based on these partial orderings, it is sensible to expect that $x_2$ arise as the best alternative, consistent with the prediction by our model here.

Numerical examples, illustrated how to apply the proposed method to construct the complete IFPRs based on acceptable incomplete IFPRs. Generally speaking, the proposed approach is relatively easy for use in determining missing values. More importantly, Herrera-Viedma et al. (2004) proposed a method to construct a consistent fuzzy preference from $n - 1$ preference values $[r_{12}, r_{23}, \ldots, r_{n-1n}]$. Sometimes, this may be a challenge for the DM to provide his/her consecutive pairwise judgments $[r_{12}, r_{23}, \ldots, r_{n-1n}]$. Based on Proposition 10, if $\mathcal{R}$ is an acceptable incomplete IFPR, then there exists at least one known non-diagonal element in each row or each column of $\mathcal{R}$. Therefore, an acceptable incomplete IFPR with at least $(n - 1)$ judgments provided by the DM. $[r_{12}, r_{23}, \ldots, r_{n-1n}]$ is only one example that makes up an acceptable incomplete IFPR. The proposed approach herein is able to handle any acceptable incomplete IFPR with $n - 1$ judgment data such as $[r_{12}, r_{23}, \ldots, r_{n-1n}]$, $[r_{21}, r_{23}, \ldots, r_{n-1n}]$, $[r_{21}, r_{32}, \ldots, r_{n-1n}]$ if the DM can provide more judgments, one can use the sequence $(s)$ of known values $r_{12}, r_{13}, \ldots, r_{n-1n}$ to estimate the missing values. Therefore, our proposed approach can be applied to more general IFPRs.
6. Conclusions

This paper first extends an additive consistency property (Herrera-Viedma et al., 2004) of the fuzzy preference relations to a general case. This property is then generalized to the IFPR based on additive transitivity. After further characterizing additive consistent IFPRs, we propose a method to construct additive consistent IFPRs from acceptable incomplete IFPRs. Numerical studies illustrate that the proposed method can handle acceptable incomplete IFPRs with as few as n − 1 judgment data. Our first illustrative example involves seven alternatives requiring six comparison judgments. Therefore, the number of comparison can be reduced by 4C2 − 6 = 15 times. Subsequently, another algorithm is developed to deal with GDM problems with acceptable incomplete IFPRs with more known judgments. This procedure is composed of two phases, the estimation of missing preference values and the selection of the best alternative (s) where missing elements are determined by the sequences of known values r_{i1}, r_{i2}, · · · , r_{i(k−1)}, r_{ijk}.

In the future, the proposed framework will be applied to other group decision making problems such as supplier selection (Chen, Lin, & Huang, 2006), e-business (Mohanty & Bhasker, 2005), technology adoption (Choudhury, Shankar, & Tiwari, 2006), broadband internet service selection (Wang & Parkan, 2008), and air-conditioning systems selection (Xu, 2009).

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Appendix A

Alonso et al. (2008) proposed the following procedure to estimate missing values for IFPRs.

An IFPR $R = (r_{ij})$ can be viewed as two “independent” fuzzy preference relations, the first one $RL$ corresponding to the left bounds of the intervals and the second one $RR$ corresponding to the right bounds of the intervals, respectively.

$$R = (r_{ij}) = ([r_{ij}, r_{ij}]) \text{ with } RL = (r_{ij}) \quad \text{and} \quad RR = (r_{ij}) \quad \text{and} \quad r_{lj} \leq r_{lj} \quad \forall i, j$$

Step 1. Determine the set $B$, the set $MV$ and the set $EV$, as follows:

$$B = \{(i, j)\mid i, j \in \{1, \ldots, n\} \land i \neq j\}$$

$$MV = \{(i, j)\in B\mid r_{ij} \text{ is unknown}\}$$

$$EV = B \setminus MV$$

where $MV$ is the set of pairs of alternatives for which the preference degree of the first alternative over the second one is unknown or missing; $EV$ is the set of pairs of alternatives for which the expert provides preference values, and the symbol “−” denotes the exclusion relation. In Alonso et al. (2008)’s method, it does not take into account the preference value of one alternative over itself as $x_i \sim x_i$ is always assumed.

Step 2. If $MV = \emptyset$, then go to Step 4. Otherwise, the following iterative procedure is used to estimate the missing value $crr_{ik}$ and $crr_{ik}$ in the set $EMV_h$ at the $h$th iteration, where $EMV_0 = \emptyset$, $(i, k) \in EMV_h$, and $h \geq 1$, shown as follows:

$$H_{ik}^{01} = \{(j \neq i, k)\mid (i, j) \in \{EV \cup EMV_h\}\}$$

$$H_{ik}^{02} = \{(j \neq i, k)\mid (j, i) \in \{EV \cup EMV_h\}\}$$

$$H_{ik}^{03} = \{(i \neq j, k)\mid (i, j) \in \{EV \cup EMV_h\}\}$$

$$EMV_h = \{(i, k)\in MV \setminus H_{ik}^{01} \cup H_{ik}^{02} \cup H_{ik}^{03}\} \setminus \{(i, k)\in EV\}$$

$$cr_{ik}^1 = \frac{1}{\pi_{ik}} \sum_{j \neq i, k} crr_{ij} , \quad cr_{ik}^2 = \frac{1}{\pi_{ik}} \sum_{j \neq i, k} crr_{jk}$$

$$cr_{ik}^{\text{in}} = \frac{1}{\pi_{ik}} \sum_{j \neq i, k} crr_{ij}^3 , \quad cr_{ik}^{\text{out}} = \frac{1}{\pi_{ik}} \sum_{j \neq i, k} crr_{jk}^3$$

$$cr_{ik} = \frac{cr_{ik}^1 + cr_{ik}^2 + cr_{ik}^{\text{in}} + cr_{ik}^{\text{out}}}{\kappa}$$

$$\kappa = \left\{ \begin{array}{ll} #H_{ik}^{01} + #H_{ik}^{02} + #H_{ik}^{03} & \text{if } #H_{ik}^{01} + #H_{ik}^{02} + #H_{ik}^{03} \neq 0 \\
1 & \text{otherwise} \end{array} \right.$$


