Fractional order [proportional derivative] controller for a class of fractional order systems

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1. Introduction

In recent years, fractional calculus has been applied in the modeling and control of various kinds of systems, as is well known and documented in many control theory or in the application literature (Bagley & Calico, 1991; Bagley & Torvik, 1984). Properties between resistance and capacitance are intermediated by a concept of fractance (Meihaute & Crepy, 1983; Nakagawa & Sorimachi, 1992; Oldham & Zoski, 1983; Westerlund, 1994). Especially in bioengineering, many real systems are modeled or fitted by fractional order systems (Caputo, 1969; Chen, Xue, & Dou, 2004; Magin, 2006; Nonnenmacher & Glockle, 1991). The dynamic model which governs the phenomenon of transposition of "fractal robustness" is a non-integer order linear differential equation, and the principle of the CRONE suspension, the synthesis method and the performance are developed for this non-integer order model (Oustaloup, Alain, Sabatier, & Lanusse, 1999).

There exists some tuning methods of fractional order controllers for integer order systems. For example, in Li, Luo, and Chen (in press), a fractional order proportional derivative controller (FO-PD) was proposed for a class of motion control systems, and a fractional order [proportional derivative] (FO-PD) controller was designed for a special type of motion control systems in Luo and Chen (under revision); fractional order PI controller tuning rules for robustness to plant uncertainties were given in Monje, Calderon, Vinagre, Chen, and Feliu (2004). However, in this paper, we focus on certain type of simplified model of FOS which is a generalization of the control systems in Luo, Li, and Chen (revised). A fractional order FO-PD controller is proposed for this class of FOS, and a practical and systematic tuning procedure has been developed for the FO-PD controller synthesis.

In fact, the design of feedback controllers for the fractional order systems (FOS) has been approached from the generalizations of some classical design methods. For example, the fractional order PID controller was proposed in Podlubny (1999) as a generalization of traditional PID controller, and some numerical examples of the fractional order controllers were presented in Zhao, Xue, and Chen (2005). However, due to the complexity of the fractional order systems, these control design techniques available for the ubiquitous fractional order systems suffer from a lack of direct systematic approaches based on the fair comparison with the traditional integer order controllers.

The fairness issue in comparing with other controllers such as the traditional integer order PID (IO-PID) controller and the FO-
PD (Luo et al., revised) controller has been addressed under the same number of design parameters and the same specifications. Fair comparisons of the three controllers (i.e., IO-PID, FO-PD and FO-[PD]) via the simulation tests illustrate that, the IO-PID controller designed may not always be stabilizing to achieve flat-phase specification while both FO-PD and FO-[PD] controllers designed are always stabilizing. Furthermore, the proposed FO-[PD] controller outperforms FO-PD controller (Luo et al., revised) for the class of FOS.

2. The fractional order system, fractional order controllers and control specifications

2.1. The fractional order system considered

The FOS discussed in this paper is based on the fractional calculus model of Membrane Charging (Luo et al., revised; Magin, 2006), which is a generalization of the control systems in Luo et al. (revised) and has the following form:

\[ P(s) = \frac{1}{s^{\alpha_p} + 1}. \]  

(1)

Note that, the plant gain is normalized to 1 without loss of generality since the proportional factor in the transfer function (1) can be incorporated in \( K_p \) of the system controller.

As mentioned in Luo et al. (revised), this generalized fractional capacitor membrane model does play an important role in describing the dielectric behavior of membranes, cells, tissues and a variety of biological material, for example, nerve, muscle, skin and so on (Magin, 2006).

2.2. Three controllers for comparisons

Three controllers are considered for fair comparison. Each has three controller parameters. The first controller is the following traditional integer order PID controller \( C_1(s) \):

\[ C_1(s) = K_p \left( 1 + \frac{K_i}{s} + K_d s \right). \]  

(2)

The second controller is the following FO-PD controller:

\[ C_2(s) = K_p (1 + K_d s^\lambda), \quad \lambda \in (0, 2). \]  

(3)

Clearly, this is a specific form of the most common \( P^\gamma D^\lambda \) controller which involves an integrator of order \( \gamma \) (\( \gamma = 0 \), in this paper) and a differentiator of order \( \lambda \). The third controller is the proposed FO-[PD] controller in this paper defined as

\[ C_3(s) = K_p (1 + K_d s + K_3 s^{\mu}), \quad \mu \in (0, 2). \]  

(4)

2.3. Three specifications for controller design

With the FOS model \( P(s) \) in (1) and the generalized form \( C(s) \) of the three controllers in 2.2, the open-loop transfer function \( G(s) \) has the form below, \( G(s) = C(s)P(s) \).

Here, the same three specifications to be met respectively by each of the above three controllers are given in the following.

2.3.1. Phase margin specification

\[ \text{Arg}[G(j\omega_c)] = \text{Arg}[C(j\omega_c)P(j\omega_c)] = -\pi + \phi_m, \]

where \( \omega_c \) is the gain crossover frequency interested, and \( \phi_m \) is the phase margin required.

2.3.2. Robustness to gain variations

\[ \frac{d(\text{Arg}(C(j\omega)P(j\omega)))}{d\omega} \bigg|_{\omega=\omega_c} = 0, \]

with the condition that the phase derivative w.r.t. the frequency is zero, i.e., the phase Bode plot is flat, at the gain crossover frequency. It means that the system is more robust to gain changes and the overshoots of the response are almost the same.

2.3.3. Gain crossover frequency specification

At the gain crossover frequency point, the amplitude of the open-loop transfer function should be zero, \( |G(j\omega_c)|_{ab} = |C(j\omega_c)P(j\omega_c)|_{ab} = 0 \).

3. Fractional order [PD] controller design

In this section, the FOS model \( P(s) \) described by (1) are considered. The proposed FO-[PD] controller has the transfer function form of (4). The phase and gain of the FOS model in frequency domain can be given as follows,

\[ \text{Arg}[P(j\omega)] = -\tan^{-1} \left( \frac{T_\omega \tan \left( \frac{\alpha \pi}{2} \right)}{1 + T_\omega \tan \left( \frac{\alpha \pi}{2} \right)} - \frac{\pi}{2} \right), \]  

(6)

\[ |P(j\omega)| = \frac{1}{M}, \]  

(7)

where

\[ M = \sqrt{\left( T_\omega \tan \left( \frac{\alpha \pi}{2} \right) \right)^2 + \left( \omega + T_\omega \tan \left( \frac{\alpha \pi}{2} \right) \right)^2}. \]

The FO-[PD] controller (4) can be written as

\[ C_3(j\omega) = K_p (1 + K_3 (s^{\mu})). \]  

(8)

The phase and gain are as follows,

\[ \text{Arg}[C_3(j\omega)] = \mu \tan^{-1}(\omega K_3), \]  

(9)

\[ |C_3(j\omega)| = K_p (1 + (K_3 s^{\mu})^{\frac{\alpha}{2}}). \]  

(10)

The open-loop transfer function \( G_3(s) \) is that

\[ G_3(s) = C_3(s)P(s), \]  

(11)

from (6) and (9), we can get the phase of \( G_3(s) \),

\[ \text{Arg}[G_3(j\omega)] = \mu \tan^{-1}(\omega K_3) \]

\[ - \tan^{-1} \left( \frac{T_\omega \tan \left( \frac{\alpha \pi}{2} \right)}{1 + T_\omega \tan \left( \frac{\alpha \pi}{2} \right)} - \frac{\pi}{2} \right). \]  

(12)

3.1. Numerical computation process

According to Specification 2.3.1, the phase of \( G_3(s) \) can be expressed as

\[ \text{Arg}[G_3(j\omega_c)]_{\omega=\omega_c} = \mu \tan^{-1}(\omega_c K_3) \]

\[- \tan^{-1} \left( \frac{T_\omega \tan \left( \frac{\alpha \pi}{2} \right)}{1 + T_\omega \tan \left( \frac{\alpha \pi}{2} \right)} - \frac{\pi}{2} \right) \]

\[ = -\pi + \phi_m, \]  

(13)

so, we can establish one relationship between \( K_3 \) and \( \mu \).

\[ K_3 = \frac{1}{\omega_c} \tan \left( \frac{1}{\mu} \left( \phi_m - \frac{\pi}{2} + \tan^{-1} \left( \frac{T_\omega \tan \left( \frac{\alpha \pi}{2} \right)}{1 + T_\omega \tan \left( \frac{\alpha \pi}{2} \right)} \right) \right) \). \]  

(14)
According to Specification 2.3.2 about the robustness to gain variations in the plant,
\[
\frac{d(\text{Arg}(G_3(j\omega)))}{d\omega} = \frac{\mu K_{d3}}{1 + (K_{d3}\omega_c)^2} \alpha T \omega_c^{\alpha-1} \sin \frac{\alpha \pi}{2} - \frac{\alpha T \omega_c^{\alpha-1} \sin \frac{\alpha \pi}{2}}{(T \omega_c^{\alpha} \sin \frac{\alpha \pi}{2})^2 + (1 + T \omega_c^{\alpha} \cos \frac{\alpha \pi}{2})^2} = 0, \tag{15}
\]
another relationship between \(K_{d3}\) and \(\mu\) can be obtained in the following form,
\[
A_3 \omega_c^2 K_{d3}^2 = \mu K_{d3} + A_3 = 0, \tag{16}
\]
that is
\[
K_{d3} = \frac{\mu \pm \sqrt{\mu^2 - 4A_3^2 \omega_c^2}}{2A_3 \omega_c^2}, \tag{17}
\]
where
\[
A_3 = \frac{\alpha T \omega_c^{\alpha-1} \sin \frac{\alpha \pi}{2}}{(T \omega_c^{\alpha} \sin \frac{\alpha \pi}{2})^2 + (1 + T \omega_c^{\alpha} \cos \frac{\alpha \pi}{2})^2}.
\]
According to Specification 2.3.3, we can establish an equation about \(K_p, K_{d3}\) and \(\mu\).
\[
|G_3(j\omega)| = |C_3(j\omega)| |P(j\omega_c)| = \frac{K_p(1 + (K_{d3}\omega_c)^2)^{\frac{\lambda}{2}}}{N} = 1, \tag{18}
\]
where
\[
N = \sqrt{(T \omega_c^{1+\alpha} \sin \frac{\alpha \pi}{2})^2 + (\omega_c + T \omega_c^{1+\alpha} \cos \frac{\alpha \pi}{2})^2}.
\]
Clearly, we can solve Eqs. (14), (17) and (18) to get \(K_p, K_{d3}\) and \(\mu\).

3.2. Design procedure summary

The graphical method can be used as a practical and simple way to get \(K_{d3}\) and \(\mu\). The procedure to tune the parameters of the FO-[PD] controller is as follows:

1. Given parameters of the fractional order system to be controlled \(\alpha\) and \(T\).
2. Given \(\omega_c\), the gain crossover frequency;
3. Given \(\phi_m\), the desired phase margin;
4. Plot the curve 1, \(K_{d3}\) w.r.t \(\mu\), according to (14);
5. Plot the curve 2, \(K_{d3}\) w.r.t \(\mu\), according to (17);
6. Obtain the \(K_{d3}\) and \(\mu\) from the intersection point on the above two curves;

3.3. Design example and Bode plot validation of the FO-[PD] controller

The time constant \(T\) in (1) is 0.4 s and the fractional order is \(\alpha = 1.4\). The control design specifications of interest are set as \(\omega_c = 10\) (rad/s), \(\phi_m = 70^\circ\). According to (14) and (17), the desired values of \(K_{d3}\) and \(\mu\) can be obtained obviously from the intersection point on the two curves, \(K_{d3} = 0.9435\) and \(\mu = 1.205\). Then \(K_{p3}\) can be calculated from (18), that is \(K_{p3} = 6.3092\). The Bode diagram of system designed is shown in Fig. 1. As can be seen, the gain crossover frequency specification, \(\omega_c = 10\) (rad/s), and phase margin specification, \(\phi_m = 70^\circ\), are fulfilled, and the phase is forced to be flat at \(\omega_c\) based on the proposed design.

4. Integer order PID controller and fractional order PD controller designs

The IO-PID controller and FO-PD controller designs also follow the same three specifications in Section 2.3 with the similar procedure as in Section 3.2 for the FO-[PD] controller design. The FOS to be controlled is the same with \(T = 0.4\) s and \(\alpha = 1.4\).

Following the FO-[PD] tuning formula derivation, similar tuning formulae for both IO-PID and FO-PD can be obtained. The three parameters of the IO-PID controller in (2) are designed as \(K_p = 18.2984, K_i = 42.45, K_d = -0.0846\); and the three parameters of the FO-PD controller in (3) are \(K_{p2} = 10.916, K_{d2} = 0.6138, \lambda = 1.189\). The Bode plots of systems designed using the IO-PID controller and FO-PD controller are shown in Figs. 2 and 3 respectively. As also can be seen, in both of the two Bode plots, the three specifications in Section 2.3 are fulfilled.

5. Simulation comparisons

The tuning procedures above for the three controllers are illustrated via numerical simulation in this section.

The fractional order operators \(s^\alpha\) for FO-PD controller and \((1 + rs)^\alpha\) for FO-[PD] controller are implemented by the impulse response invariant discretization methods (Chen, 2008a,b) in time domain, as introduced in Luo and Chen (under revision).

The approximate finite dimensional discretized (z) transfer function of \(s^\lambda\) (\(\lambda = 1.189\)) has the following form with sampling period 0.01 s,
\[
s^\lambda 189 \approx \frac{N_1}{D_1},
\]
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\]
where
\[ \begin{align*}
N_1 &= z^5 - 3.423z^4 + 4.428z^3 - 2.642z^2 + 0.6928z - 0.05711, \\
D_1 &= 0.003573z^5 - 0.007686z^4 + 0.005441z^3 - 0.00145z^2 \\
&\quad + 0.0001734z - 2.468 \times 10^{-5},
\end{align*} \]
and the approximate finite dimensional discretized (z) transfer function of the operator \((1 + Kz^\mu)^\mu (K \equiv 0.9435, \mu = 1.205)\) has the form below with sampling period 0.01 s,
\[
(1 + 0.9435z) \approx \frac{N_2}{D_2}.
\]

where
\[
\begin{align*}
N_2 &= z^5 - 3.392z^4 + 4.348z^3 - 2.572z^2 + 0.6685z - 0.05469 \\
D_2 &= 0.003513z^5 - 0.007414z^4 + 0.005137z^3 - 0.001348z^2 \\
&\quad + 0.0001721z - 2.599 \times 10^{-5}.
\end{align*}
\]

The FO system parameters and the control design specifications are the same as in the previous section. The parameters in the IO-PID, FO-PD and FO-[PD] controllers are already calculated in Sections 3 and 4, respectively. However, note that, since \(K_d = -0.0846 < 0\), the system using the designed IO-PID controller is unstable. So, we cannot obtain a properly designed IO-PID controller which can guarantee the closed-loop stability and achieve the flat-phase specification at the interested gain crossover frequency. This signifies the potential benefit of using a fractional order controller over an integer order controller.

In Fig. 4, applying the FO-PD controller, the unit step responses are plotted with the open-loop plant gain varying from 8.733 to 13.099 (±20% variations from the desired value 10.916). In Fig. 5, applying FO-[PD] controller, the unit step responses are plotted with open-loop plant gain changing from 5.0474 to 7.5710 (±20% variations from desired value 6.3092).

It can be seen from Figs. 4 and 5 that both the FO-PD and the FO-[PD] controllers designed following the proposed method in this paper are effective. The overshoots of the step responses remain almost constant under gain variations, i.e. the iso-damping property is exhibited, that means the system is robust to gain changes in the system. Furthermore, from Fig. 6, it can be seen obviously that the over shoot of the red line with the proposed FO-[PD] controller is almost zero, and much smaller than that of the blue line with the FO-PD controller. So, we can see that the FO-[PD] controller outperforms the FO-PD controller.

6. Conclusions

In this paper, we focus on a certain type of simplified model of fractional order systems. A fractional order [proportional derivative] controller is proposed for this class of FOS, and a practical and systematic tuning procedure has been developed for the proposed FO-[PD] controller synthesis. The fairness issue in comparing with other controllers such as the traditional integer order PID controller and the fractional order proportional derivative controller, all with three controller parameters, has been addressed under the same design specifications. Fair comparisons of the three controllers (i.e., IO-PID, FO-PD and FO-[PD]) via the simulation tests illustrate that, the IO-PID controller designed may not always be stabilizing to achieve flat-phase specification while both FO-PD and FO-[PD] controllers...
designed are always stabilizing. Furthermore, the proposed FO-PD controller outperforms FO-PD controller for the class of fractional order systems.

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