Abstract

Absolute deviation is utilized as a measure of risk and a new function is provided for it. We consider the mean-absolute deviation (MAD) portfolio optimization problem in a frictional market with additional constraints representing the so-called short sales. An algorithm for solving the optimization problem is thus presented, which uses the special structure of the original problem to reduce to a linear programming. The numerical test shows the validity of the method.

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Keywords: Portfolio selection; MAD model; Optimization

1. Introduction

In securities market, the distribution of return sometimes appears ‘Lepto-kurtosis’ and ‘fat-tails’. For this purpose, some functions have been proposed to measure the risk of fluctuation of the rate of return. Konno and Yamazaki [3] proposed MAD model and absolute deviation was utilized as a measure of risk, which is rather anti-interference. That is, it is insensitive to some extremes, which can be the source of errors. And MAD model can be suitable under any distributions. Although MAD model cannot give an analysis expression as MV model which was proposed by Markowitz [6] does, it can measure risk appropriately, and is now widely used in large-scale portfolio optimization because it can be reduced to a linear programming problem instead of a quadratic programming problem in the case of MV model [3,4].

In this paper, we investigate a frictional market and consider a portfolio selection problem under MAD model. In practice, the frictions are sometimes described as nonsmooth functions, which can not be given the efficient frontier and an obvious analysis. For searching for the optimal solution, the special structure of the problem is utilized and be reduced to a linear programming. Thus the simplex method can be used and be performed easily.

In the following section, we will formulate the portfolio optimization problem in a frictional market assuming that the risk function be given by the absolute deviation of the rate of return. In Section 3, we present an algorithm to the model given in Section 2. Section 4 is devoted to the comparison between MAD and MV, and Section 5 gives a numerical analysis which tests the validity of our method.
The MAD model without frictions was first proposed by [3] as an alternative to the classical MV model [6,7]

\[
\begin{align*}
(P_1) \quad &\max f(x) = \int x r P(r) \, dr, \\
&\min g(x) = \int x |r - m| P(r) \, dr, \\
&\text{s.t. } \sum_{i=1}^n x_i = 1,
\end{align*}
\]

where \( x \) represents a portfolio held by an investor, \( r \) the rate of return of assets, and \( m \) the expected value of \( r \), \( P(r) \) the probability of \( r \).

Recently, the methods for solving \((P_1)\) have been the topic of intense researches, seeing for instance [2,4,10,12]. It has been demonstrated in [3] that MAD model can generate an optimal portfolio similar with MV model when the rate of return obey Gaussian Distribution. In this paper, the transaction costs were expressed as a nonlinear function of \( x \), and considering the cluster effect of the volatility, a new function for absolute deviation was presented.

2. MAD model in a frictional market

We consider a frictional capital market with \( n \) risky assets, which is defined to be a security or a feasible portfolio of securities whose return over the period is known with certainty. An investor allocates his/her wealth among \( n \) risky assets. Throughout the paper, we denote

- \( X_i \) the amount of investment of the \( i \)th asset, \( i = 1, \ldots, n \)
- \( r_{it} \) the random variable representing the rate of return of \( i \)th asset at observation point \( t, \ t = 1, \ldots, T \)
- \( r_i \) the expected rate of return of \( i \)th asset, \( r_i = \frac{1}{\tau} \sum_{t=1}^{\tau} r_{it}, i = 1, \ldots, n \)
- \( x_i \) the proportion of the fund to be allocated to \( i \)th asset, \( x_i = X_i / \sum_i X_i, i = 1, \ldots, n \)
- \( x^0 \) the optimal portfolio held by the investor during the exchange, \( x = (x_1, \ldots, x_n) \)
- \( x^0 \) the initial portfolio held by an investor
- \( t_g \) tax rate of marginal enterprise income
- \( t_t \) commission rate of transaction, which is supposed to be constant during the exchange
- \( t_s \) stamp tax rate of transaction

We now give some assumptions.

**Assumption 2.1.** The investor is the representative of an enterprise.

**Assumption 2.2.** There exit two frictions: enterprise income tax and transaction costs. The latter include commission \( c_1(x) \) and stamp tax \( c_2(x) \).

**Assumption 2.3.** Both transactions costs are the \( V \) function of the volume of transactions

\[
\begin{align*}
c_1(x) &= t_f \sum_{i=1}^n |x_i - x_i^0|, \\
c_2(x) &= t_s \sum_{i=1}^n |x_i - x_i^0|.
\end{align*}
\]

**Assumption 2.4.** All assets are infinitely divisible and short sales be allowed.

**Assumption 2.5.** The investor makes his decision at the beginning of the period and is not allowed to revise his decisions until the end of the period.

**Assumption 2.6.** The stamp tax can be eliminated pre-tax as period expense, while commission cannot since it be regarded as investment cost.
Then the total pre-tax profit \( f_1(x) \) of portfolio \( x \) in a frictional market is given by
\[
\begin{align*}
  f_1(x) &= \sum_{i=1}^{n} r_{it}x_i - (c_1(x) + c_2(x)), \quad t = 1, \ldots, T.
\end{align*}
\]

By Assumption 2.4, we obtain the after-tax profit \( f_2(x) \) as the following:
\[
\begin{align*}
  f_2(x) &= (1 - t_g) \sum_{i=1}^{n} (r_{it}x_i - c_2(x)) - c_1(x) \\
  &= (1 - t_g) \sum_{i=1}^{n} r_{it}x_i - [(1 - t_g)t_s + t_f] \sum_{i=1}^{n} |x_i - x_i^0|, \quad t = 1, \ldots, T.
\end{align*}
\]

**Definition 2.1.** Let \( t_0 = (1 - t_g)t_s + t_f \) be a friction factor. By Definition 2.1, it yields
\[
\begin{align*}
  f_2(x) &= (1 - t_g) \sum_{i=1}^{n} r_{it}x_i - t_0 \sum_{i=1}^{n} |x_i - x_i^0|, \quad t = 1, \ldots, T.
\end{align*}
\]

The expected value of after-tax profit \( f(x) \) is as follows:
\[
\begin{align*}
  f(x) &= E(f_2(x)) = \frac{1}{T} \sum_{t=1}^{T} f_2(x) = (1 - t_g) \sum_{i=1}^{n} r_{it}x_i - t_0 \sum_{i=1}^{n} |x_i - x_i^0|.
\end{align*}
\]

Intense researches have been made on the function of absolute deviation. Cai et al. [1] proposed a \( l_{\infty} \) function to measure risk, while Teo and Yang [8] presented \( H^T_{\infty} \) function. Both their models minimized the expected absolute deviation of the future returns from their mean. However the latter divided the observation point into \( T \) periods. Based on \( H^T_{\infty} \) function, and considering the cluster effect of the volatility, we divide \( T \) into \( H \) period, for example, weekly, monthly, or seasonally. That is, the rate of returns in the same period have similar volatility. Then another absolute deviation \( g(x) \) is employed as to measure the risk.

**Definition 2.2.** The absolute deviation of the rate of return is defined as
\[
\begin{align*}
  g(x) &= \frac{1 - t_g}{H} \sum_{h=1}^{H} \frac{1}{K} \sum_{k=1}^{K} \left| \sum_{i=1}^{n} (r_{ihk} - r_{ikh})x_i \right| = \frac{1 - t_g}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ikh})x_i, \quad (2.5)
\end{align*}
\]
where \( T = H \times K \). The efficient frontier of an optimal portfolio has maximum expected value and minimum risk. Thus the mean-absolute deviation (MAD) model in a frictional market is defined as the following bi-objective problem
\[
\begin{align*}
  (P_2) \quad & \max f(x) = (1 - t_g) \sum_{i=1}^{n} r_{it}x_i - t_0 \sum_{i=1}^{n} |x_i - x_i^0| \\
  & \min g(x) = \frac{1 - t_g}{K} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ikh})x_i \\
  & \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1, \\
  & \sum_{i=1}^{n} |x_i| \leq 1 + 2a,
\end{align*}
\]
where \( a \geq 0 \). The second constraint has the restriction on the total sum of all short sales [9]. Denote
\[
\begin{align*}
  X = \left\{ x : \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} |x_i| \leq 1 + 2a \right\}.
\end{align*}
\]
Then the above problem \((P_2)\) is equivalent to the following problem
\[
(P_3) \quad \max_{x \in X} (f(x), -g(x)).
\]

To choose an optimal portfolio strategy, we need to find an optimal solution of \((P_3)\).

3. Algorithm and related theorems

For solving \((P_3)\), we introduce a weight \(\omega\) to make a tradeoff between the two objectives
\[
(P_4) \quad \max_{x \in X} (1 - \omega)f(x) - \omega g(x),
\]
where the parameter \(\omega(0 \leq \omega \leq 1)\) can be interpreted as the risk aversion factor of the investor. The greater the factor \(\omega\) is, the more risk aversion the investor has. For solving \((P_4)\), we define
\[
(P_5) \quad \max \quad (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega z \right] - t_0 y
\]
\[
s.t. \quad \sum_{i=1}^{n} x_i = 1,
\]
\[
\sum_{i=1}^{n} |x_i| \leq 1 + 2\alpha,
\]
\[
y \geq \sum_{i=1}^{n} |x_i - x_i^0|,
\]
\[
z \geq \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ih}) x_i.
\]

**Theorem 3.1.** Let \((x_1^*, \ldots, x_n^*)\) be an optimal solution to \((P_4)\). It has the sufficient and necessary condition: \((x_1^*, \ldots, x_n^*, y^*, z^*)\) is the optimal solution to \((P_5)\).

**Proof.** If \((x_1^*, \ldots, x_n^*)\) is an optimal solution to \((P_4)\), evidently, \((x_1^*, \ldots, x_n^*, y^*, z^*)\) be a feasible solution to \((P_5)\), where
\[
y^* = \sum_{i=1}^{n} |x_i^* - x_i^0|,
\]
\[
z^* = \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ih}) x_i^*.
\]

If \((x_1^*, \ldots, x_n^*, y^*, z^*)\) is not the optimal solution to \((P_3)\), then we can find a feasible solution \((x_1, \ldots, x_n, y, z)\) satisfying
\[
(1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega z \right] - t_0 y \leq (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i^* - \omega z^* \right] - t_0 y^*.
\]

By \(y \geq \sum_{i=1}^{n} |x_i - x_i^0|\), \(z \geq \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ih}) x_i\), we have
\[
(1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ih}) x_i \right] - t_0 \sum_{i=1}^{n} |x_i - x_i^0| > (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ihk} - r_{ih}) x_i \right] - t_0 \sum_{i=1}^{n} |x_i^* - x_i^0|,
\]

which conflicts to \((x_1^*, \ldots, x_n^*)\) being an optimal solution to \((P_4)\).

If \((x_1^*, \ldots, x_n^*, y^*, z^*)\) is an optimal solution to \((P_5)\), obviously \((x_1^*, \ldots, x_n^*)\) be a feasible solution to \((P_4)\). If it is not the optimal solution, thus there exists another feasible solution \((x_1, \ldots, x_n)\) to \((P_4)\) satisfying
\[
(1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \frac{\alpha}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i \right] - t_0 \sum_{i=1}^{n} |x_i - x_i^0| \\
> (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i^* - \frac{\alpha}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i^* \right] - t_0 \sum_{i=1}^{n} |x_i^* - x_i^0|.
\]

Define \( y = \sum_{i=1}^{n} |x_i - x_i^0|, \quad z = \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i^0, \) and note \( y^* \geq \sum_{i=1}^{n} |x_i^* - x_i^0|, \quad z^* \geq \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i^0, \) then
\[
(1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega z \right] - t_0 y > (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i^* - \omega z^* \right] - t_0 y^*,
\]

which contradicts to \((x_1^*, \ldots, x_n^*, y^*, z^*)\) being an optimal solution to \((P_5)\). The proof is completed. For simplicity, we first introduce three sets of nonnegative variables \( x_i^+, x_i^-(i = 1, \ldots, n), y_i^+, y_i^-(i = 1, \ldots, n) \) and \( z_{hk}, z_{hk}^{-}(t = 1, \ldots, T) \) satisfying the following conditions:

\[
\begin{align*}
  x_i^+ - x_i^- & = x_i; & x_i^+ x_i^- & = 0; & x_i^+ & \geq 0; & x_i^- & \geq 0; & i &= 1, \ldots, n, \\
  y_i^+ - y_i^- & = y_i; & y_i^+ y_i^- & = 0; & y_i^+ & \geq 0; & y_i^- & \geq 0; & i &= 1, \ldots, n, \\
  z_{hk}^- - z_{hk}^+ & = \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i; & z_{hk}^+ z_{hk}^- & = 0; & z_{hk}^+ & \geq 0; & z_{hk}^- & \geq 0; & h = 1, \ldots, H, & k = 1, \ldots, K.
\end{align*}
\]

The nonsmooth functions in \((P_5)\) are equivalent to

\[
\begin{align}
  |x_i| & = x_i^+ + x_i^-, & i & = 1, \ldots, n, \\
  |x_i - x_i^0| & = y_i^+ + y_i^-, & i & = 1, \ldots, n, \\
  \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i & = z_{hk}^+ + z_{hk}^-; & h = 1, \ldots, H, & k = 1, \ldots, K.
\end{align}
\]

Therefore, \((P_5)\) can be rewritten as follows:

\[
(P_6) \quad \max \quad (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega z \right] - t_0 y \\
\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1, \\
\sum_{i=1}^{n} (x_i^+ + x_i^-) \leq 1 + 2a, \\
x_i^+ - x_i^- = x_i; \quad x_i^+ x_i^- = 0; \quad x_i^+ \geq 0; \quad x_i^- \geq 0; \quad i = 1, \ldots, n, \\
y \geq \sum_{i=1}^{n} (y_i^+ + y_i^-), \\
y_i^+ - y_i^- = y_i; \quad y_i^+ y_i^- = 0; \quad y_i^+ \geq 0; \quad y_i^- \geq 0; \quad i = 1, \ldots, n, \\
z \geq \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} (z_{hk}^+ + z_{hk}^-), \\
z_{hk}^+ - z_{hk}^- = \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i; \quad z_{hk}^+ z_{hk}^- = 0; \quad z_{hk}^+ \geq 0; \quad z_{hk}^- \geq 0, \\
h = 1, \ldots, H, & k = 1, \ldots, K.
\]
Eliminating the complementarity constraints from (P₆), we have

\begin{align*}
(P₇) \quad \max & \quad (1 - t_\gamma)(1 - \omega)\sum_{i=1}^{n} r_i x_i - \omega z - t_0 y \\
\text{s.t.} & \quad \sum_{i=1}^{n} x_i = 1, \\
& \quad \sum_{i=1}^{n} (x_i^+ + x_i^-) \leq 1 + 2a, \\
& \quad x_i^+ - x_i^- = x_i; \quad x_i^+ \geq 0; \quad x_i^- \geq 0, \quad i = 1, \ldots, n, \\
& \quad y \geq \sum_{i=1}^{n} (y_i^+ + y_i^-), \\
& \quad y_i^+ - y_i^- = x_i - x_i^0; \quad y_i^+ \geq 0; \quad y_i^- \geq 0, \quad i = 1, \ldots, n, \\
& \quad z \geq \frac{1}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} (z_{hk}^+ + z_{hk}^-), \\
& \quad z_{hk}^+ - z_{hk}^- = \sum_{i=1}^{n} (r_{ihk} - r_{ik}) x_i; \quad z_{hk}^+ \geq 0; \quad z_{hk}^- \geq 0, \\
& \quad h = 1, \ldots, H, \quad k = 1, \ldots, K. \quad \Box
\end{align*}

**Theorem 3.2.** Let \( \Omega' = (x^*, x_i^+, x_i^-, y^*, y_i^+, y_i^-, z^*, z_{hk}^+, z_{hk}^-) \) be an optimal solution to (P₇). Then we can have \( \Omega^* = (x^*, x_i^*, x_i^+, y^*, y_i^+, y_i^-, z^*, z_{hk}^+, z_{hk}^-) \) be the optimal solution to (P₆), where

\begin{align*}
(x_i^*, x_i^-) &= (x_i^{\infty} - x_i^0, 0), \quad \text{if } x_i^{\infty} \geq x_i^0 \geq 0, \quad i = 1, \ldots, n, \\
(x_i^*, x_i^-) &= (0, x_i^{\infty} - x_i^0), \quad \text{if } x_i^0 \geq x_i^{\infty} \geq 0, \quad i = 1, \ldots, n, \\
(y_i^*, y_i^-) &= (y_i^{\infty} - y_i^0, 0), \quad \text{if } y_i^{\infty} \geq y_i^0 \geq 0, \quad i = 1, \ldots, n, \\
(y_i^*, y_i^-) &= (0, y_i^{\infty} - y_i^0), \quad \text{if } y_i^0 \geq y_i^{\infty} \geq 0, \quad i = 1, \ldots, n, \\
(z_{hk}^+, z_{hk}^-) &= (z_{hk}^{\infty} - z_{hk}^0, 0), \quad \text{if } z_{hk}^{\infty} \geq z_{hk}^0 \geq 0, \quad h = 1, \ldots, H, \quad k = 1, \ldots, K, \\
(z_{hk}^+, z_{hk}^-) &= (0, z_{hk}^{\infty} - z_{hk}^0), \quad \text{if } z_{hk}^0 \geq z_{hk}^{\infty} \geq 0, \quad h = 1, \ldots, H, \quad k = 1, \ldots, K.
\end{align*}

**Proof.** Evidently, both the objective functions in (P₆) and (P₇) are same. However, \( \Omega^* \) satisfies all the complementarity constraints automatically. If \( \Omega' \) is not the optimal solution to (P₆), then there exists another feasible solution \( \Omega = (x, x_i^*, x_i^+, y, y_i^+, y_i^-, z, z_{hk}^+, z_{hk}^-) \) to (P₆), satisfying

\begin{align*}
(1 - t_\gamma)(1 - \omega)\sum_{i=1}^{n} r_i x_i - \omega z - t_0 y > (1 - t_\gamma)(1 - \omega)\sum_{i=1}^{n} r_i x_i^* - \omega z^* - t_0 y^*.
\end{align*}

Obviously, \( \Omega \) is a feasible solution to (P₇), thus we have

\begin{align*}
(1 - t_\gamma)(1 - \omega)\sum_{i=1}^{n} r_i x_i - \omega z - t_0 y \leq (1 - t_\gamma)(1 - \omega)\sum_{i=1}^{n} r_i x_i^* - \omega z^* - t_0 y^*,
\end{align*}

which yields a contradiction. Then \( \Omega^* \) be the optimal solution to (P₆). This completes the proof of the theorem. \( \Box \)

**Theorem 3.3.** The nonnegative variables \( z_{hk}^+, h = 1, \ldots, H, \quad k = 1, \ldots, K \) can be eliminated from (P₇).

**Proof.** From above, we have

\begin{align*}
\sum_{h=1}^{H} \sum_{k=1}^{K} (z_{hk}^+ - z_{hk}^-) &= \sum_{h=1}^{H} \sum_{k=1}^{K} (r_{ihk} - r_{ik}) x_i = \sum_{i=1}^{n} x_i \sum_{h=1}^{H} \sum_{k=1}^{K} (r_{ihk} - r_{ik}) = 0.
\end{align*}
Thus, \( \sum_{h=1}^{H} \sum_{k=1}^{K} z_{hk}^+ = \sum_{h=1}^{H} \sum_{k=1}^{K} z_{hk}^- \). Since \( z_{hk}^- \) is nonnegative variables, and based on some transformation, we can eliminate it from \((P_7)\). The proof is completed. By Theorems 3.2 and 3.3, \((P_7)\) can be reduced to a linear programming as the following:

\[
(P_8) \quad \max \quad (1 - t_g) \left[ (1 - \omega) \sum_{i=1}^{n} r_i x_i - \omega z \right] - t_0 y \\
\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1, \\
\sum_{i=1}^{n} (x_i^+ + x_i^-) \leqslant 1 + 2a, \\
x_i^+ - x_i^- = x_i; \quad x_i^+ \geqslant 0; \quad x_i^- \geqslant 0, \quad i = 1, \ldots, n, \\
y \geqslant \sum_{i=1}^{n} (y_i^+ + y_i^-), \\
y_i^+ - y_i^- = x_i - x_i^0; \quad y_i^+ \geqslant 0; \quad y_i^- \geqslant 0, \quad i = 1, \ldots, n, \\
z \geqslant \frac{2}{T} \sum_{h=1}^{H} \sum_{k=1}^{K} z_{hk}^+, \\
z_{hk}^+ \geqslant \sum_{i=1}^{n} (r_{ih} - r_{ih}) x_i; \quad z_{hk}^- \geqslant 0, \quad h = 1, \ldots, H, \quad k = 1, \ldots, K. \quad \square
\]

4. Comparison to MV model

The MV model (no short sales allowed) in a frictional market [5] is as follows:

\[
(P_9) \quad \max \quad f(x) = (1 - t_g) \sum_{i=1}^{n} r_i x_i - t_0 \sum_{i=1}^{n} |x_i - x_i^0| \\
\min \quad g(x) = (1 - t_g)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_i x_j, \\
\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1, \\
\sum_{i=1}^{n} |x_i| \leqslant 1 + 2a,
\]

where \( C_{ij} \) be the covariance between variables \( r_i \) and \( r_j \) \((i, j = 1, \ldots, n)\). Evidently, by doing the same transformation as above, \((P_9)\) can be written as the following:

\[
(P_{10}) \quad \max \quad f(x) = (1 - \omega) \left[ (1 - t_g) \sum_{i=1}^{n} r_i x_i - t_0 \sum_{i=1}^{n} y \right] - \omega (1 - t_g)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_i x_j \\
\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1, \\
\sum_{i=1}^{n} (x_i^+ + x_i^-) \leqslant 1 + 2a, \\
x_i^+ - x_i^- = x_i; \quad x_i^+ \geqslant 0; \quad x_i^- \geqslant 0, \quad i = 1, \ldots, n, \\
y \geqslant \sum_{i=1}^{n} (y_i^+ + y_i^-), \\
y_i^+ - y_i^- = x_i - x_i^0; \quad y_i^+ \geqslant 0; \quad y_i^- \geqslant 0, \quad i = 1, \ldots, n,
\]

which can be solved by some algorithms for quadratic programming.
Comparing MAD model with MV, we can acknowledge that although MAD model has larger variables than the latter, it can be reduced to a linear programming which can be solved by simplex method for large scale problem. While MV model needs computing the covariance of returns which be rather complex. Furthermore, it has been proved that the efficient frontier under MAD model be much better than MV, and for details, see [11].

5. Numerical test

Consider a frictional market with tax and commission. Given $t_g = 0.3$, $t_s = 0.001$, $t_f = 0.0035$ and set $x^0 = 0$. We conducted numerical tests of the algorithm proposed in this paper using yearly data of 9 stocks from 1937 to 1954, which is listed in Table 1.

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<tbody>
<tr>
<td>1937</td>
<td>-0.0305</td>
<td>-0.173</td>
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Fig. 1. Efficient frontier for MAD and MV.
The parameter in the constraint on short sale is given as $a = 0.5$, and considering the cluster effect of the volatility, the 18 points are divided into six periods, i.e. $T = 18$, $H = 6$, $K = 3$. The efficient frontiers for MAD and MV model are depicted in Fig. 1. It is evidently that the MAD model has a similar efficient frontier with MV. Under some cases, such as the investor risk aversion, the MAD exhibits better, and vice versa.

Figs. 2 and 3 exhibit the return of portfolio for history data under different risk averse factor $\omega$, from which we can see that when the investor is risk aversion, i.e. $\omega = 0.9$, the MAD model has rather lower volatility than MV. However, if $\omega = 0.1$, which means the investor preferring risk, the MAD model has a similar appearance with MV.

6. Conclusion

It was shown in this paper that the portfolio construction problem in a frictional market with limitation on short sales can be solved in a practical way. The success depends upon the use of MAD model and the problem
reduction strategy using the special structure of the problem. Considering the cluster effect of volatility, we provide a new function for absolute deviation as to measure risk. Instead of computing the covariance of asset returns under MV model, we can use simplex method for our algorithm to solve the problem of practical size in an efficient manner. From the numerical test, we can acknowledge that the MAD model has a better efficient frontier.

Acknowledgements

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References