Interference Alignment: Precoding Subspaces Design

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Abstract—Recently, the capacity region of the Single Input Single Output (SISO) fading interference channel has been characterized. Its degree of freedom is upper bounded by \( K/2 \).

This upperbound can be asymptotically achieved using the Interference Alignment approach (IA), proposed by Cadambe et al.. Our work aims to maximize the network sum rate of the IA scheme through the precoding subspaces design. We propose two designs. One is obtained as a closed-form solution of a constrained optimization problem when a Zero-Forcing decoding scheme is employed. The other is a generalized version of the optimized 3-user beamforming vectors design proposed by Kim et al., where a Minimum Mean Square Error receiver is utilized. This design presents a high computational complexity compared to the analytical solution obtained when assuming a ZF receiver. Simulation results illustrate the data rate performance and highlight the comparison between the different designs.

I. INTRODUCTION

A few years ago, the capacity region of the interference channels was an open problem. It was unknown except for some special cases, among which [1]–[3]. For the two-user channel case, the authors in [3] have shown a new upperbound on the channel capacity. The achievability of this bound has been proven using a new approach of interference management, known as Interference Alignment (IA).

The key idea of IA is to design transmit signals such that interfering signals at each receiver overlap while the desired signal remains distinct from interferences. This approach was exploited by Cadambe-Jafar (CJ) to show that the maximum achievable Degrees of Freedom (DoF) of the \( K \)-user time-varying interference channels, in the \( n \)-dimensional Euclidean space, is \( \frac{n}{2} \) [4]. However, when the channel is not time-varying or frequency selective, the optimal DoF is not achieved. Therefore, a part of researches have been concentrated on the capacity region of the quasi-static interference channels [5], [6].

In [7], the authors have extended the idea of IA from space/time/frequency dimensions to the signal level dimensions. Based on the field of Diophantine approximation in number theory, it has been proven that interference can be aligned in the rational space using the properties of the rational and irrational numbers, thus, the full DoF, \( \frac{K}{2} \), can be achieved. This demonstration holds when the users are synchronous and the received signal at each receiver node is a synchronized linear combination of the transmitted signals. In practice, such an assumption is not realistic. Therefore, in [8] the asynchronous transmission in the \( K \)-user fading interference channels with quasi-static coefficients is considered. It has been shown that the total DoF of this channel is the same as that of the corresponding synchronous channel.

In the IA schemes described above, the full DoF is achieved for large dimensional precoding matrices, and for infinite signal-to-noise ratio (SNR) values. The maximum achievable DoF is equal to

\[
\lim_{\text{snr} \to \infty} \frac{C(\text{snr})}{\log_2(\text{snr})} = \frac{K}{2},
\]

where \( K \) is the total user number in the fading interference channel, and \( C(\text{snr}) \) represents the channel capacity.

The first proposed IA scheme by CJ is only asymptotically optimal [4]. In order to optimize the precoding subspaces that improve the data rate performance, three optimization methods have been proposed [9]–[11]. The first attempts to orthogonalize the desired signal subspaces and the interference subspaces in the 3-user IA scheme at all receiver nodes [9]. The second tries to design the beamforming vectors in order to maximize the network sum-rate. The optimized design is obtained as a solution of a constrained convex problem, and it is numerically computed [10]. The third improved design aims to reduce the computational complexity due to the numerical computation and seeks for an analytical suboptimal solution. These two last solutions are obtained using the Minimum Mean Square Error (MMSE) decoding scheme.

In fact, the IA scheme can be optimized in two steps. The first, in which we are interested in the remaining of this paper, consists in optimizing the precoding subspace at each transmitter. The second consists in optimizing the precoding vectors of each precoding subspace as described in [12], [13]. In this paper we aim to maximize the network rate of the IA scheme through the precoding subspace optimization. The main contributions of this work are

- The generalization of the improved design proposed in [10] for a 3-user IA scheme to a \( K \)-user IA scheme employing an MMSE decoding scheme.
- The demonstration of a closed-form solution of the \( K \)-user IA precoding subspaces design for high SNR values when a ZF receiver is employed. It is shown that this solution also maximizes the individual information rate.
This paper is organized as follows. In Section II, we describe the system model for the $K$-user fading interference channel. Section III generalizes the improvements proposed for a 3-user network to a $K$-user network ($K>3$). A closed-form solution is demonstrated when employing a ZF decoding scheme in Section IV. Numerical results are shown in the $K$-user fading interference channel in Section V. Finally, Section VI concludes the paper.

Notations: boldface upper case letters and boldface lower case letters denote matrices and vectors, respectively. For the transpose, transpose conjugate and conjugate matrix we use $(.)^T$, $(.)^H$ and $(.)^*$, respectively. $|.|$ and $tr(.)$ denote the determinant and the trace of a matrix, respectively.

II. SYSTEM MODEL

Let us consider the $K$-user SISO block fading time-varying interference channel as illustrated in Fig.1, with $K$ transmit receive pairs. A wireless channel links each receiver to each transmitter, but a given transmitter only intends to have its signal decoded by a single receiver. In this work, we are interested in the IA scheme proposed by Choi et al. in [14]. This IA scheme is more efficient than the CJ scheme for $K>3$ and it achieves a higher DoF for a reduced channel use number. The DoF per user is obtained using the following combinations

$$d_1 = \binom{m^*+M+1}{M} \quad \text{and} \quad d_3 = \binom{m^*+M}{M}$$

where $m^*$ is a given nonnegative integer, $M$ is a parameter depending on the user number $M = (K-1)(K-2)-1$, and $d_i$ is the DoF of the $i^{th}$ user. Provided $d_1 = d_3$, $d_1 > d_4$, $i \in K \setminus \{1, 3\}$, the IA can be then satisfied. $K$ represents the set of the user indices $\{1, 2, ..., K\}$. The precoding matrix length, obtained using the channel extensions (channel uses), is given as $N = d_1 + d_2$. The channel is supposed to be frequency selective, thus an Orthogonal Frequency Division Multiplexing (OFDM) transmission technique can be applied. At the $k^{th}$ destination, the channel output is given by

$$y_k = \sum_{j=1}^{K} H_{kj} V_j x_j + z_k, \forall k \in K,$$  

where $H_{kj}$ is the $N \times N$ diagonal channel fading matrix between the $j^{th}$ transmitter and the $k^{th}$ receiver. $V_j$ is the $N \times d_j$ precoding matrix of the $j^{th}$ transmitter. The $j^{th}$ transmitted information $x_j$, is defined as a $d_j \times 1$ vector. $z_k$ is the $N \times 1$ circular symmetric complex Gaussian noise vector at the receiver $k$, with independent and identically distributed (i.i.d.) components; i.e. $z_k \sim \mathcal{CN}(0, I_N)$. According to [14], the precoding matrices can be designed as

$$V_1 = \left\{ \prod_{k,l \in K \setminus \{1, k \neq l, (k,l) \neq (2,3)\}} ((T_{23})^{-1} T_{kl})^{n_{kl}} w \quad \right\}$$

$$\left| \sum_{k,l \in K \setminus \{1, k \neq l, (k,l) \neq (2,3)\}} n_{kl} \leq m^* + 1 \right\}$$

(3)

and

$$V_3 = \left\{ \prod_{k,l \in K \setminus \{1, k \neq l, (k,l) \neq (2,3)\}} ((T_{23})^{-1} T_{kl})^{n_{kl}} w \quad \right\}$$

$$\left| \sum_{k,l \in K \setminus \{1, k \neq l, (k,l) \neq (2,3)\}} n_{kl} \leq m^* \right\},$$

(4)

where $w$ is the $N \times 1$ vector determining the precoding subspace of each user. The $N \times N$ diagonal matrix $T_{kl}$ is expressed as

$$T_{kl} = (H_{kl})^{-1} H_{kl} (H_{kl})^{-1} H_{kl}.$$  

(5)

Improving the data rate requires a judicious selection of $w$. To simplify the problem in upcoming sections, $w$ is factored out from $V_k$ as

$$V_k(w) = W V_k(w = 1_{N \times 1}),$$

(6)

where $W$ is a diagonal matrix with elements $(W)_{ii} = w_i$. Also the precoding matrix $V_k(w = 1_{N \times 1})$ is, henceforth, replaced by $V_k$. In the following sections, the optimization problems seek to find out the vector $w$ that maximizes the network sum rate of the IA scheme described above.

III. OPTIMIZATION OF THE IA PRECODING SUBSPACES ASSUMING AN MMSE DECODER

Assuming an MMSE decoder (as defined in [9]), the authors in [10] have proposed to optimize the performance of the 3-user IA scheme via the precoding subspaces. The obtained design is the solution of the sum rate maximization problem with respect to (w.r.t.) $w$. The concavity of this optimization problem has been proven, thus, the lagrangian based solution is a global maximum.

In this section, we aim to generalize this design to the $K$-user IA scheme. We also propose a projected gradient method to converge to the optimal solution. Let us begin with the sum rate function defined as

$$R = \frac{1}{N} \sum_{k=1}^{K} R_k(w)$$

(7)
where

\[ R_k(\bar{w}) = \log_2 \frac{|I + p \sum_j H_{kj} \bar{W}_j \bar{V}_j^H \bar{W}_j^H |}{|I + p \sum_j H_{kj} \bar{W}_j \bar{V}_j^H \bar{W}_j^H |}. \]

Using Sylvester’s determinant theorem [15], (8) can be expressed in the following compact form

\[ R_k(\bar{w}) = \log_2 \frac{|I + p \mathbf{W} A_k|}{|I + p \mathbf{W} B_k|}, \]

where \( \bar{W} = \mathbf{W} \mathbf{W}^H \) is a diagonal matrix with positive elements and \( p \) is the average power of each transmitted symbol. The matrices \( \mathbf{B}_k \) and \( \mathbf{A}_k \) are defined as

\[ \mathbf{B}_k = \sum_{j=1, j \neq k}^{K} H_{kj} V_j V_j^H H_{kj}^H, \]

\[ \mathbf{A}_k = \mathbf{B}_k + H_{kk} \mathbf{V}_k^H \mathbf{H}_{kk}^H. \]

In this equation, \( \mathbf{B}_k \) is a positive-definite matrix. Therefore, applying the Cholesky decomposition [16] to (10), we obtain \( \mathbf{A}_k \) written in terms of \( \mathbf{B}_k \) as

\[ \mathbf{A}_k = \mathbf{L}_{Ak} \mathbf{L}_{Ak}^H \] \[ \mathbf{B}_k = \mathbf{L}_{Bk} \mathbf{L}_{Bk}^H. \]

Substituting (11) into (8) yields the \( k \)th user rate

\[ R_k(\bar{w}) = \log_2 \frac{|I + p \mathbf{L}_{Ak}^H \bar{W} \mathbf{L}_{Ak}|}{|I + p \mathbf{L}_{Bk}^H \bar{W} \mathbf{L}_{Bk}|}. \]

The new optimization problem that can improve the \( K \)-user IA scheme is obtained as follows

\[ \arg \max_{\bar{w}} \frac{1}{N} \sum_{k=1}^{K} \log_2 \frac{|I + p \mathbf{L}_{Ak}^H \bar{W} \mathbf{L}_{Ak}|}{|I + p \mathbf{L}_{Bk}^H \bar{W} \mathbf{L}_{Bk}|}, \]

subject to

\[ \sum_{k=1}^{K} \text{tr}(\mathbf{V}_k(\bar{w}) \mathbf{V}_k(\bar{w})^H) = KN, \quad \bar{w}_i \geq 0 \quad i \in \{1, ..., N\}. \]

The concavity of the cost function given in (13), w.r.t. \( \bar{w} \), is proven by demonstrating that it is twice differentiable and its Hessian matrix is negative semi-definite [17]. This demonstration is given in [10] (c.f. Appendix B). The global optimum of such a constrained concave problem can be obtained using the projected gradient method with an optimized variable step size.

Applying the projected gradient descent algorithm requires the computation of the gradient w.r.t. \( \bar{w} \). Referring to [18] and using that \( \frac{\partial f(X)}{\partial x_i} = \text{tr}(f^{-1}(X) \frac{\partial X}{\partial x_i}) \), the gradient is obtained as

\[ \frac{\partial R(\bar{w})}{\partial \bar{w}_i} = \sum_{k=1}^{K} (X_{ki} - Y_{ki}), \]

where the matrices \( X_{ki} \) and \( Y_{ki} \) are defined as

\[ X_{ki} = \frac{p}{N} \mathbf{L}_{Ak}^H \mathbf{L}_{Ak}^{-1} \mathbf{H}_{Ak}^H, \]

\[ Y_{ki} = \frac{p}{N} \mathbf{I}_{Bki} \left( I + p \mathbf{L}_{Ak}^H \bar{W} \mathbf{L}_{Ak} \right)^{-1} \mathbf{H}_{Ak}^H. \]

It is important to note that the Cholesky decomposition, originally defined for a positive definite matrix, can be extended to the positive semi-definite case.

\[ l_{Ak_i} \text{ and } l_{Bki} \text{ are the } i \text{th rows of the matrices } \mathbf{L}_{Ak} \text{ and } \mathbf{L}_{Bk}, \text{ respectively. The constraint, defined in (13), can be formulated as } \]

\[ \sum_{k=1}^{K} \text{tr}(\mathbf{V}_k \bar{W}_k \mathbf{V}_k^H) = \sum_{i=1}^{N} c_i \bar{w}_i, \]

with \( c_i \) the \( i \)th component of the vector \( c \), \( c_i = \sum_k ||v_{ki}||^2 \), and \( v_{ki} \) the \( i \)th row of the matrix \( \mathbf{V}_k \).

Equation (16) defines the set of \( \bar{w} \) satisfying the constraint, thus, projecting the gradient on this subspace and updating \( \bar{w} \) at the \((l+1)^{th}\) iteration as

\[ \bar{w}_{l+1} = \bar{w}_l + \mu p(\bar{w}_l), \]

with \( \mu \) a variable step size and \( p(\bar{w}_l) \) is the projected gradient defined as

\[ p(\bar{w}_l) = \nabla_w R(\bar{w}_l) - (\epsilon \nabla_w R(\bar{w}_l))^T \frac{c}{||c||^2}, \]

the convergence to a steady state is obtained as soon as

\[ ||p(\bar{w}_l)|| < \epsilon, \]

with \( \epsilon \) the tolerance factor for stopping the iterations. Choosing \( \epsilon \) small enough ensures that the solution is in a narrow neighborhood of the optimal one. In this algorithm, the step size \( \mu \) is a determining factor to ensure a faster convergence, thus, it must be judiciously selected. In [17], two line search methods are proposed: exact line search and inexact line search methods. In practice, most line searches are inexact, and many methods have been proposed. One is the backtracking line search method, which is used in our scheme. It is very simple to implement and quite effective. On the other hand, the step size is updated at each iteration to satisfy \( \bar{w}_l > 0 \).

### IV. Optimized Design when the Receivers are Zero-Forcing

As described above, the optimal solution of \( w \) can be approached numerically when the MMSE is utilized. However, the use of a numerical solver increases the complexity of the implementation. In this section, an improved design of the precoding subspaces is obtained for high SNR assuming a ZF receiver. This design results from a closed-form solution, \( w \), of the network sum-rate maximization.

Employing a ZF decoding scheme, the \( k \)th user rate is then expressed as

\[ R_k = \log_2 |I + p M_k H_{kk} W_k V_k^H W^H H_{kk}^H M_k^H|, \]

where \( M_k \) is the decoder of the \( k \)th user. Assuming well-conditioned channel matrices and using Sylvester’s determinant theorem [15], the \( k \)th user rate, for high snr, can be approximated by

\[ R_k \approx \log_2 |p H_{kk} W_k V_k^H W^H H_{kk}^H M_k^H M_k|. \]
Our goal is to maximize $\sum_k R_k$ for high SNR values w.r.t. $w$ under the total transmit power constraint. Using the following relationship

$$ C = \arg \max_{\tilde{w}} \sum_{k=1}^K R_k(\tilde{w}), $$

$$ C \equiv \arg \max_{\tilde{w}} |\tilde{W}|^2 \prod_{k=1}^K |H_{kk}V_kV_k^H H_{kk}^H| |M_k^H M_k| $$

(22)

the optimization problem is obtained as follows

$$ \arg \max_{\tilde{w}} |\tilde{W}| $$

subject to

$$ tr(\sum_k \tilde{W}V_kV_k^H) = KN, \quad \tilde{w}_i > 0 $$

(23)

where the total transmit power limitation constraint is considered. It is obvious that this problem is equivalent to maximizing the individual information rate, where each user aims to maximize its own data rate. We notice that the (23) is independent of the channel matrices, and the cost function is a simple determinant of a diagonal matrix, thus, it is concave. Hence, introducing Lagrange multiplier $\lambda$ [17], the convex dual of this problem is formulated as follows

$$ \arg \max_{\tilde{w}} \arg \min_{\lambda} |\tilde{W}| - \lambda \left( tr(\sum_k \tilde{W}V_kV_k^H) - KN \right). $$

(24)

In order to obtain the solution of the above problem, we define the Lagrangian taken from (24) as

$$ L(\tilde{w}, \lambda) = |\tilde{W}| - \lambda \left( \sum_{i=1}^N c_i \tilde{w}_i - KN \right). $$

(25)

where

$$ tr(\sum_k \tilde{W}V_kV_k^H) = \sum_{i=1}^N c_i \tilde{w}_i, \quad \text{and} \quad c_i = \sum_{k=1}^K ||v_{ki}||^2 $$

(26)

with $v_{ki}$ the $i^{th}$ row of the matrix $V_k$. Since the cost function is concave, the Karush Kuhn-Tucker (KKT) conditions are sufficient to determine the global optimum. The KKT conditions of the problem in (24) are given by

$$ \nabla_{\tilde{w}} L(\tilde{w}, \lambda)|_{\tilde{w}=\tilde{w}^*} = 0, \quad \lambda > 0, $$

(27)

$$ \sum_{i=1}^N c_i \tilde{w}_i = KN $$

(28)

By solving (27), the following relationship is obtained

$$ \lambda = \frac{|\tilde{W}|}{c_1 \tilde{w}_1} = \cdots = \frac{|\tilde{W}|}{c_N \tilde{w}_N}. $$

(29)

substituting (29) into (28) we obtain the solution as

$$ \tilde{w}_i = \frac{K}{c_i}, \quad i \in \{1, \ldots, N\}. $$

(30)

The optimal $w$ is then obtained as $w_i = \sqrt{\tilde{w}_i}$.

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V. NUMERICAL RESULTS

In this section, we present the simulated performance of the designs derived in sections III and IV, in the 3-user and 4-user SISO fading interference channel. As demonstrated in [13], the IA scheme described in section II can be improved by optimizing, firstly, the precoding subspaces and then, the precoding vectors of each optimized subspace. Our results focus only on the precoding subspaces improvement. They are based on 1000 channel realizations. Channel coefficients have an i.i.d. circular symmetric complex Gaussian distribution with zero mean and unit variance, and are supposed fully known to all users. In the next figures and interpretations, the following abbreviations are used

- Choi- The IA scheme proposed by Choi et al. [14]
- DND- The generalized version of the 3-user optimized design obtained in [10]
- Closed-form$^2$- The design with closed-form solution obtained in Section IV
- $D_n \log_2(\text{snr})$- The maximum DoF for $n = m^* + 1$

Depending on $n$, the achieved DoF $D_n$ is computed as

$$ D_n = \frac{d_1 + d_2}{\sum_{k=1}^K d_k}, $$

(31)

where $d_k$ is given in (2) (cf. Section II). For the generalized design IA$_{GD}$, the convergence is supposed achieved either when

$$ ||p(\tilde{w}^*)|| < \epsilon, $$

(32)

where $\epsilon$ is the tolerance factor for stopping the iterations, or when a maximum number of iterations is achieved. The step size is chosen using the backtracking line search method to provide faster convergence [17]. In Fig. 2, we illustrate the data rate performance of the two derived designs, IA$_{GD}$ and Closed-form, using an MMSE receiver in the 4-user network.

$^2$This solution is shown to be optimal for a ZF receiver. Nevertheless, in our simulation it is applied when assuming an MMSE; as this last presents a much better performance than a ZF for low-medium SNR values. There is no evidence of the Closed-form solution optimality for MMSE receivers, unless that the ZF performs similarly as an MMSE for very high SNR values from analytical point of view.
In this paper we have generalized to the $K$-user network ($K > 3$) the design proposed in [10], which optimize the precoding subspaces of the IA scheme when an MMSE is assumed at all receiver nodes. This solution is achieved iteratively through a projected gradient algorithm with a variable step size. We have also provided, for high SNR values, another optimized design resulting from a closed-form solution when the ZF receiver is utilized. This closed-form solution shows a simplicity in implementation while the first design (IA$_{CP}$) shows a high complexity level for a slightly better data rate performance. Numerical results have illustrated the data rate performance of the different obtained designs. These designs can be extended easily to IA scheme proposed in the asynchronous quasi-static Gaussian interference channels, proposed in [8].

**VI. CONCLUSION**

and for $N = 7$. As expected, these two designs provide a significant gain over the Choi scheme, where $\mathbf{w}$ was fixed to the unitary vector. At 20dB, the IA$_{GD}$ and Closed-form designs achieve a gain of 2.5 bits/s/Hz and 2.4 bits/s/Hz over the Choi scheme, respectively. On the other hand, compared to Closed-form, the IA$_{GD}$ design achieves a gain of about 0.1 bits/s/Hz at the medium-high SNR region, however, a significant complexity cost appears due to the numerical computation method. Next, the network sum-rate evolution is shown in Fig. 3. It can be noticed that the two proposed optimization designs increase the gain over the Choi scheme as long as $N$ increases. However, because of a no optimized form of the precoding vectors (e.g. [12], [13]), a degradation of the average sum-rate of the proposed designs is followed with the increase of $N$. In Fig.4, the rate of convergence of the projected gradient method is illustrated. As shown, the convergence to the optimal solution takes too many iterations. The slow convergence rate of the gradient projection method is an obvious drawback (needs more than 2000 iterations in our case), and thus current research has been aimed at developing a superlinearly convergent version of the gradient projection algorithm. Many of these algorithms, however, do not have the simplicity and elegance of the gradient projection method [19].

**REFERENCES**


