Analytical impulse response of a fractional second order filter and its impulse response invariant discretization

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Abstract

In this paper, we derive the impulse response of a fractional second order filter of the form $\frac{s^2 + as + b}{g}$, where $a, b \geq 0$ and $g > 0$. The asymptotic properties of the impulse responses are obtained. Moreover, based on the derived analytical impulse response, we show how to perform the discretization of the above fractional second order filter. Finally, a number of illustrated examples in time and frequency domains are provided as proofs of concepts.

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1. Introduction

Fractional calculus is a mathematical discipline which deals with derivatives and integrals of arbitrary real or complex orders [1–4]. It was proposed more than 300 years ago and the theory was developed mainly in the 19th century. Several books [1–4] provide a good source of references on fractional calculus. However, the applications of fractional calculus are just a recent focus of research. For pioneering works, we cite [5–8].

The fractional order filter can be applied in signal modeling, filter design, controller design and nonlinear system identification [9,10]. The key step toward the application of the fractional order filter is its numerical discretization. The conventional discretization method for fractional order filter is the frequency-domain fitting technique (indirect method). In indirect discretization methods [11], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization [12]. In this paper, the direct discretization method will be used. An effective impulse response invariant discretization method was discussed in [12–15]. The method is a technique for designing discrete-time infinite impulse response (IIR) filters from continuous-time fractional order filters in which the impulse response of the continuous-time fractional order filter is sampled to produce the impulse response of the discrete-time filter. For more discussions of discretization methods, we cite [12–21].

The physical realization of $\frac{s^2 + as + b}{g}$ can be illustrated as the type II fractional Langevin equation describing the fractional oscillator process with two indices [22]. The centered stationary formula discussed in [22] can be partly extended by using the discussions in this paper. For other previous works, we cite [19–21].

In this paper, we first focus on the inverse Laplace transform of $(s^2 + as + b)^{-\gamma}$ by cutting the complex plane and computing the complex integrals. The derived results...
can be easily computed in Matlab and applied to obtain the asymptotic properties of the continuous impulse responses. Moreover, a direct discretization method is used to get the digital impulse responses. The results are compared in both of the time and frequency domains. Lastly, several figures are provided as proof of concepts.

The following part of this paper is organized as: In Section 2, the basic mathematical tools are introduced. In Section 3, the time domain analysis is derived, in which the continuous impulse response and its asymptotic properties are obtained. The time and frequency responses are shown in Section 4. The conclusions and future works are discussed in Section 5.

2. Mathematical preliminaries

2.1. Laplace transform and Z transform

The Laplace transform of a function \( f(t) \), defined for all real numbers \( t \geq 0 \), is the function \( F(s) \), defined by

\[
F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt.
\]

The inverse Laplace transform is given by the following complex integral:

\[
f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}F(s)\,ds,
\]

where \( \sigma \) is a real number so that the contour path of integration is in the region of convergence of \( F(s) \) normally requiring \( \sigma > \text{Re}(s_k) \) for every singularity \( s_k \) of \( F(s) \) and \( j^2 = -1 \) [23].

The single-sided Z transform of a discrete-time signal \( x[n] \) is the function \( X(z) \) defined as

\[
X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.
\]

The inverse Z transform is

\[
x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \int_{|z|=\rho} X(z)z^{n-1}\,dz,
\]

where \( C \) is a counterclockwise closed path encircling the origin and entirely in the region of convergence. The contour \( C \) must encircle all of the poles of \( X(z) \) [24].

A special case of the contour integral occurs when \( C \) is the unit circle which is included in the region of convergence. The inverse Z transform simplifies to the inverse discrete time Fourier transform [24]:

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega.
\]

2.2. Fractional calculus

Fractional calculus plays an important role in modern science [3,7,25–27]. The Riemann–Liouville fractional integral with \( \alpha \in (0, 1) \) is defined as [3]

\[
{\alpha}_t^{\alpha}D_{t^-}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau,
\]

where \( f(t) \) is an arbitrary integrable function, \( \alpha D_t^{-\alpha} \) is the fractional integral of order \( \alpha \) on \([0,t]\), and \( \Gamma(\cdot) \) denotes the Gamma function. For an arbitrary real number \( p \), the Riemann–Liouville fractional derivative is defined as

\[
\alpha_tD_{t^-}^{\alpha}f(t) = \frac{d^{[\alpha]}\Gamma(p+1)}{d^{[\alpha]}t^{p+1}} \{\alpha D_t^{-[\alpha]p} f(t)\},
\]

where \( [\alpha] \) stands for the integer part of \( \alpha \), and \( D \) denotes the Riemann–Liouville fractional operator.

In particular,

\[
\alpha_tD_{t^-}^{\alpha}(t-t_0)^{v} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-p)} (t-t_0)^{\lambda-p},
\]

where \( \lambda, p \) are arbitrary real constants [3].

2.3. Impulse response invariant discretization method

The impulse response invariant discretization method converts analog filter transfer functions to digital filter transfer functions in such a way that their impulse responses are the same (invariant) at the sampling instants. Thus, if \( g(t) \) denotes the impulse-response of an analog (continuous-time) filter, then the digital (discrete-time) filter given by the impulse-invariant method will have impulse response \( g(nT_s) \), where \( T_s \) denotes the sampling period in seconds [28]. The basic technique is the so-called Prony method [18].

2.4. Generalized Mittag-Leffler function

The Mittag-Leffler function (Mittag-Leffler 1903, 1905) [3] is an entire function defined by the series

\[
E_{\rho}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\rho + \mu)},
\]

where \( z \in \mathbb{C} \) and \( \rho \) is an arbitrary positive constant. Moreover, the Mittag-Leffler function in two parameters has the following form:

\[
E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\rho + k\mu + \mu)},
\]

where \( z \in \mathbb{C}, \rho, \mu \) and \( \gamma \) are arbitrary positive constants [3]. When \( \mu = 1 \), \( E_{\rho,1}(z) = E_{\rho}(z) \). Finally, the generalized Mittag-Leffler function is defined as

\[
E_{\rho,\mu}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\rho + k\mu + \gamma)}
\]

where \( z \in \mathbb{C}, \rho, \mu, \gamma \) and \( \gamma \) are arbitrary positive constants, and

\[
\left\{ \begin{array}{l}
\gamma_0 = 1, \\
\gamma_1 = \gamma_0 + 1, \ldots, \gamma_{n-1} = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}
\end{array} \right.
\]

where \( n = 1, 2, 3, \ldots \) is the Pochammers symbol [29]. It can be easily seen that \( E_{\rho,\mu}^{(\gamma)}(z) = E_{\rho,\mu}(z) \) and \( E_{\rho,1}^{(\gamma)}(z) = E_{\rho}(z) \).
The Laplace transform of Generalized Mittag-Leffler function is
\[
\mathcal{L}(t^{\alpha-1}E_{\alpha,\beta}(\lambda t^\beta)) = \frac{s^\beta - \lambda}{(s^\beta + \lambda)\gamma},
\]
where \( \text{Re}(s) > |\lambda|^{1/\beta} \).

**Remark 2.1.** It can be seen that \( \mathcal{L}^{-1}(s^{\alpha}G(s)) = t^{\alpha-1}E_{\alpha,\beta}(\lambda t^\beta) \), where \( s \geq 0, \gamma > 0 \) and \( s_1, s_2 \) denote the two zeros of \( s^\alpha + \lambda s^\beta \) and \( * \) is the convolution on \([0,t]\). Especially, when \( a^2 - 4b = 0 \), \( \mathcal{L}^{-1}(s^{\alpha}G(s)) \) is a special case of generalized Mittag-Leffler function. Therefore, some properties of generalized Mittag-Leffler function can be derived from the discussions of this paper.

**3. Derivation of the analytical impulse response of \((s^\alpha + b)^{-\gamma}\)**

In this section, the inverse Laplace transform of \((s^\alpha + b)^{-\gamma}\) is derived by using the complex integral which can lead to some useful asymptotic properties of \( g(t) \).

Let
\[
G(s) = \frac{1}{(s^\alpha + b)},
\]
where \( s \geq 0, \gamma > 0 \) and \( \mathcal{L}(G(s)) = G(s) \). It can be seen that there are two poles of \( G(s) \), \( s_1 = (-\alpha - \sqrt{\alpha^2 - 4b})/2 \) and \( s_2 = (-\alpha + \sqrt{\alpha^2 - 4b})/2 \). It follows that
\[
G(s) = \frac{1}{(s-s_1)}, \quad \frac{1}{(s-s_2)}.
\]

Let \( c \in [s_1, s_2] \),
\[
\mathcal{L}^{-1}\left\{ \frac{1}{(s-c)^\gamma} \right\} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{st}}{(s-c)^\gamma} \, ds.
\]

When \( \gamma \in \{\gamma | \gamma > 0, \gamma \neq 1, 2, 3, \ldots\} \), we have \( s = c \) and \( s = \infty \) are the two branch points of \( e^{\gamma(s-c)^\gamma} \). It follows that (11) is equivalent to the complex path integral shown in Fig. 1, a curve (Handel path) which starts from \(-\infty\) along the lower side of \( s = \text{Im}(c) \), encircles the circular disc \([s-c] = \epsilon \to 0\), in the positive sense and ends at \(-\infty\) along the upper side of \( s = \text{Im}(c) \).

Along path 1, let \( s-c = xe^{-i\pi} \), where \( x \in (0, \infty) \), we have
\[
\int_{1}^{\infty} \frac{e^{st} \, ds}{(s-c)^\gamma} = \int_{0}^{\infty} \frac{e^{x(1-\gamma)t^\gamma} \, dx}{x^\gamma} = \Gamma(1-\gamma)\frac{t^{\gamma-1}e^{-\gamma t}}{\gamma}.
\]

Moreover, along path 3, let \( s-c = xe^{i\pi} \), where \( x \in (0, \infty) \), we obtain
\[
\int_{3}^{\infty} \frac{e^{st} \, ds}{(s-c)^\gamma} = -\Gamma(1-\gamma)\frac{t^{\gamma-1}e^{-\gamma t}}{\gamma}.
\]

Finally, it follows from
\[
\int_{2}^{\infty} \frac{e^{st} \, ds}{(s-c)^\gamma} = \lim_{\epsilon \to 0} \int_{-\infty}^{\epsilon} \frac{e^{x(1-\gamma)t^\gamma} \, dx}{x^\gamma} = 0.
\]

That
\[
g(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{st}}{(s-c)^\gamma} \, ds = \frac{\sin(\gamma \pi t)}{\pi} \Gamma(1-\gamma)\frac{t^{\gamma-1}e^{-\gamma t}}{\Gamma(\gamma)}.
\]

Based on the above discussion, we arrive at the following theorem.

**Theorem 3.1.** Suppose \( \gamma > 0 \) and the complex number \( c \) satisfying \( \text{Re}(c) \leq 0 \). We have
\[
\mathcal{L}^{-1}\left\{ \frac{1}{(s-c)^\gamma} \right\} = \frac{t^{\gamma-1}e^{-\gamma t}}{\Gamma(\gamma)}
\]
and
\[
\left| \mathcal{L}^{-1}\left\{ \frac{1}{(s-c)^\gamma} \right\} \right| \leq \frac{t^{\gamma-1}}{\Gamma(\gamma)},
\]
where \( t \geq 0 \) and \( |.| \) denotes the absolute value of .

**Proof.** Eq. (15) can be derived by using (14) and the frequency shifting property of Laplace transform. Moreover, it follows from \( \text{Re}(c) \leq 0 \) that
\[
\left| \mathcal{L}^{-1}\left\{ \frac{1}{(s-c)^\gamma} \right\} \right| \leq \frac{t^{\gamma-1}}{\Gamma(\gamma)},
\]
where \( t \geq 0 \). \( \square \)

**Corollary 3.2.** In (10), when \( a^2 - 4b = 0 \) and \( c = -a/2 \), we have
\[
g(t) = \frac{t^{2\gamma-1}e^{-at/2}}{\Gamma(2\gamma)},
\]
where \( \gamma > 0 \).

**Proof.** This conclusion can be proved by Theorem 3.1. \( \square \)

**Corollary 3.3.** In (10), when \( a^2 - 4b > 0 \), we have
\[
g(t) = \frac{e^{\alpha t}}{\Gamma(\gamma)} \int_{0}^{t} \frac{t^{\gamma-1}e^{\alpha s}}{\Gamma(\gamma)} \, ds \leq \frac{t^{2\gamma-1}e^{\alpha t}}{\Gamma(2\gamma)},
\]
where \( \gamma > 0 \), \( t \geq 0 \), \( s_1 = (-a - \sqrt{a^2 - 4b})/2 \), \( s_2 = (-a + \sqrt{a^2 - 4b})/2 \leq 0 \) and \( D \) denotes the Riemann–Liouville fractional operator.

**Proof.** It follows from Theorem 3.1 that
\[
g(t) = \frac{e^{\alpha t}}{\Gamma(\gamma)} \int_{0}^{t} (t-s)^{\gamma-1} e^{\alpha s} \, ds = \frac{e^{\alpha t}}{\Gamma(\gamma)} \int_{0}^{t} \left[ \frac{e^{\alpha s}}{\Gamma(\gamma)} \right] \, ds \leq \frac{t^{2\gamma-1}e^{\alpha t}}{\Gamma(2\gamma)},
\]
where \( s_1 \leq s_2 \leq 0 \) and \( t \geq 0 \). \( \square \)
Theorem 3.5. \textit{Corollaries 3.2–3.4.} \\

Proof. It follows from the same proof in Corollary 3.3 that 
\[
g(t) = e^{st}t_\gamma^g t^{-1} g(s(t^{s+as+b})^{-\gamma}}/ (2g) \]  

Applying $| \cdot |$ to the above equation yields 
\[
|g(t)| \leq |e^{st}t_\gamma^g t^{-1} g(s(t^{s+as+b})^{-\gamma}}/ (2g) | 
\]
where $t \geq 0$ and $| \cdot |$ denotes the norm of complex number $\cdot$. \hfill $\Box$

Theorem 3.5. \textit{Let $G(s) = (s^2 + as + b)^{-\gamma}$, where $a, b \geq 0$ and $\gamma > 0$, we have} 
\[
|g(t)| \leq \frac{t^{2g-1}}{(2g)^{\gamma}}, 
\]
where $t \geq 0$ and $g(t) = L^{-1}(G(s))$. 

Proof. This conclusion can be proved by using Theorem 3.1 and Corollaries 3.2–3.4. \hfill $\Box$

The impulse response $g(t)$ obtained in this section is associated with the impulse response invariant discretization method to be used in the following.

4. Impulse response invariant discretization of 

Based on the obtained analytical impulse response function $g(t)$, given sampling period $T_s$, it is straightforward to perform the inverse response invariant discretization of $(s^2 + as + b)^{-\gamma}$ by using the Prony technique [18,30,31] which is an algorithm for finding an IIR filter with a prescribed time domain impulse response. It has applications in filter design, exponential signal modeling, and system identification (parametric modeling) [30,31].

The plots of $g(t)$ for different $a, b$ and $\gamma$ are shown in Figs. 2–5. Specifically, when $a=2$ and $b=1$, it can be verified that $a^2 - 4b > 0$. The plots for different $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$ are shown in Fig. 2. When $a=3$ and $b=2$, it can be verified that $a^2 - 4b > 0$. The plots for different $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$ are shown in Fig. 3. When $a=1$ and $b=1$, it can be verified that $a^2 - 4b < 0$. The plots for different $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$ are shown in Fig. 4. It can be seen that the appearances of complex poles lead to the oscillations of $g(t)$. When $a=0$ and $b=1$, it can be verified that $a^2 - 4b < 0$. The plots for different $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$ are shown in Fig. 5.

Remark 4.1. Recall Corollary 3.3 and compare Fig. 5 with Fig. 4, we can see that the decreasing speed of $g(t)$ in Fig. 5 is much slower than it in Fig. 4. Moreover, when $a=0$, we have $g(t)$ is equivalent to a special case of generalized Mittag-Leffler function $t^{2g-1}E_{1,2}(-t^2)$. The plots of this generalized Mittag-Leffler function for different $\gamma$ are also shown in Fig. 5 which coincide with $g(t)$.

Remark 4.2. It follows from the Laplace initial value theorem that $g(0)=0$, $g(0)=+\infty$ and $g(0)=1$ are corresponding to $\gamma \in \{\frac{1}{2}, +\infty\}$, $\gamma \in \{0, \frac{1}{2}\}$ and $\gamma = \frac{1}{2}$, respectively.

Remark 4.3. The centered stationary formula and the equation (2.8) discussed in [22], where $x = \frac{1}{2}$, are special cases of this paper.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{The plots of $g(t)$, where $a=2, b=1$ and $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$.}
\end{figure}
Fig. 3. The plots of $g(t)$, where $a=3$, $b=2$ and $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$.

Fig. 4. The plots of $g(t)$, where $a=1$, $b=1$ and $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$.

Fig. 5. The comparisons of $g(t)$ and generalized Mittag-Leffler function, where $a=0$, $b=1$ and $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$.
obtain the impulse response invariant discretized version of \( G(s) \) via the well-known Prony technique \([31–35]\). In other words, the discretization impulse response can be obtained by using the continuous time impulse response as follows:

\[
g(n) = T_s g(n T_s),
\]

where \( n = 0, 1, 2, \ldots \) and \( T_s \) is the sampling period.

Figs. 6–9 show the magnitude and phase of the frequency response of the approximate discrete-time IIR filters and the continuous-time fractional order filters under four different cases, where \( \gamma \) satisfies the convergent condition \( \lim_{s \to 0} s(s^2 + as + b)^{-\gamma} = 0 \). The approximate discrete-time IIR filters can accurately portray time domain characteristic of continuous-time fractional order for any \( a, b \) and \( \gamma \). For frequency responses, the impulse response invariant discretization method works well under all the four cases for the band-limited continuous-time fractional order filters. Note here, in Fig. 9, the two curves on \( \omega \geq 10^0 \), where \( s = i\omega \) and

\[
\text{true mag. Bode} \quad \text{continuous time}
\]

\[
\text{approximated mag. Bode}
\]

\[
\text{true phase Bode} \quad \text{continuous time}
\]

\[
\text{approximated phase Bode}
\]

Fig. 6. Frequency responses for \( a=2, b=1 \) and \( \gamma = 0.8 \).

Fig. 7. Frequency responses for \( a=3, b=2 \) and \( \gamma = 0.8 \).
\(i = \sqrt{-1}\), are very different. Because, when \(a=0\) and \(b=1\), the two poles of \(1/(s^2 + 1)^\gamma\) are on the imaginary axis. In other words, the red line in Fig. 9 is not accurate for large \(\omega\) due to the direct computations of real and imaginary parts of \(1/(s^2 + 1)^\gamma\), where \(s = i\omega\). Overall, the impulse response invariant discretization method can accurately describe the fractional order filter \((s^2 + as + b)^{-\gamma}\) \([36]\).

Using the approximate discrete-time IIR filters we can make full use of the discussed fractional order filter. Moreover, in Figs. 6–9, \(\gamma = 0.8\). Table 1 lists the values of \(a_i\) for \(i = 1, 2, \ldots, 6\).

![Fig. 8. Frequency responses for \(a=1, b=1\) and \(\gamma = 0.8\).](image1)

![Fig. 9. Frequency responses for \(a=0, b=1\) and \(\gamma = 0.8\).](image2)

<table>
<thead>
<tr>
<th>(a_i)</th>
<th>(a=2, b=1)</th>
<th>(a=3, b=2)</th>
<th>(a=1, b=1)</th>
<th>(a=0, b=1)</th>
</tr>
</thead>
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<tr>
<td>(a_1)</td>
<td>0.0006991</td>
<td>0.0006956</td>
<td>0.0007028</td>
<td>0.0007044</td>
</tr>
<tr>
<td>(a_2)</td>
<td>0.002133</td>
<td>0.002155</td>
<td>0.002293</td>
<td>0.002281</td>
</tr>
<tr>
<td>(a_3)</td>
<td>0.002315</td>
<td>0.002374</td>
<td>0.002616</td>
<td>0.002397</td>
</tr>
<tr>
<td>(a_4)</td>
<td>0.0009995</td>
<td>0.001036</td>
<td>0.001084</td>
<td>0.0005399</td>
</tr>
<tr>
<td>(a_5)</td>
<td>9.904e−005</td>
<td>9.815e−005</td>
<td>2.852e−005</td>
<td>0.0005101</td>
</tr>
<tr>
<td>(a_6)</td>
<td>1.985e−005</td>
<td>2.38e−005</td>
<td>8.303e−005</td>
<td>0.0002292</td>
</tr>
</tbody>
</table>
of $G_d(z^{-1})$ is 5, $G_d(z^{-1})$ for different cases are shown below:

$$G_d(z^{-1}) = \frac{a_1 z^5 + a_2 z^4 + a_3 z^3 + a_4 z^2 + a_5 z + a_6}{z^5 + b_1 z^4 + b_2 z^3 + b_3 z^2 + b_4 z + b_5}.$$  \tag{22}

and $a_i$ ($i = 1, 2, \ldots, 6$) and $b_j$ ($i = 1, 2, \ldots, 5$) are shown in Tables 1 and 2.

### Table 2
The values of $b_i$ ($i = 1, 2, \ldots, 5$).

<table>
<thead>
<tr>
<th></th>
<th>$a=2$, $b=1$</th>
<th>$a=3$, $b=2$</th>
<th>$a=1$, $b=1$</th>
<th>$a=0$, $b=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>4.554</td>
<td>4.595</td>
<td>4.807</td>
<td>4.922</td>
</tr>
<tr>
<td>$b_2$</td>
<td>8.267</td>
<td>8.419</td>
<td>9.235</td>
<td>9.69</td>
</tr>
<tr>
<td>$b_3$</td>
<td>7.471</td>
<td>7.686</td>
<td>8.864</td>
<td>9.536</td>
</tr>
<tr>
<td>$b_4$</td>
<td>3.361</td>
<td>3.494</td>
<td>4.251</td>
<td>4.692</td>
</tr>
<tr>
<td>$b_5$</td>
<td>0.6015</td>
<td>0.6327</td>
<td>0.8145</td>
<td>0.9232</td>
</tr>
</tbody>
</table>

Moreover, the discrete and continuous impulse responses are shown in Figs. 10–13. Lastly, the above discussions are also valid for $\gamma \geq 1$.

## 5. Conclusions and future works

In this paper, we first discussed the continuous impulse responses of fractional order filter $G(s) = (s^2 + as + b)^{-\gamma}$ and their asymptotic properties. It was shown that the characters of $g(t)$ were strongly related to the poles of $G(s)$, such as the oscillations happened only for $a^2 - 4b < 0$ and the decaying speed was determined by $a$ and $b$. The Laplace final value and initial value theorems can be used to verify the limit properties of $g(t)$ for $a, b > 0$. Moreover, the impulse response invariant discretization method was used to obtain the discrete time and frequency results which were compared with the corresponding continuous cases.
Our future works include the discussions on the two indices commensurate fractional order filter \((s^2 + a s + b)^{-\gamma}\) and the distributed fractional order systems.

References