Channel Capacity Estimation in TDMS-Based MIMO Measurements

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Abstract: Time-division multiplexed switching (TDMS)-based multiple-input multiple-output (MIMO) channel sounders are widely used for wireless channel measurements due to their effective costs. However, measurement noise such as phase noise in the local oscillators as well as additive white Gaussian noise (AWGN) can result in significant errors in channel capacity estimates. This study analyzes the impact of phase noise and AWGN on channel capacity in TDMS-based MIMO measurements, with a channel capacity estimator presented that reduces the impact of noise on both the spatial multiplexing gain and on the power gain. Simulations demonstrate that the estimator consistently obtains the true capacity for various MIMO channel scenarios, even if only a limited number of observations are available.

Key words: channel capacity; multiple-input multiple-output (MIMO) systems; phase noise; spatial multiplexing gain; power gain

Introduction

Multiple-input multiple-output (MIMO) wireless systems have attracted considerable attention for many years due to their potential for very high channel capacities\(^{[1,2]}\). Accurate channel capacity estimates from measured data are essential for successful MIMO system design and deployment. Time-division multiplexed switching (TDMS) of a single radio frequency (RF) chain between transmit and receive antennas is commonly used for MIMO channel sounders since it is cost-effective\(^{[3,4]}\). However, phase noise (PN) in the local oscillators (LOs) of such channel sounders can result in serious overestimates of the channel capacity\(^{[4-6]}\). Therefore, many studies have sought to reduce the impact of PN on the measured channel capacity. The conventional average method\(^{[4]}\) requires a large number of channel observations to give accurate capacity estimates for MIMO channels with low rank (typically for line-of-sight or low scattering scenarios). However, only a few observations are normally available in many measurement applications due to the highly time varying characteristics of the channel. An eigenvalue filtering (EVF) technique\(^{[6]}\) was proposed for reducing the impact of PN on channel capacity without considering the effect of additive white Gaussian noise (AWGN). However, studies\(^{[7-9]}\) have shown that AWGN can also affect channel capacity estimates.

This paper addresses the impact of both PN and AWGN on channel capacity estimates for TDMS-based MIMO measurements. While PN has the dominate effect on channel capacity in the high signal to noise ratio (SNR) case, it can be ignored at low SNR. On the other hand, although the impact of AWGN on channel capacity is critical in the low SNR case, it can be neglected at high SNR. Moreover, neither PN nor AWGN can be ignored in the moderate SNR region. A capacity estimator is presented here to reduce the
impact of PN as well as AWGN on channel capacity estimates.

1 System Model

Consider a TDMS-based MIMO channel sounder with \(N_T\) transmit antennas and \(N_R\) receive antennas as shown in Fig. 1. The channel sounder consists of a single RF chain with LOs at the transmitter (TX) and the receiver (RX). The single RF chain is sequentially switched between different transmit/receive antennas during the measurement of one channel transfer matrix.

![Fig. 1 Architecture of a TDMS-based MIMO channel sounder](image)

According to Eq. (1), the measured MIMO channel matrix can be expressed as

\[
\hat{H} = H \odot \exp(j \Phi) + N_i
\]

where \(N_i\) is the \(N_R \times N_T\) complex AWGN matrix with zero mean and \(\sigma_n^2\) variance.

2 Channel Capacity Estimation

2.1 Channel capacity

Assume the MIMO channel is unknown at the transmitter but fully known at the receiver. The channel capacity (in bits/s \(\cdot\) Hz) can then be written as[1]

\[
C = \log_2 \det \left( I + \frac{P}{N_T} HH^H \right) = \sum_{i=1}^k \log_2 (1 + \rho \lambda_i) \quad (3)
\]

where \(\det(\cdot)\) denotes the determinant, \((\cdot)^H\) denotes the Hermitian operator, \(I\) is the \(N_R \times N_R\) identity matrix, \(\rho = 1/\sigma_n^2\) represents the average receive SNR, \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\) are the eigenvalues of the physical channel correlation matrix \(HH^H/N_T\), and \(k \leq \min(N_T, N_R)\) denotes the rank of the physical channel. The channel capacity in Eq. (3) may be conveniently interpreted as[2] \(k\) spatial parallel channels each having an SNR \(\rho\) and a power gain \(\lambda_i\) in the MIMO channel. The \(k\) spatial parallel channels and the power increase \(\lambda_i\) in each spatial channel are usually referred to as the spatial multiplexing gain and the power gain.

For TDMS-based MIMO measurements, the problem is that one can only estimate the true channel capacity based on \(L\) noisy measured data as in Eq. (2). For \(L = 1\), the measured MIMO channel capacity is given by

\[
C_{\text{meas}} = \sum_{i=1}^{\min(N_T, N_R)} \log_2 (1 + \rho \lambda_i) \quad (4)
\]

where \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\min(N_T, N_R)} > 0\) are the eigenvalues of the measured channel correlation matrix \(\hat{H}H^H/N_T\). Note that the measured MIMO channel has rank \(\min(N_T, N_R)\) with probability one due to the randomness of PN and AWGN[4-6,9], i.e., the measured channel always has a non-decreasing spatial multiplexing gain compared with the physical channel.

2.2 Impact of PN and AWGN on channel capacity

This part investigates the impact of PN as well as AWGN on the measured channel capacity. To clearly see the effect of each individual noise, the channel model without AWGN[4-6]

\[
\hat{H}_i = H \odot \exp(j \Phi)
\]
and the MIMO channel in the absence of PN\textsuperscript{[7-9]}

\[
\hat{H}_i = H + N_i
\]  

(6)

are also considered to compare with the model in Eq. (2). Note that the channel model in Eq. (6) is usually for expensive MIMO channel sounders with transmit/receive antennas connected separately to different parallel RF chains.

Consider an 8×8 MIMO channel with rank one, which corresponds to the well known keyhole channel\textsuperscript{[10,11]}. Figure 2 shows the measured capacities of the different channel models in Eqs. (2), (5), and (6) according to Eq. (4) for different SNRs with \(\sigma_v = 3.5^\circ\). The true channel capacity is shown as well according to Eq. (3). Only \(L = 1\) observations were used with 200 independent runs carried out for each channel model to give a mean result.

![Fig. 2 Measured capacities of different channel models for various SNRs for a rank-one 8×8 MIMO channel with \(\sigma_v = 3.5^\circ\) and \(L = 1\)](https://example.com/fig2.png)

The measured capacity is always more than the true capacity for all cases due to the impact of the noise. Hence, methods are needed to reduce the impact rather than ignoring the noise. In the low SNR case, e.g., \(\text{SNR} < 15 \text{ dB}\), the measured channel capacities obtained using Eqs. (2) and (6) are almost the same, which means that the AWGN has the dominate effect on the channel capacity estimates and PN in Eq. (2) can be ignored with negligible error, i.e., Eq. (2) can be approximated by Eq. (6) for low SNR. The measured channel capacity obtained from Eq. (5) is close to the true capacity in this case, which shows that accurate capacity estimates can be obtained if the AWGN is eliminated. On the other hand, the measured channel capacity calculated using Eq. (2) is close to that calculated using Eq. (5) for high SNR cases, e.g., \(\text{SNR} > 40 \text{ dB}\), which indicates that the error in the channel capacity is mainly caused by the PN and the AWGN in Eq. (2) can be neglected. The small error between the measured channel capacity obtained from Eq. (6) and the true capacity confirms this. The PN and the AWGN have comparable effects on the channel capacity for moderate SNR, e.g., \(20 \text{ dB} < \text{SNR} < 35 \text{ dB}\), and ignoring either of them will lead to excessive errors in the channel capacity estimate.

The TDMS-based channel sounders considered in this paper have both AWGN and PN, so Eq. (2) is the more appropriate channel model. The AWGN should not be neglected in the channel model, especially in the low SNR case when Eq. (2) cannot be approximated by Eq. (5). A well designed channel capacity estimator based on Eq. (5) such as the EVF technique\textsuperscript{[6]}, may fail to give accurate estimates at low SNR due to neglecting the significant impact of the AWGN.

3 General Channel Capacity Estimator

3.1 Errors in spatial multiplexing and power gains

The capacity error of the MIMO channel can be expressed as

\[
e = C_{\text{meas}} - C = \sum_{i=1}^{k} \log_2 \left( \frac{1 + \rho \hat{\lambda}_i}{1 + \rho \hat{\lambda}_i} \right) + \sum_{i=k+1}^{\text{min}(N_T, N_R)} \log_2 \left( 1 + \rho \hat{\lambda}_i \right) \quad (7)
\]

The term \(e_p\) in Eq. (7) can be thought of as the power gain error while \(e_m\) is the spatial multiplexing gain error of the physical MIMO channel. An alternative method for reducing the error, \(e_p\), in the power gain is the classical averaging technique\textsuperscript{[4]}

\[
C_{\text{avg}} = \log_2 \det \left( I + \frac{\rho}{N_T} \left( \frac{1}{L} \sum_{i=1}^{L} \hat{H}_i \right) \right) \quad (8)
\]

However, this method is not effective for reducing \(e_m\) in the spatial multiplexing gain with a small number of observations, especially for low rank MIMO channels. Nevertheless, \(e_m\) can be eliminated by neglecting the smallest \(\text{min}(N_T, N_R) - k\) eigvalues of the measured channel correlation matrix. In practice, the rank \(k\) of the physical channel is unknown, but can be well estimated based on the measured data.
3.2 Rank estimation of physical channel

Various methods can be used to estimate the rank of the physical channel[12-15]. This analysis uses the standard minimum description length (MDL) method[12] due to its simplicity and consistency. For \( L > 1 \), the estimated rank of the physical channel is given by the number \( n \) that minimizes the function

\[
f(n) = (N_R - n) \ln \left( \frac{\sum_{i=1}^{N_R} \lambda_i}{N_R - n} \right) - \ln \left( \prod_{i=1}^{N_R} \lambda_i \right) + \frac{n(2N_R - n)\ln L}{2L}
\]

where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_R} \) are the eigenvalues of the full-rank matrix \((1/L) \sum_{i=1}^{L} (\hat{H} \hat{H}_i^H / N_T)\). If the estimated rank is equal to or larger than the rank of the physical channel, i.e., \( n \geq k \), the error \( e_m \) in the spatial multiplexing gain can be completely or partially eliminated by dropping the smallest \( \min(N_T, N_R) - n \) eigenvalues of the measured channel correlation matrix. However, if \( n < k \), this will filter out the smallest \( k - n \) eigenvalues of the physical channel correlation matrix, which can result in underestimation of the true channel capacity.

Denote the probability of a correct or over rank detection as \( P(n \geq k) \). Figure 3 shows \( P(n \geq k) \) of the MDL method for 8x8 MIMO channels with various ranks for different SNRs and PN variances. Only \( L = 2 \) observations are used with 200 independent runs for each case. As \( P(n \geq k) \) decreases from one to zero as the rank increases, better results are obtained for the higher SNR case. PN degrades the performance of the MDL method, especially at high SNR.

To overcome the drawbacks of low \( P(n \geq k) \) in the high rank case, the capacity estimator can be degenerated into the averaging method in Eq. (8) if the estimated rank \( n \) is relatively high. For SNR = 25 dB and \( \sigma_\varphi = 3.5^\circ \) in Fig. 3 as an example, the physical channel can be treated as a full rank channel if \( n \geq 4 \), where 4 is used as the rank threshold, \( k_0 \). This treatment enhances the capacity estimator robustness (when the MDL method is used) with little performance degradation because it significantly reduces the probability of underestimating the rank of the physical channel, and the spatial multiplexing gain error \( e_m \) is very small in the high rank case. In practice, the proper rank threshold, \( k_{th} \), depends on the number of observations, the receive SNR, and the PN variance, as shown in Fig. 3.

Other methods[13-15] can also be used for rank detection in a similar way but this paper will not focus on choosing the best rank estimation method since this is still a fundamental problem in signal processing that is a subject of ongoing research. The MDL method is used to show the feasibility of accurately estimating the physical channel rank.

3.3 Channel capacity estimator

Step 1 Determine the rank threshold, \( k_{th} \), according to the number of observations, the receive SNR, and the PN variance.

Step 2 Estimate the rank of the physical channel using the MDL method. Denote the estimated rank as \( n \), then

\[
m = \begin{cases} 
  n, & n < k_{th}; \\
  \min(N_T, N_R), & n \geq k_{th} 
\end{cases}
\]

Step 3 Average the observed data as \( \hat{H} = (1/L) \sum_{i=1}^{L} \hat{H}_i \).

Step 4 Calculate the eigenvalues \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{\min(N_T, N_R)} > 0 \) of the matrix \( \hat{H} \hat{H}_i^H / N_T \) and estimate the channel capacity as

\[
C_p = \sum_{i=1}^{\min(N_T, N_R)} \log_2(1 + \rho \hat{\lambda}_i)
\]

4 Simulation Results

The results of the classical averaging estimator, \( C_{avg} \), in Eq. (8), the EVF technique[6], \( C_{EVF} \), and the estimator \( C_p \) in Eq. (11) were compared based on simulations along an 8x8 MIMO channel with \( \sigma_\varphi = 3.5^\circ \) with the mean relative error as the performance measure. All
results were based on 200 independent runs. The mean relative error was defined as
\[ e_r = \frac{\text{E}[(C_a - C)/C]}{\text{E}[(C_a - C)/C]} \times 100\% \]
where subscript $a = \text{avg}$, EVF, p, and $\text{E}[\cdot]$ represents the expectation operator. Figure 4 shows the estimation results of the various capacity estimators for different numbers of observations for rank $k = 1$ and SNR = 10 dB. The rank threshold, $k_{th}$, is not necessary in the test since $P(n \geq 1) = 1$. $C_{\text{EVF}}$ has a large bias in this case since it ignores the important effect of AWGN at low SNR. $C_{\text{avg}}$ converges very slowly to the true capacity due to the large spatial multiplexing gain error $e_m$ for the keyhole channel while $C_p$ gives accurate estimation of the true capacity even for $L = 2$ channel observations.

![Fig. 4 Comparison of various channel capacity estimators for different numbers of observations for a rank-one 8×8 MIMO channel with SNR=10 dB and $\sigma_\sigma=3.5°$](image)

The estimation results for various SNRs are shown in Fig. 5 for the same MIMO channel as in Fig. 4. Only $L = 5$ observations were used. Here, $C_{\text{EVF}}$ accurately estimates the capacity in the high SNR case, but it is seriously biased at low SNR due to the additional impact of the AWGN. $C_{\text{avg}}$ overestimates the true capacity when only 5 samples are provided, with smaller error in the moderate SNR region than in the low or high SNR cases. The $C_p$ consistently obtains the true capacity for all SNRs.

![Fig. 5 Comparison of various channel capacity estimators for different SNRs for a rank-one 8×8 MIMO channel with $L=5$ and $\sigma_\sigma=3.5°$](image)

Figure 6 shows the estimation results of the three capacity estimators for different ranks for SNR = 30 dB. Only $L = 5$ observations were used and the rank threshold, $k_{th}$, was set to 4. The estimation error of $C_{\text{avg}}$ decreases as the rank increases because of the reduced spatial multiplexing gain error $e_m$. Although $C_{\text{EVF}}$ gives accurate capacity estimates in the low rank case, it underestimates the true capacity for channels with high rank because some small eigenvectors of the physical channel correlation matrix are filtered out. The $C_p$ estimator gives accurate estimates of the channel capacity for rank $k < k_{th} = 4$, since the estimated rank $n$ is equal to $k$ with very high probability in this case, while for rank $k \geq k_{th}$, the estimator has almost the same estimation accuracy as $C_{\text{avg}}$ because $C_p$ degenerates into $C_{\text{avg}}$ in this situation. The estimation results of the $C_p$ estimator with correct rank estimation, i.e., let $m$ be equal to the true channel rank in Steps 2 and 4, are also plotted in Fig. 6. The estimation error can be further reduced by completely eliminating the spatial multiplexing gain error $e_m$. However, the accuracy improvement is very small (<5%) because $e_m$ can be neglected in the high rank case. These results agree well with previous analysis and show the effectiveness of the $C_p$ capacity estimator for different MIMO channel scenarios.

![Fig. 6 Comparison of various channel capacity estimators for different ranks for all 8×8 MIMO channel with SNR=30 dB, $L=5$, $\sigma_\sigma=3.5°$, and $k_{th}=4$.](image)
5 Conclusions

This study analyzes the impact of PN as well as AWGN on the channel capacity in TDMS-based MIMO measurements with a capacity estimator presented for such conditions. The estimator obtains the true channel capacity even with a limited number of observations and is suitable for line-of-sight or low scattering scenarios.

References


